Chapter 2

Dual phase synchronization of chaotic systems using nonlinear observer based technique

2.1 Introduction

Chaos theory is a developing field since 1970 and still the theory has not yet been understood very well. If a dynamical system is bounded and has infinite recurrences with dependency on initial conditions, then it is known as chaotic (Azar and Vaidyanathan (2015)). Effect of chaos in nonlinear dynamics is studied during last few years. Several researchers have studied chaotic dynamical systems in various fields from different parts of the world. This effect is most common and has been detected in a number of dynamical systems of various types of physical nature. In the development of non-equilibrium problems and mathematical computing, chaos theory plays very important role. Chaos theory is also used to analyse the problems of dynamical and non-linear dynamical systems related with complex networks which is generally used in biological and social systems including ecology, medicine and in the field of business strategy.

The most important achievement in the research of chaos is that chaotic systems can be made to synchronize with each other. Synchronization of chaotic systems is a naturally occurring phenomenon where one chaotic dynamical system mimics the dynamical behavior of another chaotic system. The first idea of synchronizing of two identical chaotic systems was analyzed by Pecora and Carroll (1990). Runzi et al. (2011) discussed synchronization using two master and one slave systems. Before this, the synchronization was confined to one master and one slave systems. Lämmer et al. (2006) used phase

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synchronization in the study of control of materials or traffic flows in networks. Bagheri et al. (2015) obtained fruitful results about adaptive observer-based synchronization of two non-identical chaotic systems with unknown parameters with the assumption that the both master and slave systems' parameters are unknown. Louodop et al. (2014) explained the adaptive time-delay synchronization of chaotic systems with uncertainties via nonlinear feedback coupling method. Zemouche and Boutayeb (2009) studied nonlinear observer based H^{∞} synchronization using observed design method for nonlinear Lipschitz discrete-time systems. Kaddoum (2016) has presented a survey of wireless chaos-based communication systems. Mahmud et al. (2012) have discussed nonlinear observer-based excitation controller design for interconnected power systems via exact linearization approach and used single machine infinite bus (SMIB) system as a test system. Yadav et al. (2016) have obtained dual function projective synchronization of fractional order complex chaotic systems.

Juan and Yuan (2007) has discussed nonlinear observer based phase synchronization of chaotic systems. Sun et al. (2013) have analysed combination-combination synchronization among four identical or different chaotic systems. Sun et al. (2016) have also studied dual combination synchronization of six chaotic systems. Ohata and Saito (2015) analyzed multi-phase synchronization in paralleled buck converters with two different kinds of switching rules. Boubaker et al. (2017) have studied time delay Systems and applied it in Modeling, Analysis, Estimation, Control and Synchronization. Khan et al. (2017) have proposed a novel idea of dual combination-combination multi switching synchronization involving eight chaotic systems. Singh et al. (2017) have explained dual combination synchronization of the fractional order complex chaotic systems.

The purpose of this chapter is the investigation of dual phase synchronization of chaotic systems with nonlinear observer controllers. Dual synchronization is a special circumstance in synchronization in which two identical/non-identical pairs of chaotic systems (two master systems and two slave systems) are synchronized. The dual synchronization of systems plays an important role in many fields including chaotic secure communication. Recently the dual synchronization of chaotic systems has received less attention. There are only a few results available in the literature on dual synchronization between chaotic systems. In phase synchronization, the coupled chaotic systems keep their phase difference bounded by a constant while their amplitudes remain uncorrelated. The phase synchronization usually applied upon two waveforms of the same frequency with identical phase angles with each cycle. However, it can be applied if there is an integer relationship of frequency such that the cyclic signals share a repeating sequence of phase angles over consecutive cycles.

Observer design, having vital importance in the area of systems and control theory, arises whenever some components of the state are not directly measured. After giving the solution of multivariate problems in the linear time invariant case by Luenberger (1996), many researchers had been motivated to extend the basic ideas of his work to the nonlinear context. Though applications of linear observer theory to nonlinear problems had received success, still the researchers were involved to construct nonlinear observers using tools from nonlinear systems theory. A brief introduction to some of these nonlinear approaches to the problem of observer design can be found in the article of Primbs (1996). Beikzadeh and Taghirad (2012) presented a novel nonlinear continuous time observer based on differential state dependent Riccati equation filter with guaranteed exponential stability of the estimation error dynamics utilising Lyapunov stability analysyis which is utilized in order to obtain the required conditions for exponential stability of the estimation error dynamics. Two different observer design techniques for linear and nonlinear systems based on the so-called observability function had been proposed by Carnevale et al. (2013).

These have motivated me to study on the dual phase synchronization between two identical pair of different chaotic systems with nonlinear state observer algorithm using stability theory. The numerical example is provided to illustrate the obtained results. Numerical simulation results demonstrate the effectiveness and feasibility of the method.

The organization of this chapter is as follows. In Section 2.2, problem formulation of the dual phase synchronization scheme of two different types of chaotic systems is presented. In section 2.3, the descriptions of Qi and Newton-Leipnik systems are given. In Section 2.4, dual phase synchronization of chaotic systems is discussed. The numerical simulations are presented through Section 2.5 to verify the effectiveness of the proposed method. In Section 2.6, the conclusion of the work is presented.

2.2 Problem formulation

Let us consider the following two chaotic systems as

$$\dot{x} = Ax + Bf(x), \tag{2.1}$$

$$\dot{y} = Cy + Dg(y), \tag{2.2}$$

where $x, y \in R^n$ are the state vectors of the systems (2.1) and (2.2), $A, C \in R^{n\times n}$, $B, D \in R^{n\times m}$ are the constant matrices and $f, g : R^n \to R^m$ are the nonlinear functions.

Suppose the dynamical systems (2.1) and (2.2) with the output

$$s(x) = f(x) + K_j x, \tag{2.3}$$

$$S(y) = g(y) + K'_j y, \tag{2.4}$$

where K_j, $K'_j \in R^{m\times n}$ denote the feedback gain matrices.

Let us define the observer as

$$\dot{\hat{x}} = A\hat{x} + Bf(\hat{x}) + B[s(x) - s(\hat{x})], \tag{2.5}$$

$$\dot{\hat{y}} = C\hat{y} + Dg(\hat{y}) + D[S(y) - S(\hat{y})]. \tag{2.6}$$

The synchronization errors among the systems (2.1), (2.2) and (2.5), (2.6) are defined as

$$e_{x\hat{x}} = x - \hat{x}, \tag{2.7}$$

$$e_{y\hat{y}} = y - \hat{y}, \tag{2.8}$$

then the error systems can be obtained as

$$\dot{e}_{x\hat{x}} = \dot{x} - \dot{\hat{x}} = Ae_{x\hat{x}} + Bf(x) - Bf(\hat{x}) - B[s(x) - s(\hat{x})],$$

$$\dot{e}_{y\hat{y}} = \dot{y} - \dot{\hat{y}} = Ce_{y\hat{y}} + Dg(y) - Dg(\hat{y}) - D[S(y) - S(\hat{y})].$$

From equations (2.3) and (2.4), the error systems reduce in following form

$$\dot{e}_{x\hat{x}} = [A - BK_j]e_{x\hat{x}}, \tag{2.9}$$

$$\dot{e}_{y\hat{y}} = [C - DK'_j]e_{y\hat{y}}. \tag{2.10}$$

In order to make systems (2.9) and (2.10) controllable with the controllable matrices $[B, AB, ...A^{n-1}B]$ and $[D, CD, ...C^{n-1}D]$ of full ranks, the choices of the feedback gain matrices K_j, K'_j will be in such a way that the characteristic polynomials of the matrices $[A - BK_j]$ and $[C - DK'_j]$ must have all the eigenvalues with negative real parts. Then the error systems will be stabilized and the dual synchronization among considered systems is achieved.

If there is any eigen value of the error system is equal to zero, then another type of synchronization phenomenon called phase synchronization occurs, in which the difference between various states of synchronized systems may not necessarily converge to zero, but is less than or equal to a constant.

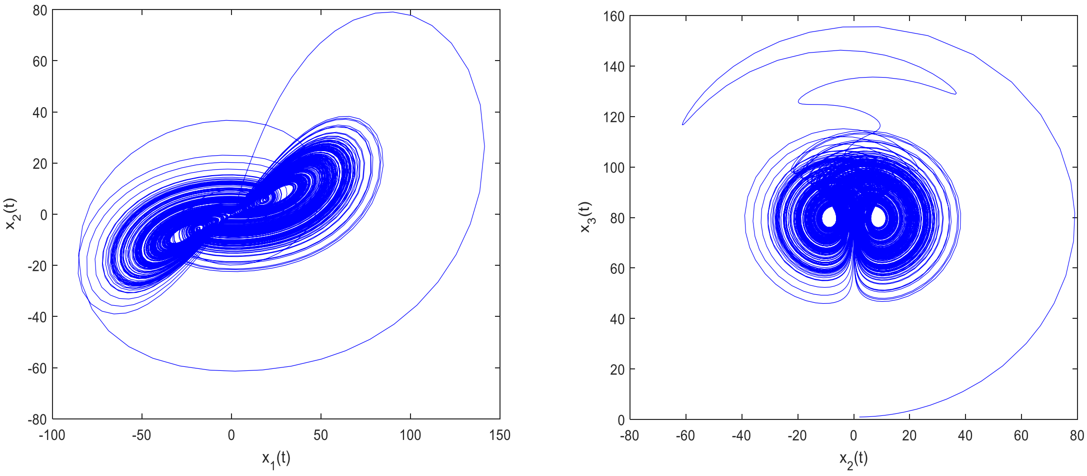
2.3 Systems' Descriptions

2.3.1 Qi chaotic system

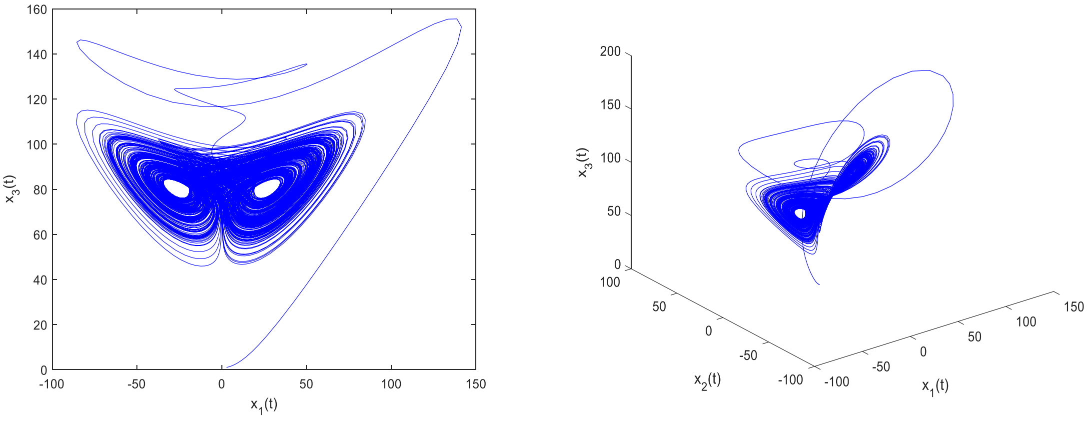
Consider the following Qi system (Yadav et al. (2016)) as

$$\begin{aligned}\dot{x}_1 &= a_1(x_2 - x_1) + x_2 x_3,\\ \dot{x}_2 &= a_3 x_1 - x_2 - x_1 x_3,\\ \dot{x}_3 &= -a_2 x_3 + x_1 x_2,\end{aligned} \tag{2.11}$$

where x_1, x_2, x_3 are the state variables. The phase portraits of the system (2.11) in $x_1(t) - x_2(t)$, $x_2(t) - x_3(t)$, $x_1(t) - x_3(t)$ planes and $x_1(t) - x_2(t) - x_3(t)$ space for the parameters' values $a_1 = 35$, $a_2 = 8/3$, $a_3 = 80$ and initial condition is (3, 2, 1) are depicted through Fig. 2.1.



(a) (b)



(c) (d)

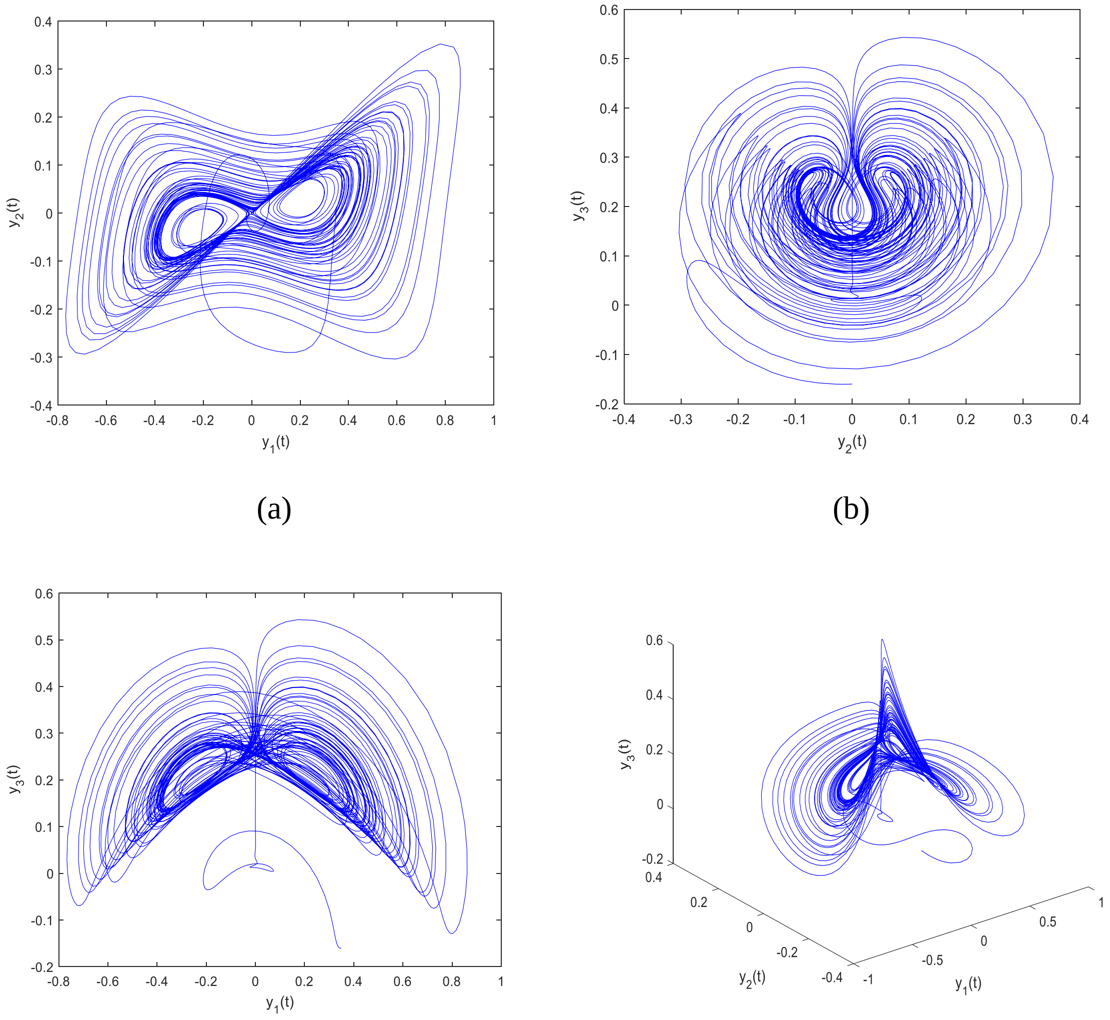
Fig. 2.1 Phase portraits of the Qi system: (a) In $x_1(t)-x_2(t)$ plane, (b) In $x_2(t)-x_3(t)$ plane, (c) In $x_1(t)-x_3(t)$ plane, (d) In $x_1(t)-x_2(t)-x_3(t)$ space.

2.3.2 Newton-Leipnik system

The Newton-Leipnik system (Jia et al. (2008)) is defined as

$$\begin{aligned}
\dot{y}_1 &= -b_1 y_1 + y_2 + 10 y_2 y_3, \\
\dot{y}_2 &= -y_1 - 0.4 y_2 + 5 y_1 y_3, \qquad (2.12) \\
\dot{y}_3 &= b_2 y_3 - 5 y_1 y_2,
\end{aligned}$$

where y_1, y_2, y_3 are the state variables. The phase portraits of the Newton-Leipnik system ((2.12) in $y_1(t)-y_2(t)$, $y_2(t)-y_3(t)$, $y_1(t)-y_3(t)$ planes and $y_1(t)-y_2(t)-y_3(t)$ space are depicted through Fig. 2.2 for the values of the parameters as $b_1 = 0.4$, $b_2 = 0.175$ and initial condition $(0.349, 0, -0.16)$.



(a) (b)

(c) (d)

Fig. 2.2 Phase portraits of the Newton-Leipnik system: (a) In $y_1(t)-y_2(t)$ plane, (b) In $y_2(t)-y_3(t)$ plane, (c) In $y_1(t)-y_3(t)$ plane, (d) In $y_1(t)-y_2(t)-y_3(t)$ space.

2.4 Dual phase synchronization of chaotic systems

In this section we are taking two systems viz., Qi and Newton-Leipnik to perform dual phase synchronization.

The systems (2.11) and (2.12) can be rewritten as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & a_1 & 0 \\ a_3 & -1 & 0 \\ 0 & 0 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 x_3 \\ x_1 x_3 \\ x_1 x_2 \end{bmatrix} \tag{2.13}$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} = \begin{bmatrix} -b_1 & 1 & 0 \\ -1 & -0.4 & 0 \\ 0 & 0 & b_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} y_2 y_3 \\ y_1 y_3 \\ y_1 y_2 \end{bmatrix} \tag{2.14}$$

Comparing equations (2.13) and (2.14) with equations (2.1) and (2.2), we get

$$A = \begin{bmatrix} -a_1 & a_1 & 0 \\ a_3 & -1 & 0 \\ 0 & 0 & -a_2 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ C = \begin{bmatrix} -b_1 & 1 & 0 \\ -1 & -0.4 & 0 \\ 0 & 0 & b_2 \end{bmatrix},\ D = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

The observer of the systems (2.11) and (2.12) are designed as

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \dot{\hat{x}}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & a_1 & 0 \\ a_3 & -1 & 0 \\ 0 & 0 & -a_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_2 \hat{x}_3 \\ \hat{x}_1 \hat{x}_3 \\ \hat{x}_1 \hat{x}_2 \end{bmatrix} + B[s(x) - s(\hat{x})]\ , \tag{2.15}$$

$$\begin{bmatrix} \dot{\hat{y}}_1 \\ \dot{\hat{y}}_2 \\ \dot{\hat{y}}_3 \end{bmatrix} = \begin{bmatrix} -b_1 & 1 & 0 \\ -1 & -0.4 & 0 \\ 0 & 0 & b_2 \end{bmatrix} \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \end{bmatrix} + \begin{bmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} \hat{y}_2 \hat{y}_3 \\ \hat{y}_1 \hat{y}_3 \\ \hat{y}_1 \hat{y}_2 \end{bmatrix} + D[S(y) - S(\hat{y})], \tag{2.16}$$

where $B[s(x) - s(\hat{x})],\ D[S(y) - S(\hat{y})]$ output of the systems.

Now defining the error function towards dual synchronization as

$e_{x_1\hat{x}_1} = x_1 - \hat{x}_1$, $e_{x_2\hat{x}_2} = x_2 - \hat{x}_2$, $e_{x_3\hat{x}_3} = x_3 - \hat{x}_3$, $e_{y_1\hat{y}_1} = y_1 - \hat{y}_1$, $e_{y_2\hat{y}_2} = y_2 - \hat{y}_2$, $e_{y_3\hat{y}_3} = y_3 - \hat{y}_3$.

Then the error systems can be obtained as

$$\begin{bmatrix} \dot{e}_{x_1\hat{x}_1} \\ \dot{e}_{x_2\hat{x}_2} \\ \dot{e}_{x_3\hat{x}_3} \end{bmatrix} = \left(\begin{bmatrix} -a_1 & a_1 & 0 \\ a_3 & -1 & 0 \\ 0 & 0 & -a_2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} K_1 \right) \begin{bmatrix} e_{x_1\hat{x}_1} \\ e_{x_2\hat{x}_2} \\ e_{x_2\hat{x}_3} \end{bmatrix}, \tag{2.17}$$

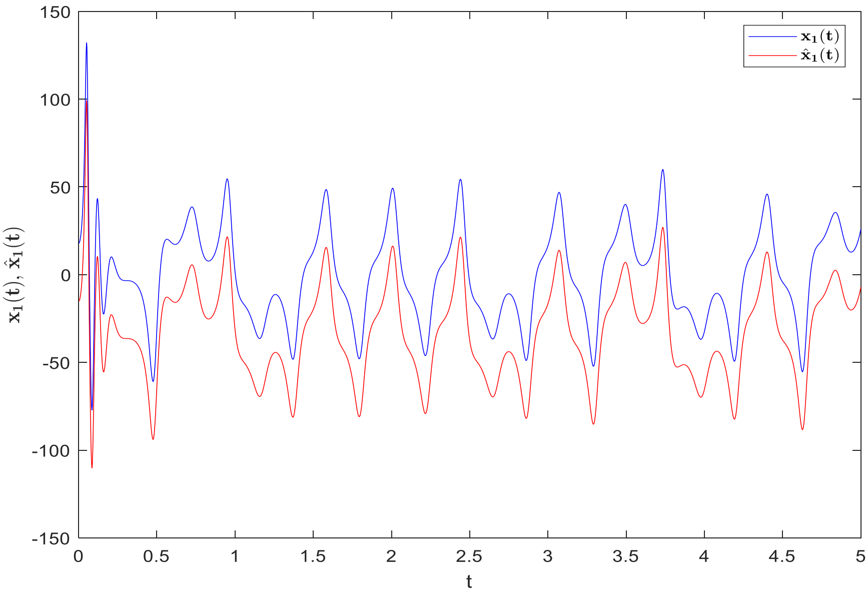
$$\begin{bmatrix} \dot{e}_{y_1\hat{y}_1} \\ \dot{e}_{y_2\hat{y}_2} \\ \dot{e}_{y_3\hat{y}_3} \end{bmatrix} = \left(\begin{bmatrix} -b_1 & 1 & 0 \\ -1 & -0.4 & 0 \\ 0 & 0 & b_2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{bmatrix} K_1' \right) \begin{bmatrix} e_{y_1\hat{y}_1} \\ e_{y_2\hat{y}_2} \\ e_{y_3\hat{y}_3} \end{bmatrix}. \tag{2.18}$$

The matrices $[B, AB, A^2B]$ and $[D, CD, C^2D]$ are in full ranks, so systems (2.15) and (2.16) are the global observer of systems (2.13) and (2.14) through proper choices of the feedback gain matrices towards synchronization as

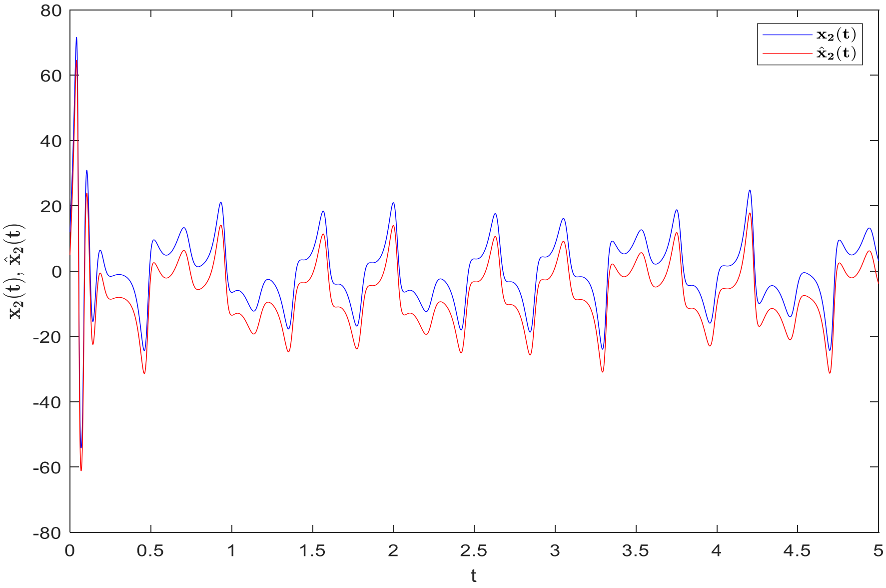
$$K_1 = \begin{bmatrix} -34 & 35 & 0 \\ -80 & 0 & 0 \\ 0 & 0 & -5/3 \end{bmatrix}, \ K_1' = \begin{bmatrix} 3/50 & 1/10 & 0 \\ -1/5 & 3/25 & 0 \\ 0 & 0 & -0.235 \end{bmatrix}.$$

For phase synchronization of the said systems, the feedback gain matrices are taken as

$$K_1 = \begin{bmatrix} -35 & 35 & 0 \\ -80 & 1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix}, \ K_1' = \begin{bmatrix} -2/50 & 1/10 & 0 \\ -1/5 & -2/25 & 0 \\ 0 & 0 & -0.035 \end{bmatrix}.$$

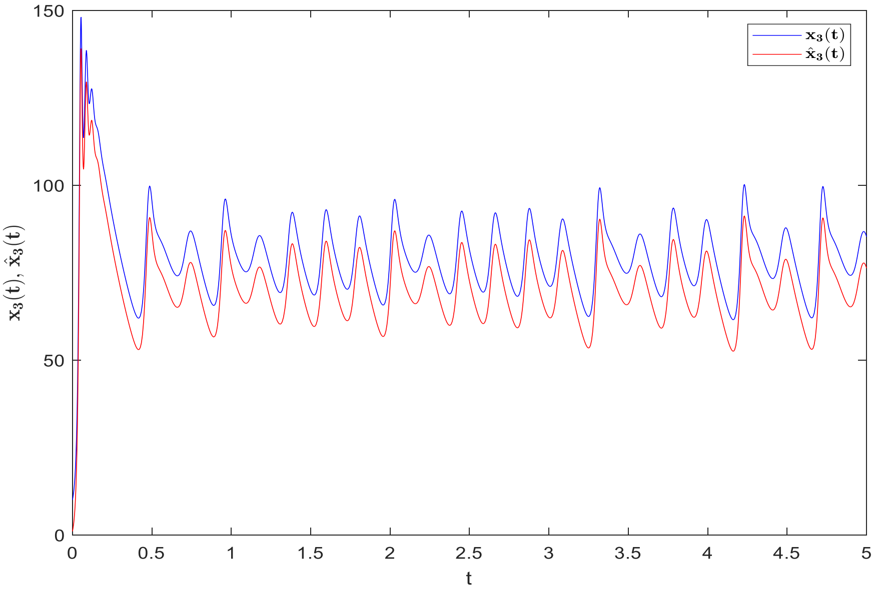


(a)



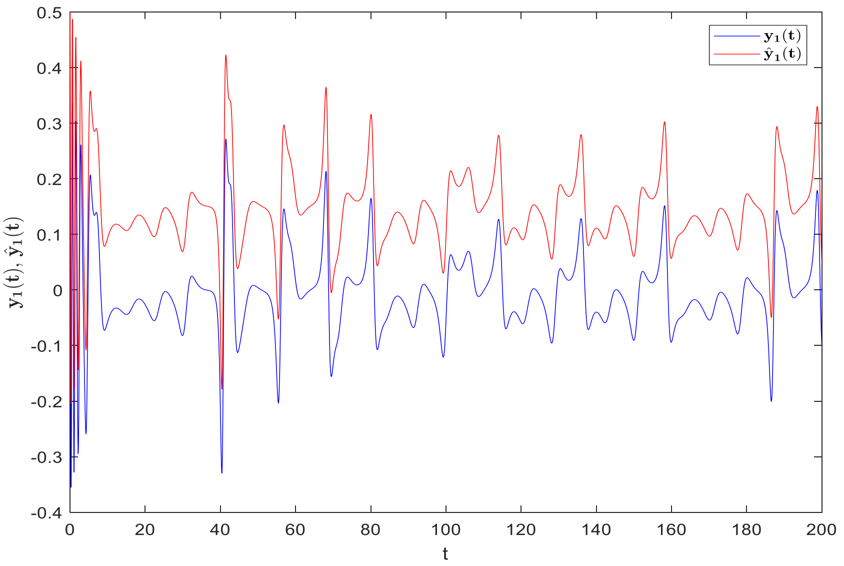
$x_2(t)$
$\hat{x}_2(t)$
$x_2(t), \hat{x}_2(t)$
80
60
40
20
0
-20
-40
-60
-80
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
t

(b)

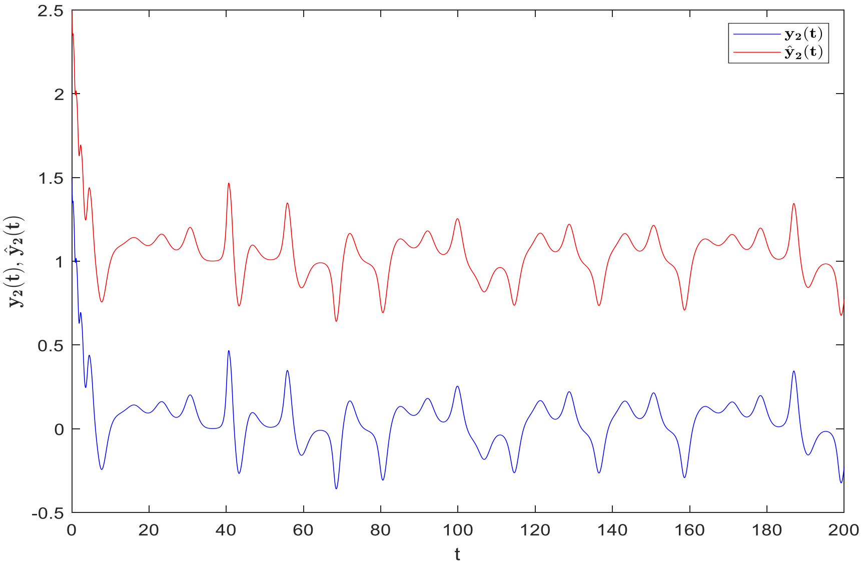


$x_3(t)$
$\hat{x}_3(t)$
$x_3(t), \hat{x}_3(t)$
150
100
50
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
t

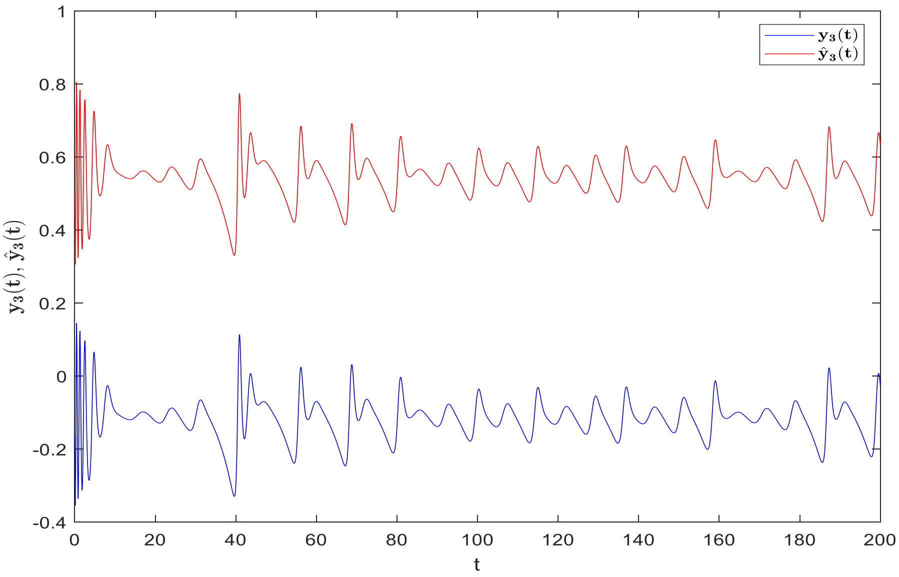
(c)


$\mathbf{y_1(t)}$
$\hat{\mathbf{y}}_1\mathbf{(t)}$
$\mathbf{y_1(t)}, \hat{\mathbf{y}}_1\mathbf{(t)}$
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0
20
40
60
80
100
120
140
160
180
200
t

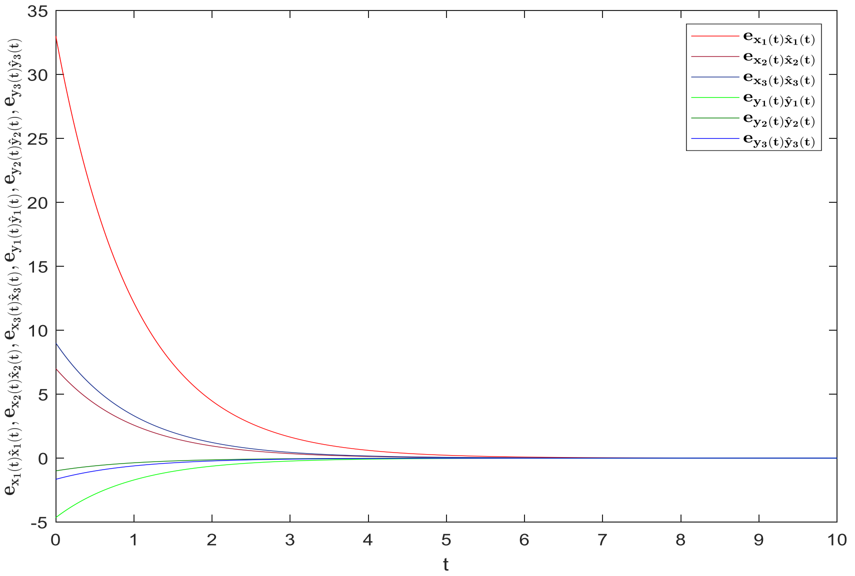
(d)


$\mathbf{y_2(t)}$
$\hat{\mathbf{y}}_2\mathbf{(t)}$
$\mathbf{y_2(t)}, \hat{\mathbf{y}}_2\mathbf{(t)}$
2.5
2
1.5
1
0.5
0
-0.5
0
20
40
60
80
100
120
140
160
180
200
t

(e)



(f)



(g)

Fig. 2.3 Phase synchronization for signals (a) between $x_1(t)$ and $\hat{x}_1(t)$, (b) between $x_2(t)$ and $\hat{x}_2(t)$, (c) between $x_3(t)$ and $\hat{x}_3(t)$, (d) between $y_1(t)$ and $\hat{y}_1(t)$, (e) between $y_2(t)$ and $\hat{y}_2(t)$, (f) between $y_3(t)$ and $\hat{y}_3(t)$, (g) The evolution of the error functions of chaotic systems during synchronization.

2.5 Numerical Simulation and results

During numerical simulation the earlier considered parameters of the chaotic systems are taken. For the dual phase synchronization the initial conditions of the master systems I, II and slave systems I, II are taken as $(x_1(0), x_2(0), x_3(0)) = (18, 12, 10)$, $(y_1(0), y_2(0), y_3(0)) = (0.349, 1.5, -0.16)$ and $(\hat{x}_1(0), \hat{x}_2(0), \hat{x}_3(0)) = (-15, 5, 1)$, $(\hat{y}_1(0), \hat{y}_2(0), \hat{y}_3(0)) = (0.5, 2.5, 0.5)$ respectively. Hence the initial conditions of error system for dual phase synchronization will be $(33, 7, 9, -0.151, -1, -0.66)$. During dual synchronization of the systems, the time step size is taken as 0.005. Now choosing $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -1$, $\lambda_4 = -1$, $\lambda_5 = -1$, $\lambda_6 = -1$, the phase synchronization between signals x_1 and $\hat{x}_1$ is achieved. It should be noted that, when $\lambda_1 = 0$, $\lambda_2 = -1$, $\lambda_3 = -1$, $\lambda_4 = -1$, $\lambda_5 = -1$, $\lambda_6 = -1$, signals x_2 and $\hat{x}_2$, x_3 and $\hat{x}_3$, y_1 and $\hat{y}_1$, y_2 and $\hat{y}_2$, y_3 and $\hat{y}_3$ become synchronized. Similarly, If $\lambda_1 = -1$, $\lambda_2 = 0$, $\lambda_3 = -1$, $\lambda_4 = -1$, $\lambda_5 = -1$, $\lambda_6 = -1$; $\lambda_1 = -1$, $\lambda_2 = -1$, $\lambda_3 = 0$, $\lambda_4 = -1$, $\lambda_5 = -1$, $\lambda_6 = -1$; $\lambda_1 = -1$, $\lambda_2 = -1$, $\lambda_3 = -1$, $\lambda_4 = 0$, $\lambda_5 = -1$, $\lambda_6 = -1$; $\lambda_1 = -1$, $\lambda_2 = -1$, $\lambda_3 = -1$, $\lambda_4 = -1$, $\lambda_5 = 0$, $\lambda_6 = -1$ and $\lambda_1 = -1$, $\lambda_2 = -1$, $\lambda_3 = -1$, $\lambda_4 = -1$, $\lambda_5 = -1$, $\lambda_6 = 0$ are taken, phase synchronizations between signals x_2 and $\hat{x}_2$, x_3 and $\hat{x}_3$, y_1 and $\hat{y}_1$, y_2 and $\hat{y}_2$, y_3 and $\hat{y}_3$ are obtained respectively. State trajectories of the dual phase synchronization of chaotic systems are depicted through Fig. 2.3 (a)-(f). The plot of the error function for dual synchronization is depicted through in Fig. 2.3(g), which shows that error states converge to zero when time becomes large. This implies that the dual phase synchronization between identical pair of different chaotic systems consisting of Qi and Newton-Leipnik systems occurs with the help of nonlinear observers.

2.6 Conclusion

The present chapter has successfully demonstrated the dual phase synchronization between Qi and Newton-Leipnik systems using nonlinear observer based technique. Based on the stability analysis, the dual phase synchronization of chaotic systems through nonlinear controller input parameters on the respective systems has been achieved and the components of the error system tend to zero as time becomes large, which helps to find the time required for dual phase synchronization between different chaotic systems. Numerical simulations are given to exhibit the reliability and effectiveness of the proposed dual combination synchronization scheme towards predicting the accuracy of the theory. The author is optimist that the outcome of this chapter will be utilized by the researchers involved in the field of chaotic systems.

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