Chapter 8

Prescribed-time optimal control of nonlinear systems

8.1 Introduction

When designing a controller for a nonlinear system, our primary concerns are stability and fast convergence. However, using a large control input can be costly and cause saturation of the actuators, leading to instability in the closed-loop system. To address these issues, various techniques have been developed to minimize control efforts while maintaining stability margins. One such technique is discussed in [94], which proposes an infinite-horizon optimal stabilization strategy for a nonlinear system with asymptotic stability. The general approach offers the groundwork for expanding linear quadratic control to nonlinear nonquadratic control issues. Currently, the maximum principle is the sole way to generate optimal finite-time controllers, which typically does not result in feedback controllers [15]. As an extension of [94], [99] studies the optimal finite-time stability and stabilization problem in this sense. The designed controllers are feedback controllers since the results were obtained on the framework established in [94]. In particular, a nonlinear, nonquadratic performance functional-based feedback control problem with an indefinite horizon is taken into account.

As a further development of the concepts discussed in [15,16,43,94,99], this chapter tackles the issue of optimal prescribed-time control design.

The main contributions of this chapter are summarized as follows:

(i) First, sufficient conditions for prescribed-time stability using continuous Lyapunov candidate functions are given.

(ii) Different from the finite-time and predefined-time optimal control formation in [99] and [16], a new optimal prescribed-time stabilization method that utilizes certain differential inequalities, which satisfies the Hamilton-Jacobi-Bellman steady-state equation and ensures both optimality and prescribed-time stability is proposed.

(iii) Specifically the challenge of identifying a control law that minimizes a particular performance index while simultaneously ensuring stability of the given nonlinear system within a priori chosen time.

(iv) We look at how our methodology relates to the inverse optimal control design, where we parameterize a set of optimal prescribed time stabilizing control laws which minimize a derived cost functional containing subquadratic components.

8.2 Preliminaries

8.2.1 On optimal nonlinear control

Let us consider the controlled nonlinear system described by (2.1). Our objective is to derive a feedback control law that minimizes the given performance functional, which is expressed as

$$J\left(t_0, z_0, u(\cdot)\right) = \int_{t_0}^{\infty} \mathcal{L}(t, z(t), u(t)) \mathrm{dt} \tag{8.1}$$

where, $u(\cdot) \in \mathcal{U}$ represent the set of admissible control and $\mathcal{L} : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is convex in $u(t)$ and continuous in $z(t)$.

Now, let us define the minimum cost function as $J^*(z(t))$, which is given by:

$$J^*(z(t)) = \min_{u(\cdot)\in\mathcal{U}} J\left(t_0, z_0, u(\cdot)\right)$$

Then, the Hamiltonian, for the costate $p \in \mathbb{R}^n$ is given as

$$\mathcal{H}(t, z(t), u(t), p) = \mathcal{L}(t, z(t), u(t)) + p^\top f(t, z(t), u(t))$$

Finally, the steady-state equation of the HJB equation can be expressed as follows:

$$\min_{u(\cdot)\in\mathcal{U}} \mathcal{H}\left(t, z(t), u(t), \frac{\partial {J^*}^\top(z(t))}{\partial z}\right) = 0$$

which gives sufficient conditions for optimality [15].

8.3 Main results

8.3.1 Optimal prescribed-time stabilization

In this section, an optimal prescribed time feedback controller is proposed in order to minimize the amount of control input required to maintain the stability of (2.1) within a priori chosen time.

We consider a restriction on the control $u(\cdot)$ in (2.1), limiting it to the class of admissible controls. These admissible controls consist of measurable functions $u(\cdot)$ such that for all $t \geq t_0$, $u(t) \in \mathcal{U}$, where $\mathcal{U}$ represents the set of admissible control values. Next, we define the set of prescribed-time feedback controllers denoted by $\mathcal{S}(z_0)$. This set includes all admissible controls $u(\cdot)$ that satisfy the following two conditions: (1) the control function $u(\cdot)$ is admissible as per the defined class of admissible controls, and (2) the state trajectory $z(\cdot)$ governed by (2.1) converges to zero as t approaches $t_p + t_0$.

The following theorem presents sufficient conditions for a controller to address the aforementioned issue:

Theorem 8.1 *Consider the nonlinear system* (2.1) *with the performance functional* (8.1)*. Suppose there exists a positive definite continuously differentiable function* $V : [t_o, \infty] \times \mathcal{D} \to \mathbb{R}{\geq}\, 0$*, a design constant* $\gamma \in \mathbb{R}{>}\, 1$*, and* $t_p > t_0$ *such that the following conditions hold:*

$$V(t,z) > 0, \quad z(t) \in \mathcal{D} \setminus 0 \tag{8.2}$$

$$V(t,0) = 0 \tag{8.3}$$

$$\psi(0) = 0 \tag{8.4}$$

$$\frac{\partial V}{\partial z} f(t, z(t), \psi(z)) \leq -\gamma \frac{V(z)}{(t_p + t_0 - t)}, \quad z(t) \in \mathcal{D} \setminus 0 \tag{8.5}$$

$$\mathcal{H}\left(t, z(t), \psi(z), \frac{\partial V}{\partial z}^{\top}\right) = \mathcal{L}(t, z(t), \psi(z)) + \frac{\partial V}{\partial z} f(t, z(t), \psi(z)) = 0 \tag{8.6}$$

$$\mathcal{H}\left(t, z(t), u(t), \frac{\partial V}{\partial z}^{\top}\right) = \mathcal{L}(t, z(t), u(t)) + \frac{\partial V}{\partial z} f(t, z(t), u(t)) \geq 0, \quad \forall u(t) \in \mathcal{U} \tag{8.7}$$

Then, by utilizing the control law:

$$u(\cdot) = \psi(z(\cdot)) = \underset{u \in \mathcal{U}}{\operatorname{argmin}}\, \mathcal{H}\left(t, z(t), u(t), \frac{\partial V}{\partial z}^{\top}\right), \tag{8.8}$$

the zero solution $z(t) \equiv 0$ *of the closed-loop nonlinear system:*

$$\dot{z}(t) = f(t, z(t), \psi(z)) \tag{8.9}$$

is prescribed-time stable, and there exists a vicinity of the origin such that:

$$J\left(t_0, z_0, \psi(\cdot)\right) = V(t_0, z_0). \tag{8.10}$$

Moreover, the proposed control law (8.8) *minimizes the performance functional in the sense that:*

$$J\left(t_0, z_0, \psi(\cdot)\right) = \min_{u \in \mathcal{U}} J\left(t_0, z_0, u(\cdot)\right). \tag{8.11}$$

Furthermore, the control law proposed in equation (8.8) *successfully addresses the problem of optimal prescribed-time stabilization for the nonlinear system described by equation* (2.1)*.*

Proof. By applying Lemma 4.2 to the closed-loop system (8.9), we can directly establish the prescribed-time stability based on the conditions (8.2), (8.3), and (8.5).

To prove (8.10), let $z(t)$ be the solution of the closed-loop system (8.9). For all $t \geq t_0$, we have $\frac{\partial V}{\partial z} f(t, z(t), \psi(z)) - \dot{V}(t, z) = 0$. Further, using (8.7), we can express it as:

$$\begin{aligned} \mathcal{L}(t, z(t), \psi(z)) &= \frac{\partial V}{\partial z} f(t, z(t), \psi(z)) - \dot{V}(t, z) + \mathcal{L}(t, z(t), \psi(z)) \\ &= -\dot{V}(t, z) \end{aligned} \tag{8.12}$$

Hence,

$$J\left(t_0, z_0, \psi(\cdot)\right) = -\int_{t_0}^{\infty} \dot{V}(t, z)\mathrm{dt} \tag{8.13}$$

Using (8.3) it follows from (8.12) that $J\left(t_0, z_0, \psi(\cdot)\right) = -V(\lim_{t\to\infty} z(t)) + V(t_0, z_0)$. Since $\lim_{t\to t_p+t_0} z(t) = \lim_{t\to\infty} z(t) = 0$, further, we get $J\left(t_0, z_0, \psi(\cdot)\right) = V(t_0, z_0)$.

Next, to prove (8.11), for all $t \geq t_0$, let $z(t)$ be the solution of (2.1) and let $u(\cdot) \in \mathcal{U}$, such that $\forall t \geq t_0$, we have

$$\frac{\partial V}{\partial z} f(t, z(t), u(t)) - \dot{V}(t, z) = 0 \tag{8.14}$$

Hence

$$\begin{aligned} \mathcal{L}(t, z(t), u(t)) =& \frac{\partial V}{\partial z} f(t, z(t), u(t)) - \dot{V}(t, z) + \mathcal{L}(t, z(t), u(t)) \\ =& \mathcal{H}\left(t, z(t), u(t), \frac{\partial V}{\partial z}^{\top}\right) - \dot{V}(t, z) \end{aligned} \tag{8.15}$$

Now, integrating (8.15) over infinite-horizon

$$\begin{aligned} \int_{t_0}^{\infty} \mathcal{L}(t, z(t), u(t)) &= \int_{t_0}^{\infty} \left[\mathcal{H}\left(t, z(t), u(t), \frac{\partial V}{\partial z}^{\top}\right) - \dot{V}(t, z)\right]\mathrm{dt} \\ J(t_0, z_0, u(\cdot)) &= \int_{t_0}^{\infty} \mathcal{H}\left(t, z(t), u(t), \frac{\partial V}{\partial z}^{\top}\right)\mathrm{dt} - \lim_{t\to\infty} V(t, z) + V(t_0, z_0) \\ &= \int_{t_0}^{\infty} \mathcal{H}\left(t, z(t), u(t), \frac{\partial V}{\partial z}^{\top}\right)\mathrm{dt} + V(t_0, z_0) \end{aligned} \tag{8.16}$$

Thus, it follows from (8.6), (8.7) and (8.16), that $J(t_0, z_0, u(\cdot)) \geq V(t_0, z_0) = J(t_0, z_0, \psi(\cdot))$, which results in (8.11). ■

Remark 8.2 *It is worth noting that the conditions* (8.2)-(8.7) *stated in Theorem 8.1 guarantee prescribed-time optimality for the closed-loop system* (8.9). *However, these conditions are somewhat restrictive as they do not provide a closed-form expression for the optimal prescribed-time feedback control. Despite this limitation, these conditions are valuable in deriving an inverse optimal prescribed-time stabilizing controller for a specific class of affine dynamical systems.*

8.4 Optimal prescribed-time stabilization for nonlinear affine dynamical systems

In this section, we apply the findings from the previous section to a specific class of affine nonlinear dynamical systems represented in the form of

$$\dot{z}(t) = f(t, z(t)) + \mathcal{G}(t, z(t))u(t), \quad z(t_0) = z_0 \tag{8.17}$$

where $z(t) \in \mathbb{R}^n$ represents the state, $u(t) \in \mathbb{R}^m$ represents the control input, and the functions $f : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{G} : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous in $z(t)$. Additionally, we have $f(t, 0) = 0$, indicating that f evaluates to zero when the state $z(t)$ is zero.

Moreover, we define the performance integrand $\mathcal{L}(t, z(t), u(t))$ in a specialized form as follows:

$$\mathcal{L}(t, z(t), u(t)) = \ \mathcal{L}_1(z) + \mathcal{L}_2(z)u(t) + u^\top(t)R_2(z)u(t) \tag{8.18}$$

where, $R_2 : \mathbb{R}^n \to \mathbb{R}^{m\times m}$ is a positive definite matrix function, and $\mathcal{L}_1 : \mathbb{R}^n \to \mathbb{R}$ and $\mathcal{L}_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$ are continuous functions defined on $\mathbb{R}^n$.

Now, we can express the performance functional (8.1) in the following form:

$$J\left(t_0, z_0, u(\cdot)\right) = \int_{t_0}^{\infty} \left[\mathcal{L}_1(z(t)) + \mathcal{L}_2(z(t))u(t) + u^\top(t)R_2(z)u(t)\right] \mathrm{dt} \tag{8.19}$$

The subsequent theorem offers an inverse optimal control law that addresses the optimal prescribed-time control issue for the system (8.17) using performance functional of the form (8.19).

Theorem 8.3 *Consider the affine nonlinear dynamical system* (8.17) *with the performance functional* (8.19). *Suppose there exists a continuously differentiable positive definite function*

$V : [t_0, \infty] \times \mathbb{R}^n \to \mathbb{R}$ and design constants $\gamma \in \mathbb{R}{>}\ 1$ and $t_p > t_0$ such that:

$$V(t, 0) = 0 \tag{8.20}$$

$$V(t, z) > 0, \quad z(t) \in \mathbb{R}^n \backslash \{0\} \tag{8.21}$$

$$\frac{\partial V}{\partial z}\left[f(t, z) - \frac{1}{2}\mathcal{G}(t, z)R_2^{-1}(z)\mathcal{L}_2^{\top}(z) \ -\frac{1}{2}\mathcal{G}(t, z)R_2^{-1}(z)\mathcal{G}^{\mathrm{T}}(t, z)\frac{\partial V}{\partial z}^{\top}\right]$$
$$\leq -\gamma \frac{V(z)}{(t_p + t_0 - t)}, \quad z(t) \in \mathbb{R}^n \tag{8.22}$$

$$\mathcal{L}_2(0) = 0 \tag{8.23}$$

$$0 = \mathcal{L}_1(z) + \frac{\partial V}{\partial z} f(t, z) - \frac{1}{4}\left[\frac{\partial V}{\partial z}\mathcal{G}(t, z) + \mathcal{L}_2(z)\right]$$
$$R_2^{-1}(z)\left[\frac{\partial V}{\partial z}\mathcal{G}(t, z) + \mathcal{L}_2(z)\right]^{\top}, (t, z(t)) \in [t_o, \infty) \times \mathbb{R}^n \tag{8.24}$$

Then, using the subsequent feedback control

$$u(t) = \psi(z) = -\frac{1}{2}R_2^{-1}(z)\left[\frac{\partial V}{\partial z}\mathcal{G}(t, z) + \mathcal{L}_2(z)\right]^{\top} \tag{8.25}$$

the origin of the closed-loop system

$$\dot{z}(t) = f(t, z(t)) + \mathcal{G}(t, z(t))\psi(z), \quad z(t_0) = z_0 \tag{8.26}$$

is prescribed-time stable and convergence can be guaranteed within the a priori chosen time t_p. Furthermore, the performance measure (8.19), *with*

$$\mathcal{L}_1(z) = \psi^{\top}(z)R_2(z)\psi(z) - \frac{\partial V}{\partial z} f(t, z) \geq 0 \tag{8.27}$$

is minimized in the sense of (8.11). *Moreover, the control input $u(t) = \psi(z(\cdot))$ minimizes $J(t_0, z_0, u(\cdot))$ such that*

$$J(t_0, z_0, \psi(z(\cdot))) = \min_{u \in \mathcal{U}} J(t_0, z_0, u(\cdot)) \tag{8.28}$$

Finally

$$J(t_0, z_0, \psi(z(\cdot)) = V(t_0, z_0), \quad z_0 \in \mathbb{R}^n \tag{8.29}$$

Proof. The outcome can be directly inferred from Theorem 8.1. Specifically, with the performance integrand $\mathcal{L}(t, z(t), u(t))$ of the form (8.18), the Hamiltonian takes the form:

$$\mathcal{H}(t, z(t), u(t)) = \mathcal{L}_1(z) + \mathcal{L}_2(z)u(t) + u^{\top}(t)R_2(z)u(t) + \frac{\partial V}{\partial z}(f(t, z) + \mathcal{G}(t, z)u(t))$$

Since the optimal control law (8.25) is obtained by solving

$$\frac{\partial \mathcal{H}}{\partial u} = \frac{\partial}{\partial u}\left[\mathcal{L}_1(z) + \mathcal{L}_2(z)u(t) + u^{\top}(t)R_2(z)u(t) + \frac{\partial V}{\partial z}(f(t, z) + \mathcal{G}(t, z)u(t))\right] = 0 \tag{8.30}$$

Now, with the control law $u(t) = \psi(z)$ given by (8.25), the conditions (8.20), (8.21), (8.22) and (8.23) are a direct consequence of (8.2), (8.3), (8.4) and (8.5) respectively.

Furthermore, given that $V(\cdot)$ is continuously differentiable and satisfies (8.20) and (8.21), the fact that $V(0)$ is a local minimum implies that $\frac{\partial V}{\partial z}\big|_{z=0} = 0$. Consequently, combining (8.23) and (8.25), we can conclude that $\psi(0) = 0$, which corresponds to the condition stated in (8.4). Therefore, the optimal control law $\psi(z)$ satisfies $\frac{\partial \mathcal{H}}{\partial u}\big|_{u(t)=\psi(z)} = 0$. Then, we can rewrite the expression (8.24) in terms of $\psi(z)$ as:

$$\mathcal{L}_1(z) + \frac{\partial V}{\partial z} f(t,z) - \psi^\top(z) R_2(z) \psi(z) = 0 \tag{8.31}$$

This directly verifies the hypothesis (8.6).

Finally, based on (8.6), (8.18), (8.25), and the positive definiteness of $R_2(z)$, we can express the following relationship:

$$\begin{aligned}
\mathcal{L}(t, z(t), u(t)) + \frac{\partial V}{\partial z}\Big[f(t,z) + \mathcal{G}(t,z)u(t)\Big]& \\
=&\mathcal{L}(t, z(t), u(t)) + \frac{\partial V}{\partial z}\Big[f(t,z) + \mathcal{G}(t,z)u(t)\Big] \\
&- \mathcal{L}(t, z(t), \psi(z)) \\
&- \frac{\partial V}{\partial z}\Big[f(t,z) + \mathcal{G}(t,z)\psi(z)\Big] \\
=&\Big[\mathcal{L}_2(z) + \frac{\partial V}{\partial z}\mathcal{G}(t,z)\Big](u(t) - \psi(z)) \\
&+ u^\top(t) R_2(z) u(t) - \psi^\top(z) R_2(z) \psi(z) \\
=& - 2\psi^\top(z) R_2(z)(u(t) - \psi(z)) \\
&+ u^\top(t) R_2(z) u(t) - \psi^\top(z) R_2(z) \psi(z) \\
=& \, [u(t) - \psi(z)]^\top R_2(z) [u(t) - \psi(z)] \\
\geq& \, 0, \quad z(t) \in \mathbb{R}^n
\end{aligned}$$

which implies the hypothesis (8.7). Now, as a direct outcome of Theorem 8.1, the conclusions follow. ■

Remark 8.4 *Theorem 2 offered a family of prescribed-time controllers that minimizes a specific cost function rather than directly solving the HJB steady-state equation to minimize a provided performance functional. In this scenario, one can specify $R_2(z)$ and $\mathcal{L}_2(z)$ in a flexible manner, while from (8.31), $\mathcal{L}_1(z)$ is parameterized as (8.27).*

8.5 Application to the coupled tank system

This section comprises computer simulations and experiments aimed at validating the effectiveness and accuracy of the designed optimal prescribed-time feedback control law (8.25), which simultaneously minimizes the cost function (8.19).

We consider the coupled tank system as a test bench. The schematic diagram in Figure 8.1 represents the model of a coupled tank system and the task is to maintain the desired water level in tank 1. Since it is running in configuration 1, the control input appears directly at the inlet of tank 1. The mathematical model for the configuration, as mentioned above, is given by

$$\frac{d}{dt}L_1 = \frac{1}{A_{t1}}\left[K_pV_p - A_{o1}\sqrt{2gL_1}\right] \tag{8.32}$$

where L_1 represents tank one water level, V_p is the pump input voltage, K_p is the constant of proportionality, A_{t1} and A_{o1} are cross-section area of tank one and its outlet hole respectively and g denotes the acceleration caused by gravity.

Let us assume the state variable as $z(t) = L_1$ and the control input variable as $u(t) = V_p$. Then the system dynamics can be expressed as

$$\dot{z}(t) = \frac{1}{A_{t1}}\left[K_pu(t) - A_{o1}\sqrt{2gz(t)}\right] \tag{8.33}$$

Let us say tank 1's water level is intended to be kept at $z_d(t)$. Now we define the tracking error as $e(t) = z(t) - z_d(t)$, then the dynamics in the error coordinate given by

$$\dot{e}(t) = \frac{1}{A_{t1}}\left[K_pu(t) - A_{o1}\sqrt{2ge(t)}\right] \tag{8.34}$$

Next, the affine form of the error dynamics (8.34) can be written as

$$\dot{e}(t) = f(e(t)) + \mathcal{G}(e(t))u(t) \tag{8.35}$$

where $f(e(t)) = -\frac{A_{o1}}{A_{t1}}\sqrt{2ge(t)}$ and $\mathcal{G}(e(t)) = \frac{K_p}{A_{t1}}$.

For the considered coupled tank system error dynamics (8.35), we utilize Theorem 8.3 to obtain an inverse optimal prescribed-time control law $u(t) = \psi(\cdot)$, for $t_0 \leq t < t_p + t_0$.

Next, we consider the following Lyapunov function

$$V(e) = \frac{1}{2}e^2(t)$$

According to Theorem 8.3, we take $R_2 = 1$ and $\mathcal{L}_2 = -2\frac{A_{t1}}{K_p}\left[\frac{A_{o1}}{A_{t1}}\sqrt{2ge(t)} + \frac{1}{2}\frac{K_p^2}{A_{t1}^2}e(t) - \gamma\frac{e(t)}{t_p+t_0-t}\right]$.

Now, the inverse optimal prescribed-time control law (8.25) for $t_0 \leq t < t_p + t_0$ is given by

$$u(t) = -\frac{K_p}{2A_{t1}}e(t) + \frac{A_{t1}}{K_p}\left[\frac{A_{o1}}{A_{t1}}\sqrt{2ge(t)} + \frac{1}{2}\frac{K_p^2}{A_{t1}^2}e(t) - \gamma\frac{e(t)}{t_p + t_0 - t}\right] \tag{8.36}$$

and the performance functional (8.19), with $\mathcal{L}_1 = \left(\frac{A_{o1}}{K_p}\sqrt{2ge(t)} - \frac{A_{t1}\gamma e(t)}{K_p(t_p+t_0-t)}\right)^2 + e(t)\frac{A_{o1}}{A_{t1}}\sqrt{2ge(t)}$ is minimized in the sense of (8.11).

Furthermore, since (8.20) and (8.21) holds and $\frac{\partial V}{\partial e} = e(t)$, then

$$\frac{\partial V}{\partial e}\left[f(e) - \frac{1}{2}\mathcal{G}(e)R_2^{-1}(e)\mathcal{L}_2^\top(e) - \frac{1}{2}\mathcal{G}(e)R_2^{-1}(e)\mathcal{G}^{\mathrm{T}}(e)\frac{\partial V}{\partial e}^\top\right] \leq -\gamma\frac{e^2(t)}{(t_p+t_0-t)} \tag{8.37}$$

$$= -\gamma'\frac{V(e)}{(t_p+t_0-t)} \tag{8.38}$$

where $\gamma' = 2\gamma$, which verify (8.22). Since the above proposed optimal control law is valid for all $t < t_p + t_0$, we adopted the switching gain strategy to maintain tracking after terminal time. Here for all $t \geq t_p + t_0$, the optimal control law proposed in [100] is utilized. The numerical values of the coupled tank system parameters were set according to the official datasheet of the Quanser coupled tank system [120].

Simulation results were obtained by applying the suggested optimal prescribed-time stabilization scheme (8.36) to the coupled tank system (8.35), with a predetermined time $t_p = 8s$ and a desired water level of $z_d(t) = 15cm$. The results are presented in Figure 8.2 and 8.3, where Figure 8.2(a) shows the trajectory of water level in tank 1 with the user-defined time t_p = 8s, and Figure 8.2(b) depicts the corresponding control input $u(t)$. To demonstrate that the convergence time can be chosen according to the designer's preference, simulation results with $t_p = 10s$ were also obtained. Figure 8.3(a) shows the trajectory tracking with the desired convergence time of 10 seconds, and the corresponding required control input $u(t)$ is shown in Figure 8.3(b).

To illustrate the effectiveness and advantages of the proposed optimal prescribed-time stabilizing control law (8.36), experiments were conducted using the Quanser coupled tank experimental setup, depicted in Figure 8.4. The experiments were conducted with a sampling period of $t = 0.001s$, while other parameters remained the same as those used in simulations. Figure 8.5 and 8.6 present the experimental findings, where Figure 8.5(a) shows the trajectory of water level in tank 1 with a user-defined time $t_p = 8s$, and Figure 8.5(b) depicts the corresponding control input $u(t)$. Moreover, to demonstrate the flexibility of the convergence time, experimental results with t_p = 10 seconds were also obtained. Figure 8.6(a) represents the trajectory tracking with the desired convergence time of 10 seconds, and Figure 8.6(b) shows the corresponding required control input $u(t)$. The experimental results indicate that the designed optimal prescribed-time stabilizing control law (8.36) is highly accurate and efficient.

Furthermore, a comparative analysis was conducted to demonstrate the superiority of the proposed scheme over existing results, which include the optimal nonlinear control scheme

[94], finite-time optimal nonlinear control scheme [99], and optimal predefined-time stabilization scheme [16]. Figure 8.7 illustrates that all four schemes successfully achieved water level tracking in tank 1. As shown in Figure 8.7(a), the method proposed in [94] guarantees only asymptotic stability and stabilization for the considered coupled tank experimental setup. The optimal nonlinear control scheme [99] with finite-time stability achieved the desired water level tracking in some finite time. However, the settling time still depends on the initial error, as depicted in Figure 8.7(b). On the other hand, the optimal predefined-time stabilization scheme [16] offers a new type of fixed time stabilization with the upper bound of the settling time defined in advance (i.e., $t_p = 10s$), as shown in Figure 8.7(c). However, it does not provide users with the flexibility to converge precisely at the user-defined time, giving only a conservative estimate of settling time. Figure 8.7(d) shows that the proposed optimal prescribed-time nonlinear control scheme (8.36) achieved the desired water level in a priori chosen time (i.e., $t_p = 10s$). The comparative results for the corresponding required control inputs are depicted in Figure 8.8. In summary, when it is necessary to maintain the desired water level in tank one at a given time, which can be prescribed in advance, only the proposed optimal prescribed-time control scheme can accomplish the task. In contrast, the optimal predefined-time scheme [16] only offers the flexibility to pre-assign the upper bound of the settling time. Therefore, one can conclude the practicality and feasibility of the proposed optimal prescribed-time stabilization scheme in real-world applications.

Remark 8.5 *In contrast to our proposed control approach, a comparison with the control scheme presented in [99] reveals that the convergence time and design parameters in the [99] scheme are not intrinsically interconnected. Consequently, in the [99] approach, it is challenging to predetermine and achieve the desired tracking performance for the coupled tank system, as it cannot be readily tailored to meet the designer's specifications. In contrast, our designed control approach establishes a direct relationship, enabling the attainment of the desired performance in accordance with the designer's requirements.*

Remark 8.6 *We compared our controller with the existing algorithm [16] in Figure 8.7(c) and Figure 8.7(d). Both aim to achieve convergence within a specified time. The existing algorithm fails to converge precisely at $t_p = 10s$, while our approach achieves this exact time when set as the predefined target. This demonstrates our control approach's ability to meet user-defined performance requirements, offering precise and reliable control of convergence time, a significant advantage in various applications.*

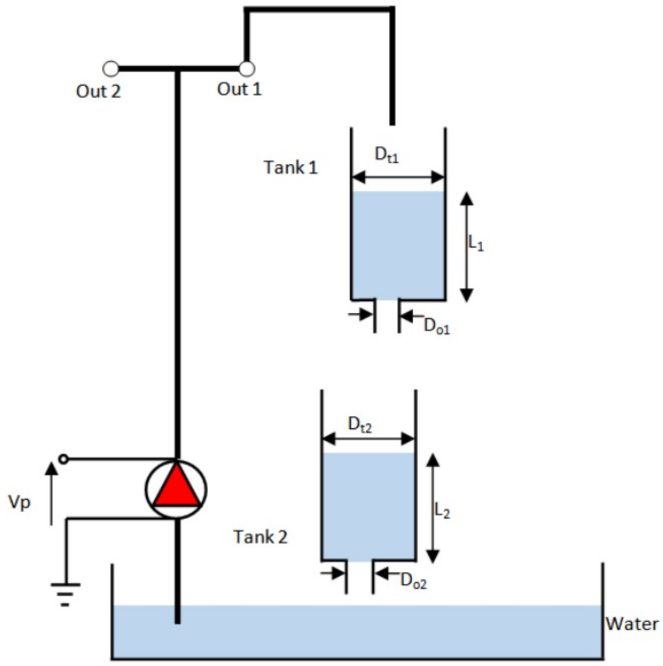
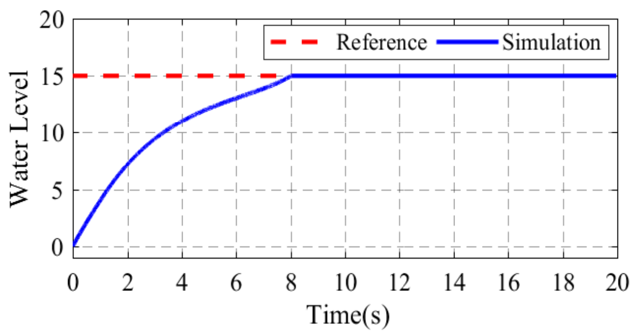
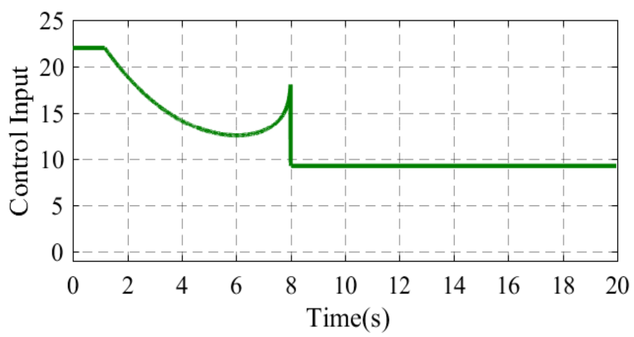


Figure 8.1: Schematic diagram of a coupled water tank system.

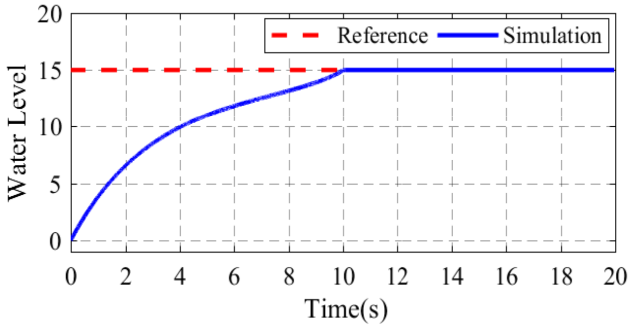


(a)

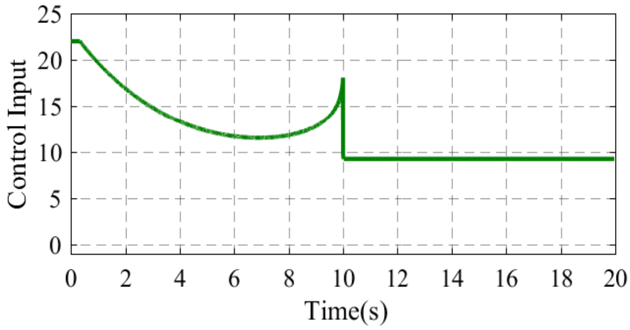


(b)

Figure 8.2: Simulation results with $t_p = 8s$: (a) Tracking response. (b) Control input.



(a)



(b)

Figure 8.3: Simulation results with $t_p = 10s$: (a) Tracking response. (b) Control input.

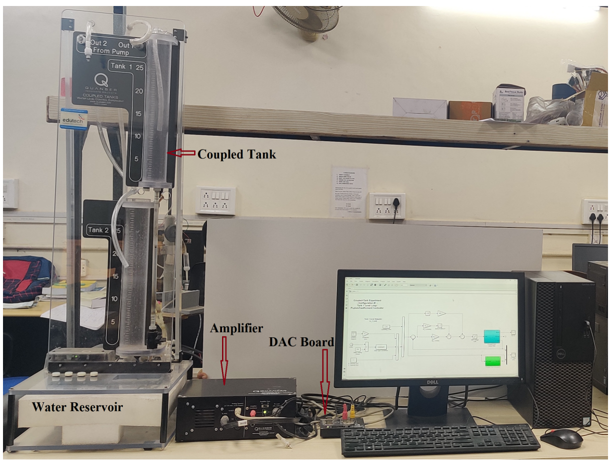
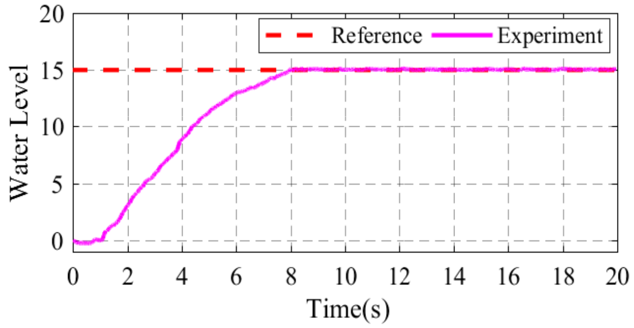
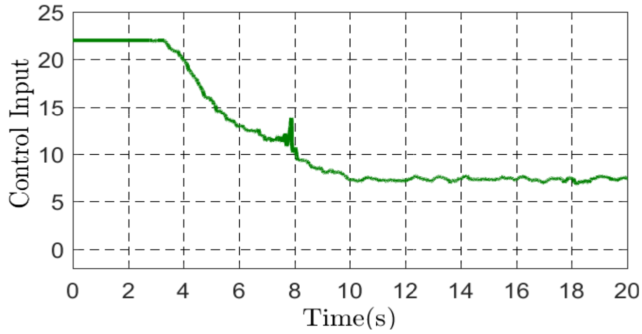


Figure 8.4: Experimental setup of a coupled water tank system.

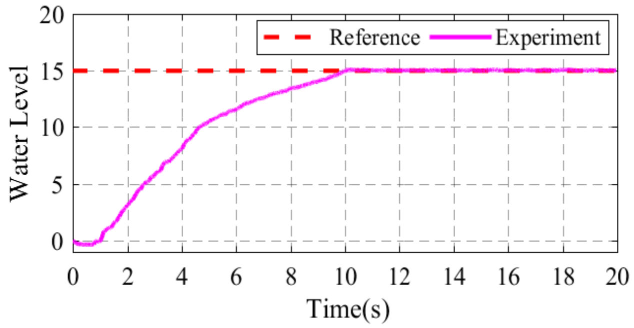


(a)

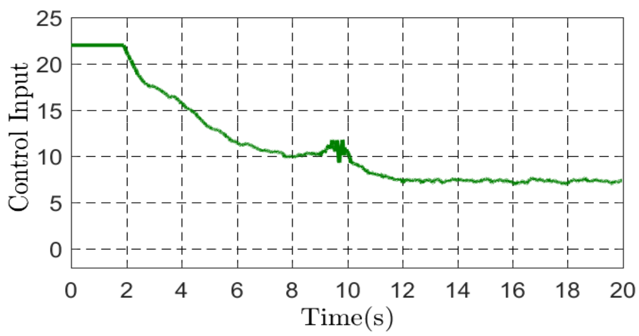


(b)

Figure 8.5: Experimental results with $t_p = 8s$: (a) Tracking response. (b) Control input.



(a)



(b)

Figure 8.6: Experimental results with $t_p = 10s$: (a) Tracking response. (b) Control input.

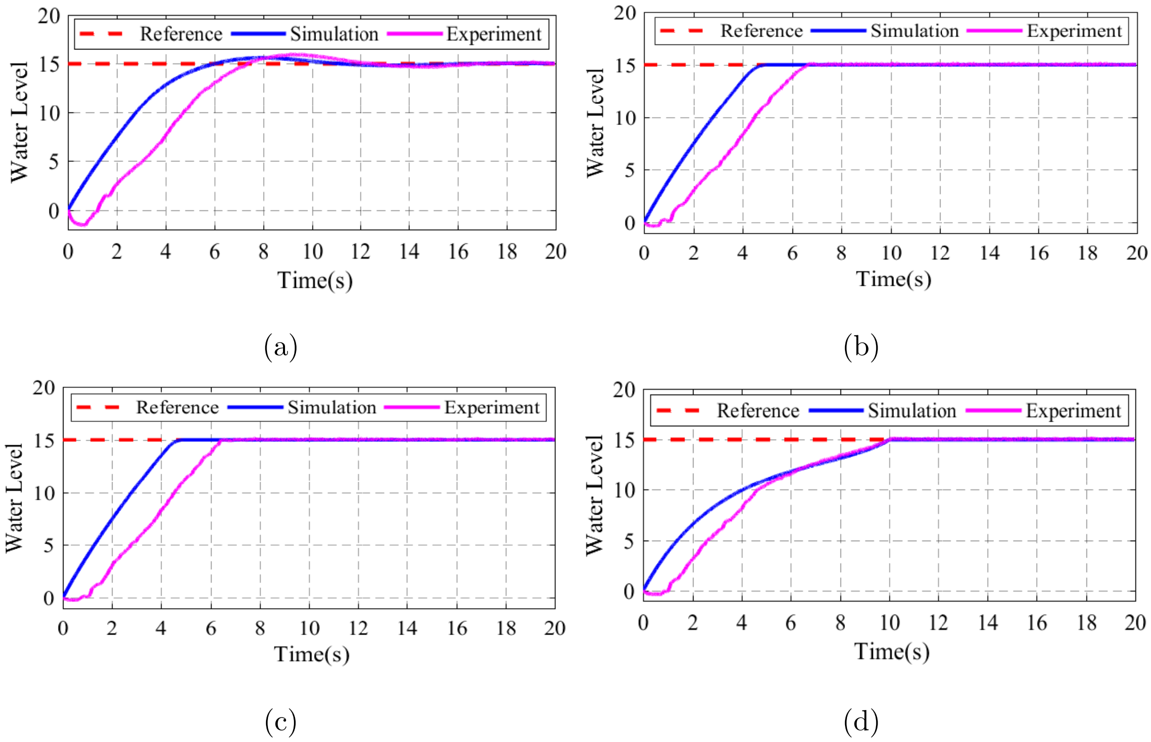


Figure 8.7: Tracking: (a) Under control proposed in [94]. (b) Under control proposed in [99]. (c) Under control proposed in [16]. (d) Under proposed control law (8.36).

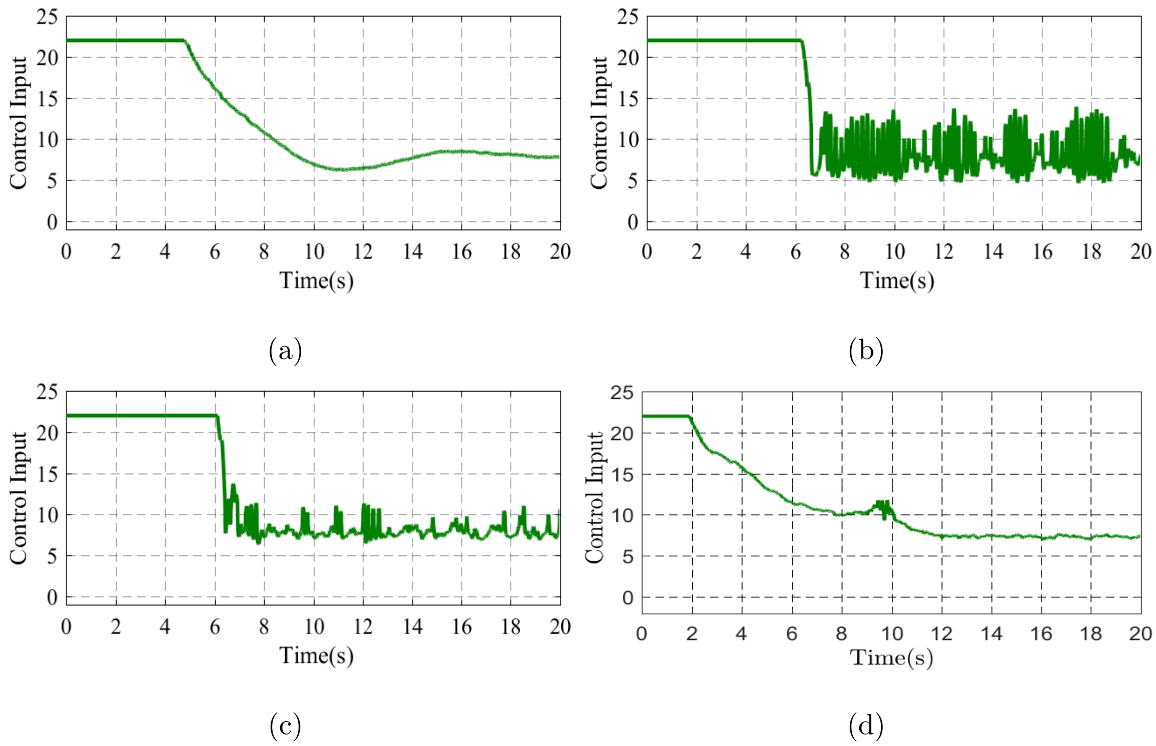


Figure 8.8: Control input: (a) Under control proposed in [94]. (b) Under control proposed in [99]. (c) Under control proposed in [16]. (d) Under proposed control law (8.36).

8.6 Conclusion

This chapter studies the prescribed-time stability and optimal feedback control problem for nonlinear dynamical systems. First, a theorem is proposed that guarantees the prescribed-time stability. The obtained result is further utilized to develop an optimal prescribed-time nonlinear feedback control law that ensures the prescribed-time stability and optimality of the closed-loop system. Sufficient conditions are derived using the Hamilton-Jacobi-Bellman steady-state equation, where a given Lyapunov function satisfies a differential inequality. Moreover, optimal prescribed-time stabilizing controllers are characterized by a class of affine nonlinear systems. Finally, extensive simulations and experiments on coupled tank systems validate the proposed stabilization method.