Chapter 4

An efficient domain decomposition method for singularly perturbed semi-linear coupled systems of parabolic problems

In this chapter, we consider the following semilinear coupled system on the domain $\Omega = D \times (0, T]$, $D = (0, 1)$:

$$\boldsymbol{\mathcal{L}}\boldsymbol{u} := \partial_t \boldsymbol{u}(x,t) - \boldsymbol{\varepsilon}\partial_x^2 \boldsymbol{u}(x,t) + \boldsymbol{f}(x,t,\boldsymbol{u}) = 0, \tag{4.1}$$

with initial and boundary conditions

$$\begin{cases} \boldsymbol{u}(x,0) = \boldsymbol{\phi}(x), \ 0 \le x \le 1, \\ \boldsymbol{u}(a,t) = \boldsymbol{\varphi}_a(t), \ 0 < t \le T, \ a = 0, 1. \end{cases}$$

Here, $\boldsymbol{\mathcal{L}} = (\mathcal{L}_1, \mathcal{L}_2)^T$, the solution $\boldsymbol{u} = (u_1, u_2)^T$, the boundary data $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)^T$ and initial data $\boldsymbol{\phi} = (\phi_1, \phi_2)^T$. Assume that the reaction term $\boldsymbol{f} = (f_1, f_2)^T$ satisfies

$$\frac{\partial f_i}{\partial u_i}(x,t,u_1,u_2) \ge \overline{\alpha} > 0, \ \frac{\partial f_i}{\partial u_j}(x,t,u_1,u_2) \le 0, \ i \ne j,$$

$$\min_{-\infty < u_i < \infty} \sum_{j=1}^{2} \frac{\partial f_i}{\partial u_j}(x,t,u_1,u_2) \ge \alpha > 0, \ i = 1, 2.$$

Assume that the diffusion parameters $\varepsilon_1, \varepsilon_2$ can have different magnitudes and that $0 < \varepsilon_1 \leq \varepsilon_2 \leq 1$. Suppose that the functions $f_i,\ \phi_i,\ \varphi_i,\ i = 1, 2$ are sufficiently smooth and satisfy the compatibility conditions to ensure that $\boldsymbol{u} \in C^{4,2}(\overline{\Omega})^2$ [94, 119]. The most of situations in the actual world are essentially nonlinear. Due to complexity of analysis, SWR approaches for these problems remains challenging still today. In particular, we are only know of one work [64] that developed and analyzed a domain decomposition algorithm of the SWR type for the scalar semilinear parabolic problem.

In this article, we consider a coupled system of semilinear reaction diffusion problems. Due to the coupling of the discrete solution's components at each time level, numerical methods used to solve singularly perturbed systems typically have high computational costs. The process of approximating the solution's components is decoupled using additive schemes [55, 107, 120]. So that we are able to solve decoupled problems separately at each level of discretization, that leads to a significant reduction in computational cost. The numerical data presented later makes it very clear. For the first time, a domain decomposition method based on additive schemes has been applied to solve a semilinear coupled system (4.1). The convergence analysis of the algorithm is demonstrated using several auxiliary problems. We proved that the proposed algorithm converge with accuracy of almost second order in space and one in time. Additionally, the numerical results demonstrate that the proposed algorithm is more effective than the traditional Euler method based algorithm. Furthermore, the solution converges to the desired accuracy in a small number of iterations.

This chapter is structured as follows: The continuous solution and its derivatives bounds are introduced in Section 4.1. In Section 4.2, we introduce the discrete method for solving problem (4.1) and analyze it in section 4.3. Finally, we give the numerical outcomes for a few test problems in Section 4.4. Then, we conclude the work in Section 4.5.

4.1 Derivative bounds

In this section, we derive the estimates for the solution of (4.1) and its derivatives. The solution $\boldsymbol{u}$ of problem (4.1) is decomposed into $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$ following [54]. The solutions $\boldsymbol{v}$ and $\boldsymbol{w}$ denote the regular and singular parts of the solution $\boldsymbol{u}$ respectively such that

$$\begin{cases} \partial_t \boldsymbol{v}(x,t) - \boldsymbol{\varepsilon}\partial_x^2 \boldsymbol{v}(x,t) + \boldsymbol{f}(x,t,\boldsymbol{v}) = \boldsymbol{0} \quad \text{in } \Omega, \\ \boldsymbol{v} = \boldsymbol{z} \quad \text{on } \{0,1\} \times (0,T], \\ \boldsymbol{v} = \boldsymbol{\phi}(x) \quad \text{on } [0,1] \times \{0\}, \end{cases} \tag{4.2}$$

and

$$\begin{cases} \partial_t \boldsymbol{w}(x,t) - \boldsymbol{\varepsilon}\partial_x^2 \boldsymbol{w}(x,t) + \boldsymbol{f}(x,t,\boldsymbol{v}+\boldsymbol{w}) - \boldsymbol{f}(x,t,\boldsymbol{v}) = \boldsymbol{0} \quad \text{in } \Omega, \\ \boldsymbol{w} = \boldsymbol{u} - \boldsymbol{v} \quad \text{on } (\{0,1\} \times (0,T]) \cup ([0,1] \times \{0\}), \end{cases} \tag{4.3}$$

where $\boldsymbol{z}$ satisfies

$$\begin{cases} \partial_t \boldsymbol{z} + \boldsymbol{f}(x,t,\boldsymbol{z}) = \boldsymbol{0}, \quad (x,t) \in \{0,1\} \times (0,T], \\ \boldsymbol{z}(x,0) = \boldsymbol{\phi}(x), \quad x \in \{0,1\}. \end{cases} \tag{4.4}$$

Then, using [65] the following results can be proved.

$$||\partial_t^s \boldsymbol{u}||_{\overline{\Omega}} \leq C, \;\; s = 0,1,2.$$

Lemma 4.1. *The regular part* $v = (v_1, v_2)^t$ *satisfies the following estimates* $||\partial_t^s \boldsymbol{v}||_{\overline{\Omega}} \leq C, \ ||\partial_x^s \boldsymbol{v}||_{\overline{\Omega}} \leq C, \quad s = 0, 1, 2, \ ||\partial_x^s v_j||_{\overline{\Omega}} \leq C\varepsilon_j^{(1-s/2)}, \quad j = 1, 2; \ s = 3, 4.$

Lemma 4.2. *The singular part* $w = (w_1, w_2)^t$ *satisfies the following estimates*

$|w_l(x,t)| \leq C\mathcal{B}_{\varepsilon_2}(x), \ l = 1, 2,$

$|\partial_x^s w_1(x,t)| \leq C(\varepsilon_1^{-s/2}\mathcal{B}_{\varepsilon_1}(x) + \varepsilon_2^{-s/2}\mathcal{B}_{\varepsilon_2}(x)), \ |\partial_x^s w_2(x,t)| \leq C(\varepsilon_2^{-s/2}\mathcal{B}_{\varepsilon_2}(x)), \quad s = 1, 2,$

$|\partial_x^s w_1(x,t)| \leq C(\varepsilon_1^{-s/2}\mathcal{B}_{\varepsilon_1}(x) + \varepsilon_2^{-s/2}\mathcal{B}_{\varepsilon_2}(x)), \quad s = 3, 4,$

$|\partial_x^s w_2(x,t)| \leq C\varepsilon_2^{-1}(\varepsilon_1^{-(s-2)/2}\mathcal{B}_{\varepsilon_1}(x) + \varepsilon_2^{-(s-2)/2}\mathcal{B}_{\varepsilon_2}(x)), \quad s = 3, 4, \quad \forall \ (x,t) \in \overline{\Omega}.$

Further, for $\varepsilon_1 < \varepsilon_2$ *and* $\varepsilon_2 \leq \alpha/2$, *the singular part* $\boldsymbol{w} = (w_1, w_2)^T$ *can be further decomposed as* $w_1 = \hat{w}_{1,\varepsilon_1} + \hat{w}_{1,\varepsilon_2}, \ w_2 = \hat{w}_{2,\varepsilon_1} + \hat{w}_{2,\varepsilon_2}$, *where*

$$|\hat{w}_{1,\varepsilon_1}(x,t)| \leq \mathcal{B}_{\varepsilon_1}(x), \quad |\partial_x^2 \hat{w}_{1,\varepsilon_1}(x,t)| \leq \varepsilon_1^{-1}\mathcal{B}_{\varepsilon_1}(x), \quad |\partial_x^4 \hat{w}_{1,\varepsilon_2}(x,t)| \leq \varepsilon_2^{-2}\mathcal{B}_{\varepsilon_2}(x),$$

$$|\hat{w}_{2,\varepsilon_1}(x,t)| \leq \mathcal{B}_{\varepsilon_1}(x), \quad |\partial_x^2 \hat{w}_{2,\varepsilon_1}(x,t)| \leq \varepsilon_2^{-1}\mathcal{B}_{\varepsilon_1}(x), \quad |\partial_x^4 \hat{w}_{2,\varepsilon_2}(x,t)| \leq \varepsilon_2^{-2}\mathcal{B}_{\varepsilon_2}(x).$$

for $(x,t) \in \overline{\Omega}$. *Here, the layer functions* $\mathcal{B}_{\varepsilon_q}(x)$ *are defined as* $\mathcal{B}_{\varepsilon_q}(x) = e^{-x\sqrt{\alpha/\varepsilon_q}} + e^{-(1-x)\sqrt{\alpha/\varepsilon_q}}, \ x \in [0,1], q = 1, 2.$

4.2 Splitting schemes based domain decomposition algorithm

From the bounds given in the above section we can observe two overlapping boundary layers near the points $x = 0, 1$. A domain decomposition method is constructed to solve problem (4.1). The original domain is divided as follows: $\Omega_p = D_p \times \omega, \ p = \ell\ell, \ell, m, r, rr$, such that $D_{\ell\ell} = (0, 4\rho_1), \ D_\ell = (\rho_1, 4\rho_2 - 3\rho_1), \ D_m = (\rho_2, 1-\rho_2), \ D_r = (1-4\rho_2+3\rho_1, 1-\rho_1), \ \ D_{rr} = (1-4\rho_1, 1)$ and $\omega = (0, T]$ with subdomain parameters

ρ_1 and ρ_2 (see[118])

$$\rho_2 = \min\left\{\frac{4}{26},\ \ 2\sqrt{\frac{\varepsilon_2}{\alpha}}\ln N\right\},\quad \rho_1 = \min\left\{\frac{\rho_2}{4},\ \ 2\sqrt{\frac{\varepsilon_1}{\alpha}}\ln N\right\}, \tag{4.5}$$

On each subdomain $\Omega_p = D_p \times \omega$, we consider the mesh $D_p^N \times \omega^M$ defined as follows. For $D_p = (c, d)$, we define a mesh $\overline{D}_p^N = \{x_i\}_{i=0}^N$ with uniform step length $h_p = (d-c)/N$, and the mesh $\overline{\omega}^M = \{t_j\}_{j=0}^M$ with uniform step length $\Delta t = T/M$ is defined on ω. Suppose $D_p^N = \overline{D}_p^N \cap D_p$ and $\omega^M = \overline{\omega}^M \cap (0, T]$. We consider the following discrete schemes on the subdomains $\Omega_p^{N,M}$:

Scheme 1 :

$$[\boldsymbol{\mathcal{L}}_p^{N,M}\mathbf{U}_p]_{i,j} = \begin{pmatrix} [\mathcal{L}_{p,1}^{N,M}\mathbf{U}_p]_{i,j} \\ [\mathcal{L}_{p,2}^{N,M}\mathbf{U}_p]_{i,j} \end{pmatrix} = \mathbf{0}, \tag{4.6}$$

where

$$[\mathcal{L}_{p,1}^{N,M}\mathbf{U}_p]_{i,j} := [\delta_t U_{p,1}]_{i,j} - \varepsilon_1[\delta_x^2 U_{p,1}]_{i,j} + f_1(x_i, t_j, [U_{p,1}]_{i,j}, [U_{p,2}]_{i,j-1}) = 0,$$

$$[\mathcal{L}_{p,2}^{N,M}\mathbf{U}_p]_{i,j} := [\delta_t U_{p,2}]_{i,j} - \varepsilon_2[\delta_x^2 U_{p,2}]_{i,j} + f_2(x_i, t_j, [U_{p,1}]_{i,j-1}, [U_{p,2}]_{i,j}) = 0.$$

Scheme 2 :

$$[\boldsymbol{\mathcal{L}}_p^{N,M}\mathbf{U}_p]_{i,j} = \begin{pmatrix} [\mathcal{L}_{p,1}^{N,M}\mathbf{U}_p]_{i,j} \\ [\mathcal{L}_{p,2}^{N,M}\mathbf{U}_p]_{i,j} \end{pmatrix} = \mathbf{0}, \tag{4.7}$$

where

$$[\mathcal{L}_{p,1}^{N,M}\mathbf{U}_p]_{i,j} := [\delta_t U_{p,1}]_{i,j} - \varepsilon_1[\delta_x^2 U_{p,1}]_{i,j} + f_1(x_i, t_j, [U_{p,1}]_{i,j}, [U_{p,2}]_{i,j-1}) = 0,$$

$$[\mathcal{L}_{p,2}^{N,M}\mathbf{U}_p]_{i,j} := [\delta_t U_{p,2}]_{i,j} - \varepsilon_2[\delta_x^2 U_{p,2}]_{i,j} + f_2(x_i, t_j, [U_{p,1}]_{i,j}, [U_{p,2}]_{i,j}) = 0.$$

Here,

$\delta_t \mathbf{W}_p(x_i,t_j) = \frac{\mathbf{W}_p(x_i,t_j)-\mathbf{W}_p(x_i,t_{j-1})}{\Delta t}$, $\delta_x^2 \mathbf{W}_p(x_i,t_j) = \frac{\mathbf{W}_p(x_i,t_{j-1})-2\mathbf{W}_p(x_i,t_j)+\mathbf{W}_p(x_i,t_{j+1})}{h_p^2}$.

Next, assuming $\overline{\Omega}^{N,M} := (\overline{\Omega}_{\ell\ell}^{N,M} \setminus \overline{\Omega}_\ell) \cup (\overline{\Omega}_\ell^{N,M} \setminus \overline{\Omega}_m) \cup \overline{\Omega}_m^{N,M} \cup (\overline{\Omega}_r^{N,M} \setminus \overline{\Omega}_m) \cup (\overline{\Omega}_{rr}^{N,M} \setminus \overline{\Omega}_r)$, and initial solution $\mathbf{U}^{[0]}$ as

$$\begin{cases} \mathbf{U}^{[0]}(x_i,t_j) = \mathbf{0} & \text{for } (x_i,t_j) \in D \times (0,T], \\ \mathbf{U}^{[0]}(x_i,t_j) = \boldsymbol{u}(x_i,t_j) & \text{for } (x_i,t_j) \in \overline{D}^N \times \{0\}, \\ \mathbf{U}^{[0]}(a,t_j) = \boldsymbol{u}(a,t_j) & \text{for } (a,t_j) \in \{0,1\} \times \omega^M, \end{cases}$$

we calculate the numerical solution $\mathbf{U}^{[k]}$, $k \geq 1$ of problem (4.1) on $\overline{\Omega}^{N,M}$ as follows

$$\mathbf{U}^{[k]} = \begin{cases} \mathbf{U}_{\ell\ell}^{[k]} & \text{in } \overline{\Omega}_{\ell\ell}^{N,M} \setminus \overline{\Omega}_\ell, \\ \mathbf{U}_{\ell}^{[k]} & \text{in } \overline{\Omega}_{\ell}^{N,M} \setminus \overline{\Omega}_m, \\ \mathbf{U}_{m}^{[k]} & \text{in } \overline{\Omega}_{m}^{N,M}, \\ \mathbf{U}_{r}^{[k]} & \text{in } \overline{\Omega}_{r}^{N,M} \setminus \overline{\Omega}_m, \\ \mathbf{U}_{rr}^{[k]} & \text{in } \in \overline{\Omega}_{rr}^{N,M} \setminus \overline{\Omega}_r, \end{cases} \tag{4.8}$$

where we solve

$$\begin{cases} [\boldsymbol{\mathcal{L}}_{\ell\ell}^{N,M} \mathbf{U}_{\ell\ell}^{[k]}]_{i,j} = \mathbf{0} \;\text{ in } \Omega_{\ell\ell}^{N,M}, \;\; \mathbf{U}_{\ell\ell}^{[k]}(x_i,0) = \boldsymbol{\phi}(x_i) \;\text{ for } x_i \in \overline{D}_{\ell\ell}^N, \\ \mathbf{U}_{\ell\ell}^{[k]}(0,t_j) = \boldsymbol{\varphi}_0(t_j) \;\text{ for } t_j \in \omega^M, \\ \mathbf{U}_{\ell\ell}^{[k]}(4\rho_1,t_j) = \mathcal{I}_j \mathbf{U}^{[k-1]}(4\rho_1,t_j) \;\text{ for } t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{\mathcal{L}}_{rr}^{N,M} \mathbf{U}_{rr}^{[k]}]_{i,j} = \mathbf{0} \;\text{ in } \Omega_{rr}^{N,M}, \;\; \mathbf{U}_{rr}^{[k]}(x_i,0) = \boldsymbol{\phi}(x_i) \;\text{ for } x_i \in \overline{D}_{rr}^N, \\ \mathbf{U}_{rr}^{[k]}(1-4\rho_1,t_j) = \mathcal{I}_j \mathbf{U}^{[k-1]}(1-4\rho_1,t_j) \;\text{ for } t_j \in \omega^M, \\ \mathbf{U}_{rr}^{[k]}(1,t_j) = \boldsymbol{\varphi}_1(t_j) \;\text{ for } t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{\mathcal{L}}_r^{N,M}\mathbf{U}_r^{[k]}]_{i,j} = \mathbf{0} \ \text{ in } \Omega_r^{N,M}, \ \ \mathbf{U}_r^{[k]}(x_i,0) = \boldsymbol{\phi}(x_i) \ \text{ for } x_i \in \overline{D}_r^N, \\ \mathbf{U}_r^{[k]}(1-4\rho_2+3\rho_1,t_j) = \mathcal{I}_j\mathbf{U}^{[k-1]}(1-4\rho_2+3\rho_1,t_j) \ \text{ for } t_j \in \omega^M, \\ \mathbf{U}_r^{[k]}(1-\rho_1,t_j) = \mathcal{I}_j\mathbf{U}_{rr}^{[k]}(1-\rho_1,t_j) \ \text{ for } t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{\mathcal{L}}_\ell^{N,M}\mathbf{U}_\ell^{[k]}]_{i,j} = \mathbf{0} \ \text{ in } \Omega_\ell^{N,M}, \ \ \mathbf{U}_\ell^{[k]}(x_i,0) = \boldsymbol{\phi}(x_i) \ \text{ for } x_i \in \overline{D}_\ell^N, \\ \mathbf{U}_\ell^{[k]}(\rho_1,t_j) = \mathcal{I}_j\mathbf{U}_{\ell\ell}^{[k]}(\rho_1,t_j) \ \text{ for } t_j \in \omega^M, \\ \mathbf{U}_\ell^{[k]}(4\rho_2-3\rho_1,t_j) = \mathcal{I}_j\mathbf{U}^{[k-1]}(4\rho_2-3\rho_1,t_j) \ \text{ for } t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{\mathcal{L}}_m^{N,M}\mathbf{U}_m^{[k]}]_{i,j} = \mathbf{0} \ \text{ in } \Omega_m^{N,M}, \ \ \mathbf{U}_m^{[k]}(x_i,0) = \boldsymbol{\phi}(x_i) \ \text{ for } x_i \in \overline{D}_m^N, \\ \mathbf{U}_m^{[k]}(\rho_2,t_j) = \mathcal{I}_j\mathbf{U}_\ell^{[k]}(\rho_2,t_j) \ \text{ for } t_j \in \omega^M, \\ \mathbf{U}_m^{[k]}(1-\rho_2,t_j) = \mathcal{I}_j\mathbf{U}_r^{[k]}(1-\rho_2,t_j) \ \text{ for } t_j \in \omega^M. \end{cases}$$

We repeat the process till $||\mathbf{U}^{[k+1]} - \mathbf{U}^{[k]}||_{\overline{\Omega}^{N,M}} \leq \tau$ (user chosen parameter).

4.3 Convergence analysis

Here, we give convergence analysis of the proposed algorithm. We define some auxiliary problems and construct the mesh function $\widetilde{\mathbf{U}}$ in a similar way as of (4.8). We define

$$\begin{cases} [\boldsymbol{\mathcal{L}}_{\ell\ell}^{N,M}\widetilde{\mathbf{U}}_{\ell\ell}]_{i,j} = \mathbf{0} \ \text{ in } \Omega_{\ell\ell}^{N,M}, \ \ \widetilde{\mathbf{U}}_{\ell\ell}(x_i,0) = \boldsymbol{\phi}(x_i) \ \text{ for } x_i \in \overline{D}_{\ell\ell}^N, \\ \widetilde{\mathbf{U}}_{\ell\ell}(0,t_j) = \boldsymbol{u}(0,t_j), \ \ \text{for } t_j \in \omega^M, \\ \widetilde{\mathbf{U}}_{\ell\ell}(4\rho_1,t_j) = \boldsymbol{u}(4\rho_1,t_j) \ \text{ for } t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{\mathcal{L}}_{rr}^{N,M}\widetilde{\mathbf{U}}_{rr}]_{i,j} = \mathbf{0} \ \text{ in } \Omega_{rr}^{N,M}, \ \ \widetilde{\mathbf{U}}_{rr}(x_i,0) = \boldsymbol{\phi}(x_i) \ \text{ for } x_i \in \overline{D}_{rr}^N, \\ \widetilde{\mathbf{U}}_{rr}(1-4\rho_1,t_j) = \boldsymbol{u}(1-4\rho_1,t_j), \ \ \text{for } t_j \in \omega^M, \\ \widetilde{\mathbf{U}}_{rr}(1,t_j) = \boldsymbol{u}(1,t_j) \ \text{ for } t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{\mathcal{L}}_r^{N,M}\widetilde{\mathbf{U}}_r]_{i,j} = \mathbf{0} \ \text{ in } \Omega_r^{N,M}, \ \ \widetilde{\mathbf{U}}_r(x_i,0) = \boldsymbol{\phi}(x_i) \ \text{ for } x_i \in \overline{D}_r^N, \\ \widetilde{\mathbf{U}}_r(1-4\rho_2+3\rho_1,t_j) = \boldsymbol{u}(1-4\rho_2+3\rho_1,t_j), \ \text{for } t_j \in \omega^M, \\ \widetilde{\mathbf{U}}_r(1-\rho_1,t_j) = \boldsymbol{u}(1-\rho_1,t_j) \ \text{ for } t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{\mathcal{L}}_\ell^{N,M}\widetilde{\mathbf{U}}_\ell]_{i,j} = \mathbf{0} \ \text{ in } \Omega_\ell^{N,M}, \ \ \widetilde{\mathbf{U}}_\ell(x_i,0) = \boldsymbol{\phi}(x_i) \ \text{ for } x_i \in \overline{D}_\ell^N, \\ \widetilde{\mathbf{U}}_\ell(\rho_1,t_j) = \boldsymbol{u}(\rho_1,t_j), \ \ \text{for } t_j \in \omega^M, \\ \widetilde{\mathbf{U}}_\ell(4\rho_2-3\rho_1,t_j) = \boldsymbol{u}(4\rho_2-3\rho_1,t_j) \ \text{ for } t_j \in \omega^M, \end{cases}$$

$$\begin{cases} [\boldsymbol{\mathcal{L}}_m^{N,M}\widetilde{\mathbf{U}}_m]_{i,j} = \mathbf{0} \ \text{ in } \Omega_m^{N,M}, \ \ \widetilde{\mathbf{U}}_m(x_i,0) = \boldsymbol{\phi}(x_i) \ \text{ for } x_i \in \overline{D}_m^N, \\ \widetilde{\mathbf{U}}_m(\rho_2,t_j) = \boldsymbol{u}(\rho_2,t_j), \ \ \text{for } t_j \in \omega^M, \\ \widetilde{\mathbf{U}}_m(1-\rho_2,t_j) = \boldsymbol{u}(1-\rho_2,t_j) \ \text{ for } t_j \in \omega^M. \end{cases}$$

Now, using the solution $\widetilde{\mathbf{U}}$ we split the global error $||\boldsymbol{u}-\mathbf{U}^{[k]}||_{\overline{\Omega}^{N,M}}$ into the discretization error $||\boldsymbol{u}-\widetilde{\mathbf{U}}||_{\overline{\Omega}^{N,M}}$ and iteration error $||\widetilde{\mathbf{U}}-\mathbf{U}^{[k]}||_{\overline{\Omega}^{N,M}}$, and calculate the error bound by using the following triangle inequality

$$||\boldsymbol{u}-\mathbf{U}^{[k]}||_{\overline{\Omega}^{N,M}} \leq ||\boldsymbol{u}-\widetilde{\mathbf{U}}||_{\overline{\Omega}^{N,M}} + ||\widetilde{\mathbf{U}}-\mathbf{U}^{[k]}||_{\overline{\Omega}^{N,M}}. \tag{4.9}$$

Now, we will introduce a few notation which we shall use to prove the convergence of the algorithm.

$$\begin{aligned}
\xi_{\rho_1} &= \max\{\max_{t_j\in\omega^M} |(\widetilde{\mathbf{U}}_\ell - \widetilde{\mathbf{U}}_{\ell\ell})(\rho_1,t_j)|, \max_{t_j\in\omega^M} |(\widetilde{\mathbf{U}}_r - \widetilde{\mathbf{U}}_{rr})(1-\rho_1,t_j)\}, \\
\xi_{\rho_2} &= \max\{\max_{t_j\in\omega^M} |(\widetilde{\mathbf{U}}_m - \widetilde{\mathbf{U}}_\ell)(\rho_2,t_j)|, \max_{t_j\in\omega^M} |(\widetilde{\mathbf{U}}_m - \widetilde{\mathbf{U}}_r)(1-\rho_2,t_j)\}, \\
\xi_{4\rho_1} &= \max\{\max_{t_j\in\omega^M} |(\widetilde{\mathbf{U}}_{\ell\ell} - \mathcal{I}_j\widetilde{\mathbf{U}}_\ell)(4\rho_1,t_j)|, \max_{t_j\in\omega^M} |(\widetilde{\mathbf{U}}_{rr} - \mathcal{I}_j\widetilde{\mathbf{U}}_r)(1-4\rho_1,t_j)\}, \\
\xi_{4\rho_2-3\rho_1} &= \max\{\max_{t_j\in\omega^M} |(\widetilde{\mathbf{U}}_\ell - \mathcal{I}_j\widetilde{\mathbf{U}}_m)(4\rho_2-3\rho_1,t_j)|, \max_{t_j\in\omega^M} |(\widetilde{\mathbf{U}}_r - \mathcal{I}_j\widetilde{\mathbf{U}}_r)(1-4\rho_2+3\rho_1,t_j)\},
\end{aligned}$$

$$\xi^{[k]} = \max\{\max_{t_j\in\omega^M}|(\widetilde{\mathbf{U}}_{\ell\ell}-\mathcal{I}_j\mathbf{U}^{[k-1]})(4\rho_1,t_j)|,\max_{t_j\in\omega^M}|(\widetilde{\mathbf{U}}_{rr}-\mathcal{I}_j\mathbf{U}^{[k-1]})(1-4\rho_1,t_j)|,$$
$$\max_{t_j\in\omega^M}|(\widetilde{\mathbf{U}}_{\ell}-\mathcal{I}_j\mathbf{U}^{[k-1]})(4\rho_2-3\rho_1,t_j)|,\max_{t_j\in\omega^M}|(\widetilde{\mathbf{U}}_{r}-\mathcal{I}_j\mathbf{U}^{[k-1]})(1-4\rho_2+3\rho_1,t_j)|\}.$$

Lemma 4.3. *Let $\boldsymbol{u}$ be the solution of (4.1) and $\widetilde{\mathbf{U}}_p$ denote the solution of auxiliary problems. Then*

$$||\boldsymbol{u}-\widetilde{\mathbf{U}}_p||_{\overline{\Omega}_p^{N,M}} \le C(\Delta t + N^{-2}\ln^2 N).$$

Proof. Consider Scheme 1 for $(x_i,t_j)\in\Omega_p^{N,M},\ p=\ell\ell,\ell,m,r,rr$. Then, for $n=1,2$, we have

$$[\delta_t e_{p,n}]_{i,j}-\varepsilon_n[\delta_x^2 e_{p,n}]_{i,j}+(f_n(x_i,t_j,u_{n;i,j},u_{3-n;i,j-1})-f_n(x_i,t_j,\widetilde{U}_{p,n;i,j},\widetilde{U}_{p,3-n;i,j-1}))$$
$$=[(\delta_t-\partial_t)u_n]_{i,j}+\varepsilon_n[(\partial_x^2-\delta_x^2)u_n]_{i,j}-(f_n(x_i,t_j,u_{n;i,j},u_{3-n;i,j})-f_n(x_i,t_j,u_{n;i,j},u_{3-n;i,j-1})), \tag{4.10}$$

where $\mathbf{e}_p(x_i,t_j)=\boldsymbol{u}(x_i,t_j)-\widetilde{\mathbf{U}}_p(x_i,t_j)$ denotes the error function and $\mathbf{e}_p=(e_{p,1},e_{p,2})^T$. The error equation (4.10) defined as

$$L_{p,n}^{N,M}(u-\widetilde{U}_p)_{i,j}:=[\delta_t e_{p,n}]_{i,j}-\varepsilon_n[\delta_x^2 e_{p,n}]_{i,j}+[s_{p,n}]_{i,j}(u_{1;i,j-n+1}-\widetilde{U}_{p,1;i,j-n+1})$$
$$+[q_{p,n}]_{i,j}(u_{2;i,j+n-2}-\widetilde{U}_{p,2;i,j+n-2})$$
$$=[(\delta_t-\partial_t)u_n]_{i,j}+\varepsilon_n[(\partial_x^2-\delta_x^2)u_n]_{i,j}-(f_n(x_i,t_j,u_{n;i,j},u_{3-n;i,j})-f_n(x_i,t_j,u_{n;i,j},u_{3-n;i,j-1})),$$

where

$$s_{p,n;i,j}=\int_0^1\frac{\partial f_n}{\partial u_1}\Big(x_i,t_j,\widetilde{U}_{p,n;i,j}+a(u_{n;i,j}-\widetilde{U}_{p,n;i,j}),\widetilde{U}_{p,3-n;i,j-1}+a(u_{3-n;i,j-1}-\widetilde{U}_{p,3-n;i,j-1})\Big)\,da$$

and

$$q_{p,n;i,j}=\int_0^1\frac{\partial f_n}{\partial u_2}\Big(x_i,t_j,\widetilde{U}_{p,n;i,j}+a(u_{n;i,j}-\widetilde{U}_{p,n;i,j}),\widetilde{U}_{p,3-n;i,j-1}+a(u_{3-n;i,j-1}-\widetilde{U}_{p,3-n;i,j-1})\Big)\,da.$$

Now, using the mean-value theorem the above expression leads to

$$L_{p,n}^{N,M}(u-\widetilde{U}_p)_{i,j} = \left(\delta_t^- - \partial_t\right) u_{n;i,j} + \varepsilon_n \left(\delta_x^2 - \partial_x^2\right) u_{n;i,j} + a_n(u_{3-n;i,j} - u_{3-n;i,j-1}), \quad (4.11)$$

where $a_1 = \dfrac{\partial f_1}{\partial u_2}(x_i, t_j, \eta_{1;i,j}, \nu_{1;i,j})$, $a_2 = \dfrac{\partial f_2}{\partial u_1}(x_i, t_j, \eta_{2;i,j}, \nu_{2;i,j})$ and $\eta_{k;i,j}$, $\nu_{k;i,j}$, $k = 1, 2$, are intermediate values. The operator $\boldsymbol{L}_p^{N,M} = (L_{p,1}^{N,M}, L_{p,2}^{N,M})^T$ satisfies the following discrete maximum principle.

Suppose the mesh function $\boldsymbol{W}_p$ satisfies $\boldsymbol{W}_p(x_i, t_j) \geq \boldsymbol{0}$, $i = 0, N$, for $t_j \in \omega^M$, and $\boldsymbol{W}_p(x_i, 0) \geq \boldsymbol{0}$ for $x_i \in \overline{D}_p^N$. If $\boldsymbol{L}_p^{N,M}\boldsymbol{W}_p \geq \boldsymbol{0}$ in $\Omega_p^{N,M}$, then $\boldsymbol{W}_p \geq \boldsymbol{0}$ in $\overline{\Omega}_p^{N,M}$ $p = \ell\ell, \ell, m, r, rr$.

For the third term of equation (4.11) we have

$$\begin{aligned} |a_n(u_{3-n;i,j} - u_{3-n;i,j-1})| &\leq C\left|(u_{3-n;i,j} - u_{3-n;i,j-1})\right| \\ &\leq C(t_j - t_{j-1}) \left\|\partial_t u_{3-n}(x_i, .)\right\|_{[t_{j-1}, t_j]} \leq C\Delta t, \end{aligned} \quad (4.12)$$

by using Taylor expansion and the bounds in Section 4.1.

Suppose $\rho_1 = (2\sqrt{\varepsilon_1}\ln N)/\sqrt{\alpha}$ and $\rho_2 = (2\sqrt{\varepsilon_2}\ln N)/\sqrt{\alpha}$. In this case, ε_1 and ε_2 are the parameters of different magnitudes and they are small. The other cases can be handled similar to this one.

Now, using Taylor expansions and bounds in Section 4.1 with $h_{\ell\ell} \leq C\sqrt{\varepsilon_1}N^{-1}\ln N$, the rest two terms of equation (4.11) can be bounded as follows

$$\begin{aligned} \left|\left(\delta_t - \partial_t\right) u_{n;i,j}\right| + \varepsilon_n \left|\left(\delta_x^2 - \partial_x^2\right) u_{n;i,j}\right| &\leq C(t_j - t_{j-1}) \left\|\partial_t^2 u_n(x_i, .)\right\|_{[t_{j-1}, t_j]} + Ch_p^2 \varepsilon_n \left\|\partial_x^4 u_n(., t_j)\right\|_{[x_{i-1}, x_{i+1}]} \\ &\leq C(\Delta t + N^{-2}\ln^2 N), \quad n = 1, 2. \end{aligned}$$

Now, applying the discrete maximum principle for $\boldsymbol{L}_{\ell\ell}^{N,M}$ to $\boldsymbol{C}(\Delta t + N^{-2}\ln^2 N) \pm (\boldsymbol{u} - \widetilde{\mathbf{U}}_{\ell\ell})(x_i, t_j)$, we obtain

$$||\boldsymbol{u} - \widetilde{\mathbf{U}}_{\ell\ell}||_{\overline{\Omega}_{\ell\ell}^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N).$$

Similarly,

$$||\boldsymbol{u} - \widetilde{\mathbf{U}}_{rr}||_{\overline{\Omega}_{rr}^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N).$$

For $(x_i, t_j) \in \Omega_\ell^{N,M}$, $n = 1, 2$,

$$\begin{aligned} L_{\ell,n}^{N,M}(u - \widetilde{U}_\ell)_{i,j} := [\delta_t e_{\ell,n}]_{i,j} - \varepsilon_n[\delta_x^2 e_{\ell,n}]_{i,j} + [s_{\ell,n}]_{i,j}(u_{1;i,j-n+1} - \widetilde{U}_{\ell,1;i,j-n+1}) \\ +[q_{\ell,n}]_{i,j}(u_{2;i,j+n-2} - \widetilde{U}_{\ell,2;i,j+n-2}) \\ = [(\delta_t - \partial_t)u_n]_{i,j} + \varepsilon_n[(\partial_x^2 - \delta_x^2)u_n]_{i,j} + a_n(u_{3-n;i,j} - u_{3-n;i,j-1}), \end{aligned}$$

Then by using equation (4.12) we get

$$|L_{\ell,n}^{N,M}(u - \widetilde{U}_\ell)_{i,j}| \leq |\,(\delta_t - \partial_t)\,u_{n;i,j}| + \varepsilon_n \left|\left(\delta_x^2 - \partial_x^2\right) u_{n;i,j}\right| + C\Delta t. \tag{4.13}$$

Further, $|\,(\delta_t - \partial_t)\,u_{n;i,j}| \leq C\Delta t$ by using Taylor expansion and the bounds in Section (4.1). To calculate the estimate for $\varepsilon_n\,|(\delta_x^2 - \partial_x^2)\,u_{n;i,j}|$, use $u_n = v_n + w_n,\ w_n = \hat{w}_{n,\varepsilon_1} + \hat{w}_{n,\varepsilon_2}$ and consider the Taylor expansions, derivatives bounds with the step length in equation (4.13) to get

$$\begin{aligned} \varepsilon_n \left|\left(\delta_x^2 - \partial_x^2\right) u_{n;i,j}\right| &\leq Ch_p^2\varepsilon_n \left\|\partial_x^4 v_n(.,t_j)\right\|_{[x_{i-1},x_{i+1}]} + C\varepsilon_n \left\|\partial_x^2 \hat{w}_{n,\varepsilon_1}(.,t_j)\right\|_{[x_{i-1},x_{i+1}]} \\ &\quad + Ch_p^2\varepsilon_n \left\|\partial_x^4 \hat{w}_{n,\varepsilon_2}(.,t_j)\right\|_{[x_{i-1},x_{i+1}]} \\ &\leq CN^{-2}\ln^2 N,\ \ n = 1, 2. \end{aligned}$$

Now, by utilizing discrete maximum principle for $\boldsymbol{L}_\ell^{N,M}$ to $\boldsymbol{C}(\Delta t+N^{-2}\ln^2 N)\pm(\boldsymbol{u}-\widetilde{\mathbf{U}}_\ell)(x_i,t_j)$, we obtain

$$||\boldsymbol{u}-\widetilde{\mathbf{U}}_\ell||_{\overline{\Omega}_\ell^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N).$$

Similarly,

$$||\boldsymbol{u}-\widetilde{\mathbf{U}}_r||_{\overline{\Omega}_r^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N).$$

For $(x_i,t_j)\in\Omega_m^{N,M}$, $n=1,2$,

$$\begin{aligned} L_{m,n}^{N,M}(u-\widetilde{U}_m)_{i,j} &:= [\delta_t e_{m,n}]_{i,j} - \varepsilon_n[\delta_x^2 e_{m,n}]_{i,j} + [s_{m,n}]_{i,j}(u_{1;i,j-n+1}-\widetilde{U}_{m,1;i,j-n+1}) \\ &\qquad + [q_{m,n}]_{i,j}(u_{2;i,j+n-2}-\widetilde{U}_{m,2;i,j+n-2}) \\ &= [(\delta_t-\partial_t)u_n]_{i,j} + \varepsilon_n[(\partial_x^2-\delta_x^2)u_n]_{i,j} + a_n(u_{3-n;i,j}-u_{3-n;i,j-1}), \end{aligned}$$

Now using equation (4.12) we get

$$|L_{m,n}^{N,M}(u-\widetilde{U}_m)_{i,j}| \leq |\left(\delta_t-\partial_t\right)u_{n;i,j}| + \varepsilon_n\left|\left(\delta_x^2-\partial_x^2\right)u_{n;i,j}\right| + C\Delta t. \tag{4.14}$$

Also, $|\left(\delta_t-\partial_t\right)u_{n;i,j}|\leq C\Delta t$ by using Taylor expansion and the bounds in Section (4.1). To calculate the estimate for $\varepsilon_n\left|(\delta_x^2-\partial_x^2)\,u_{n;i,j}\right|$, use $u_n=v_n+w_n$. Apply Taylor expansions with derivatives bounds in Section 4.1 to obtain

$$\begin{aligned} \varepsilon_n\left|\left(\delta_x^2-\partial_x^2\right)u_{n;i,j}\right| &\leq Ch_m^2\varepsilon_n\left\|\partial_x^4 v_n(.,t_j)\right\|_{[x_{i-1},x_{i+1}]} + C\varepsilon_n\left\|\partial_x^2 w_n(.,t_j)\right\|_{[x_{i-1},x_{i+1}]} \\ &\leq C(\Delta t+N^{-2}\ln^2 N),\ \ n=1,2. \end{aligned}$$

Now, by utilizing discrete maximum principle for $\boldsymbol{L}_m^{N,M}$ to $\boldsymbol{C}(\Delta t+N^{-2}\ln^2 N)\pm(\boldsymbol{u}-\widetilde{\mathbf{U}}_m)(x_i,t_j)$, we obtain

$$||\boldsymbol{u}-\widetilde{\boldsymbol{U}}_m||_{\overline{\Omega}_m^{N,M}} \leq C(\Delta t + N^{-2}\ln^2 N).$$

We can establish the same bound for the Scheme 2 by applying the preceding arguments. □

Theorem 4.4. *Suppose* $\widetilde{\mathbf{U}}$ *and* $\mathbf{U}^{[k]}$ *are the solutions of the auxiliary problems and the proposed algorithm respectively. Then*

$$||\widetilde{\mathbf{U}}-\mathbf{U}^{[k]}||_{\overline{\Omega}^{N,M}} \leq C\left(\frac{1}{2}\right)^k + C(\Delta t + N^{-2}\ln^2 N). \tag{4.15}$$

Proof. Suppose $(x_i,t_j)\in\Omega_p^{N,M},\ p=\ell\ell,\ell,m,r,rr$ and the operator $\mathbb{L}_p^{N,M}=(\mathbb{L}_{p,1}^{N,M},\mathbb{L}_{p,2}^{N,M})^T$. Then, the error equation of $\mathbf{e}_p^{[1]}(x_i,t_j)=(\widetilde{\mathbf{U}}_p-\mathbf{U}_p^{[1]})$ is defined as

$$\begin{aligned}\mathbb{L}_{p,n}^{\mathcal{N},\mathcal{M}}(\widetilde{U}_p-U_p^{[1]})_{i,j} := [\delta_t^- e_{p,n}^{[1]}]_{i,j}-\varepsilon_n[\delta_x^2 e_{p,n}^{[1]}]_{i,j}+[\hat{s}_{p,n}]_{i,j}(\widetilde{U}_{1;i,j-n+1}-U_{p,1;i,j-n+1}^{[1]})\\ +[\hat{q}_{p,n}]_{i,j}(\widetilde{U}_{2;i,j+n-2}-U_{p,2;i,j+n-2}^{[1]})=0,\ n=1,2,\end{aligned} \tag{4.16}$$

where

$$\hat{s}_{p,n;i,j}=\int_0^1\frac{\partial f_n}{\partial y_1}\left(x_i,t_j,U_{p,n;i,j}^{[1]}+a(\widetilde{U}_{n;i,j}-U_{p,n;i,j}^{[1]}),U_{p,3-n;i,j-1}^{[1]}+a(\widetilde{U}_{3-n;i,j-1}-U_{p,3-n;i,j-1}^{[1]})\right)da$$

and

$$\hat{q}_{p,n;i,j}=\int_0^1\frac{\partial f_n}{\partial y_2}\left(x_i,t_j,U_{p,n;i,j}^{[1]}+a(\widetilde{U}_{n;i,j}-U_{p,n;i,j}^{[1]}),U_{p,3-n;i,j-1}^{[1]}+a(\widetilde{U}_{3-n;i,j-1}-U_{p,3-n;i,j-1}^{[1]})\right)da.$$

For $(x_i,t_j)\in\Omega_{\ell\ell}^{N,M}$, $\widetilde{\mathbf{U}}_{\ell\ell}-\mathbf{U}_{\ell\ell}^{[1]}$ satisfies

$$\mathbb{L}_{\ell\ell}^{N,M}(\widetilde{\mathbf{U}}_{\ell\ell}-\mathbf{U}_{\ell\ell}^{[1]})=\mathbf{0},\ \text{in}\ \Omega_{\ell\ell}^{N,M},\ (\widetilde{\mathbf{U}}_{\ell\ell}-\mathbf{U}_{\ell\ell}^{[1]})(x_i,0)=\mathbf{0},\quad x_i\in\overline{D}^N,$$

$$(\widetilde{\mathbf{U}}_{\ell\ell}-\mathbf{U}_{\ell\ell}^{[1]})(0,t_j)=\mathbf{0},\ |(\widetilde{\mathbf{U}}_{\ell\ell}-\mathbf{U}_{\ell\ell}^{[1]})(4\rho_1,t_j)|\le\xi^{[1]}\mathbf{1},\ t_j\in\omega^M.$$

Assume the mesh function $\Psi_1^{\pm}(x_i,t_j)=\frac{x_i}{4\rho_1}\xi^{[1]}\mathbf{1}\pm(\widetilde{\mathbf{U}}_{\ell\ell}-\mathbf{U}_{\ell\ell}^{[1]})(x_i,t_j)$ for $(x_i,t_j)\in\Omega_{\ell\ell}^{N,M}$. Now, employing the discrete maximum principle to the mesh function $\Psi_1^{\pm}(x_i,t_j)$, we obtain

$$|(\widetilde{\mathbf{U}}_{\ell\ell}-\mathbf{U}_{\ell\ell}^{[1]})|\le\frac{x_i}{4\rho_1}\xi^{[1]}\mathbf{1},\quad (x_i,t_j)\in\overline{\Omega}_{\ell\ell}^{N,M}.$$

Therefore,

$$||\widetilde{\mathbf{U}}_{\ell\ell}-\mathbf{U}_{\ell\ell}^{[1]}||_{\overline{\Omega}_{\ell\ell}^{N,M}\backslash\overline{\Omega}_{\ell}}\le\frac{1}{4}\xi^{[1]}.\tag{4.17}$$

Similarly,

$$||\widetilde{\mathbf{U}}_{rr}-\mathbf{U}_{rr}^{[1]}||_{\overline{\Omega}_{rr}^{N,M}\backslash\overline{\Omega}_{r}}\le\frac{1}{4}\xi^{[1]}.\tag{4.18}$$

Next, for $(x_i,t_j)\in\Omega_{\ell}^{N,M},\ n=1,2,$ the error equation of $\mathbf{e}_{\ell}^{[1]}(x_i,t_j)=(\widetilde{\mathbf{U}}_{\ell}-\mathbf{U}_{\ell}^{[1]})$ is defined as

$$\mathbb{L}_{\ell,n}^{\mathcal{N},\mathcal{M}}(\widetilde{U}_\ell-U_\ell^{[1]})_{i,j}:=[\delta_t^- e_{\ell,n}^{[1]}]_{i,j}-\varepsilon_n[\delta_x^2 e_{\ell,n}^{[1]}]_{i,j}+[\hat{s}_{\ell,n}]_{i,j}(\widetilde{U}_{1;i,j-n+1}-U_{\ell,1;i,j-n+1}^{[1]})$$
$$+[\hat{q}_{\ell,n}]_{i,j}(\widetilde{U}_{2;i,j+n-2}-U_{\ell,2;i,j+n-2}^{[1]})=0.\tag{4.19}$$

Thus, $\widetilde{\mathbf{U}}_{\ell}-\mathbf{U}_{\ell}^{[1]}$ satisfies

$$\mathbb{L}_{\ell}^{N,M}(\widetilde{\mathbf{U}}_{\ell}-\mathbf{U}_{\ell}^{[1]})=\mathbf{0},\ \text{in}\ \Omega_{\ell}^{N,M},\ (\widetilde{\mathbf{U}}_{\ell}-\mathbf{U}_{\ell}^{[1]})(x_i,0)=\mathbf{0},\quad x_i\in\overline{D}^N,\tag{4.20}$$

$$|(\widetilde{\mathbf{U}}_{\ell}-\mathbf{U}_{\ell}^{[1]})(4\rho_2-3\rho_1,t_j)|\le\xi^{[1]}\mathbf{1},\quad t_j\in\omega^M,$$

$$|(\widetilde{\mathbf{U}}_{\ell}-\mathbf{U}_{\ell}^{[1]})(\rho_1,t_j)|\le|(\widetilde{\mathbf{U}}_{\ell}-\widetilde{\mathbf{U}}_{\ell\ell})(\rho_1,t_j)|+|(\widetilde{\mathbf{U}}_{\ell\ell}-\mathbf{U}_{\ell}^{[1]})(\rho_1,t_j)|$$

$$\le \frac{1}{4}\xi^{[1]}\mathbf{1} + \xi_{\rho_1}\mathbf{1}, \quad t_j \in \omega^M.$$

Now, we consider mesh function $\Psi_2^{\pm}(x_i,t_j) = \zeta(x_i)\xi^{[1]}\mathbf{1} + \xi_{\rho_1}\mathbf{1} \pm (\widetilde{\mathbf{U}}_\ell - \mathbf{U}_\ell^{[1]})(x_i,t_j)$ for $(x_i,t_j) \in \Omega_\ell^{N,M}$, where $\zeta(x) = \dfrac{-x^2 + (13\rho_2 - 11\rho_1)x + 12\rho_2^2 + 24\rho_1^2 - 37\rho_1\rho_2}{48(\rho_2-\rho_1)^2}$, is an increasing function in the domain $[\rho_1, 4\rho_2 - 3\rho_1]$ and $\zeta(\rho_2) = 1/2$.

So, on employing the discrete maximum principle to the mesh function $\Psi_2^{\pm}(x_i,t_j)$ for $(x_i,t_j) \in \overline{\Omega}_\ell^{N,M}$, we obtain

$$|(\widetilde{\mathbf{U}}_\ell - \mathbf{U}_\ell^{[1]})(x_i,t_j)| \le \zeta(x_i)\xi^{[1]}\mathbf{1} + \xi_{\rho_1}\mathbf{1}.$$

Therefore, we get

$$||\widetilde{\mathbf{U}}_\ell - \mathbf{U}_\ell^{[1]}||_{\overline{\Omega}_\ell^{N,M}\setminus\overline{\Omega}_m} \le \frac{1}{2}\xi^{[1]} + \xi_{\rho_1}. \tag{4.21}$$

Similarly,

$$||\widetilde{\mathbf{U}}_r - \mathbf{U}_r^{[1]}||_{\overline{\Omega}_r^{N,M}\setminus\overline{\Omega}_m} \le \frac{1}{2}\xi^{[1]} + \xi_{\rho_1}. \tag{4.22}$$

Next, for $(x_i,t_j) \in \Omega_m^{N,M}$, $n = 1,2$, the error equation of $\mathbf{e}_m^{[1]}(x_i,t_j) = (\widetilde{\mathbf{U}}_m - \mathbf{U}_m^{[1]})$ is defined as

$$\begin{aligned}\mathbb{L}_{m,n}^{\mathcal{N},\mathcal{M}}(\widetilde{U}_m - U_m^{[1]})_{i,j} := [\delta_t^- e_{m,n}^{[1]}]_{i,j} - \varepsilon_n[\delta_x^2 e_{m,n}^{[1]}]_{i,j} + [\hat{s}_{m,n}]_{i,j}(\widetilde{U}_{1;i,j-n+1} - U_{m,1;i,j-n+1}^{[1]})\\ +[\hat{q}_{m,n}]_{i,j}(\widetilde{U}_{2;i,j+n-2} - U_{m,2;i,j+n-2}^{[1]}) = 0.\end{aligned} \tag{4.23}$$

Thus, $\widetilde{\mathbf{U}}_m - \mathbf{U}_m^{[1]}$ satisfies

$$\mathbf{L}_m^{N,M}(\widetilde{\mathbf{U}}_m - \mathbf{U}_m^{[1]}) = \mathbf{0}, \text{ in } \Omega_m^{N,M}, (\widetilde{\mathbf{U}}_m - \mathbf{U}_m^{[1]})(x_i,0) = \mathbf{0}, \ x_i \in \overline{D}^N,$$

$$\begin{aligned}|(\widetilde{\mathbf{U}}_m - \mathbf{U}_m^{[1]})(\rho_2,t_j)| &\le |(\widetilde{\mathbf{U}}_m - \widetilde{\mathbf{U}}_\ell)(\rho_2,t_j)| + |(\widetilde{\mathbf{U}}_\ell - \mathbf{U}_m^{[1]})(\rho_2,t_j)|\\ &\le \frac{1}{2}\xi^{[1]}\mathbf{1} + \xi_{\rho_1}\mathbf{1} + \xi_{\rho_2}\mathbf{1}, \ t_j \in \omega^M,\end{aligned}$$

$$|(\widetilde{\mathbf{U}}_m - \mathbf{U}_m^{[1]})(1-\rho_2, t_j)| \le |(\widetilde{\mathbf{U}}_m - \widetilde{\mathbf{U}}_r)(1-\rho_2, t_j)| + |(\widetilde{\mathbf{U}}_r - \mathbf{U}_m^{[1]})(1-\rho_2, t_j)|$$

$$\le \frac{1}{2}\xi^{[1]}\mathbf{1} + \xi_{\rho_1}\mathbf{1} + \xi_{\rho_2}\mathbf{1}, \ t_j \in \omega^M.$$

Therefore, by employing the discrete maximum principle, we get

$$||\widetilde{\mathbf{U}}_m - \mathbf{U}_m^{[1]}||_{\overline{\Omega}_m^{N,M}} \le \frac{1}{2}\xi^{[1]} + \xi_{\rho_1} + \xi_{\rho_2}. \tag{4.24}$$

Hence,

$$||\widetilde{\mathbf{U}} - \mathbf{U}^{[1]}||_{\overline{\Omega}^{N,M}} \le \frac{1}{2}\xi^{[1]} + \xi_{\rho_1} + \xi_{\rho_2}.$$

Now, we calculate the bound for the term $||\widetilde{\mathbf{U}} - \mathbf{U}^{[2]}||$. For this first we need to find the estimate for $\xi^{[2]}$. By using equations (4.21), (4.22) and (4.24) along with a triangle inequality, the operator $\mathcal{I}_j$ stability, we get

$$\begin{aligned}
|(\widetilde{\mathbf{U}}_{\ell\ell} - \mathcal{I}_j\mathbf{U}^{[1]})(4\rho_1, t_j)| &\le \xi_{4\rho_1}\mathbf{1} + \frac{1}{2}\xi^{[1]}\mathbf{1} + \xi_{\rho_1}\mathbf{1},\\
|(\widetilde{\mathbf{U}}_{rr} - \mathcal{I}_j\mathbf{U}^{[1]})(1-4\rho_1, t_j)| &\le \xi_{4\rho_1}\mathbf{1} + \frac{1}{2}\xi^{[1]}\mathbf{1} + \xi_{\rho_1}\mathbf{1},\\
|(\widetilde{\mathbf{U}}_{\ell} - \mathcal{I}_j\mathbf{U}^{[1]})(4\rho_2 - 3\rho_1, t_j)| &\le \xi_{4\rho_2-3\rho_1}\mathbf{1} + \frac{1}{2}\xi^{[1]}\mathbf{1} + \xi_{\rho_1}\mathbf{1} + \xi_{\rho_2}\mathbf{1},\\
|(\widetilde{\mathbf{U}}_{r} - \mathcal{I}_j\mathbf{U}^{[1]})(1-4\rho_2 + 3\rho_1, t_j)| &\le \xi_{4\rho_2-3\rho_1}\mathbf{1} + \frac{1}{2}\xi^{[1]}\mathbf{1} + \xi_{\rho_1}\mathbf{1} + \xi_{\rho_2}\mathbf{1}.
\end{aligned}$$

Therefore,

$$\xi^{[2]} \le \frac{1}{2}\xi^{[1]} + \xi_{\rho_1} + \xi_{\rho_2} + \xi_{4\rho_1} + \xi_{4\rho_2-3\rho_1}.$$

Hence,

$$\max\left\{\xi^{[2]}, ||\widetilde{\mathbf{U}} - \mathbf{U}^{[1]}||_{\overline{\Omega}^{N,M}}\right\} \le \lambda + \frac{1}{2}\xi^{[1]}, \quad \lambda = \xi_{\rho_1} + \xi_{\rho_2} + \xi_{4\rho_1} + \xi_{4\rho_2-3\rho_1}.$$

Utilizing the same arguments as before will lead to

$$\max\left\{\xi^{[k+1]}, ||\widetilde{\mathbf{U}}-\mathbf{U}^{[k]}||_{\overline{\Omega}^{N,M}}\right\} \leq \lambda + \frac{1}{2}\xi^{[k]}.$$

On simplifying the above expression, we get $\xi^{[k]} \leq 2\lambda + \left(\frac{1}{2}\right)^{(k-1)}\xi^{[1]}$. Therefore,

$$||\widetilde{\mathbf{U}}-\mathbf{U}^{[k]}||_{\overline{\Omega}^{N,M}} \leq 2\lambda + \left(\frac{1}{2}\right)^{k}\xi^{[1]}. \tag{4.25}$$

We use Lemma 4.3 for calculate the bounds for ξ_{ρ_1} and ξ_{ρ_2}. Also we note that $\xi_{4\rho_1}$ and $\xi_{4\rho_2-3\rho_1}$ are the interpolation errors. Therefore, using the arguments in Lemma 4.3 we can calculate the estimate for $\xi_{4\rho_1}+\xi_{4\rho_2-3\rho_1}$. Hence, we get $\lambda \leq C(N^{-2}(\ln N)^2+\Delta t)$. Further, we have $\xi^{[1]} \leq C$ by utilizing discrete maximum principle.

Hence,

$$||\widetilde{\mathbf{U}}-\mathbf{U}^{[k]}||_{\overline{\Omega}^{N,M}} \leq C\left(\frac{1}{2}\right)^{k} + C(\Delta t + N^{-2}\ln^2 N).$$

□

Theorem 4.5. *Suppose* $\boldsymbol{u}$ *and* $\mathbf{U}^{[k]}$ *are the solutions of problem (4.1) and the proposed algorithm respectively. Then*

$$||\boldsymbol{u}-\mathbf{U}^{[k]}||_{\overline{\Omega}^{N,M}} \leq C\left(\frac{1}{2}\right)^{k} + C(\Delta t + N^{-2}\ln^2 N). \tag{4.26}$$

Proof. The proof of the theorem can be obtained by combining Lemma 4.3 and Theorem 4.4 with (4.9). □

4.4 Numerical results

To verify the theoretical findings given in the previous section we consider some test problems in this section. The user defined threshold is chosen to be $\tau = N^{-2}(\ln N)^2$

for $||\mathbf{U}^{[k+1]} - \mathbf{U}^{[k]}||_{\overline{\Omega}^{N,M}} \leq \tau$.

Example 4.1. Consider problem (4.1) with the following data

$$f_1(x,t,\boldsymbol{u}) = 3u_1 - 2u_2 + t^2(\exp\big(-u_1^2\big) + \sin(u_2)) + t(1-\exp(3t))\sin(\pi x),$$

$$f_2(x,t,\boldsymbol{u}) = -t^2 u_1 \left(1 + \frac{1}{1+u_1^2}\right) + 3u_2 - 10t^2(1-\cos(2\pi x)),$$

$$\boldsymbol{\varphi}_0 = (8t^3 - 1.5t^2 + t + 1, 20t^3 + \exp(t) - 2t^2)^T,\ \boldsymbol{\varphi}_1 = (4.5t^2 + 3t + 1, \exp(3t))^T,$$

$$\boldsymbol{\phi}(x) = (1,1)^T.$$

We use the double mesh method to compute the maximum pointwise errors because the actual solution to this test problem is unknown. So, we compute

$$\mathbf{E}^{N,\Delta t}_{\varepsilon_1,\varepsilon_2} = ||\mathbf{U}^{N,\Delta t} - \mathbf{U}^{2N,\Delta t/4}||_{\overline{\Omega}^{N,M}},$$

where the approximation $\mathbf{U}^{2N,\Delta t/4}$ is obtained by taking $2N+1$ discretization points in x direction and $\Delta t/4$ mesh width in t direction by utilizing the same transition parameters ρ_1 and ρ_2 as for the solution $\mathbf{U}^{N,\Delta t}$. The uniform errors $\mathbf{E}^{N,\Delta t}$ are calculated as

$$\mathbf{E}^{N,\Delta t} = \max_{\varepsilon_1} \mathbf{E}^{N,\Delta t}_{\varepsilon_1},$$

where $\mathbf{E}^{N,\Delta t}_{\varepsilon_1} = max\{\mathbf{E}^{N,\Delta t}_{\varepsilon_1,1}, \mathbf{E}^{N,\Delta t}_{\varepsilon_1,10^{-1}}, ..., \mathbf{E}^{N,\Delta t}_{\varepsilon_1,10^{-n}}\}$ is calculated for a constant value of $\varepsilon_1 = 10^{-n}$, $n \in \{s : 0 \leq s \leq 10\}$. Next, we use the formula below to define the uniform convergence rates

$$\mathbf{R}^{N,\Delta t} = \log_2(\mathbf{E}^{N,\Delta t}/\mathbf{E}^{2N,\Delta t/4}).$$

TABLE 4.1: Uniform errors $\mathbf{E}^{N,\Delta t}$ and uniform convergence rates $\mathbf{R}^{N,\Delta t}$ for Example 4.1.

Schemes		$N=2^5$ $M=4$	$N=2^6$ $M=4^2$	$N=2^7$ $M=4^3$	$N=2^8$ $M=4^4$	$N=2^9$ $M=4^5$
Scheme 1	$E_1^{N,\Delta t}$	5.8467e-01	1.3635e-01	3.2901e-02	8.1412e-03	2.0121e-03
	$R_1^{N,\Delta t}$	2.100	2.051	2.015	2.014	
	$E_2^{N,\Delta t}$	3.0848e-01	8.4248e-02	2.1595e-02	5.4327e-03	1.3241e-03
	$R_2^{N,\Delta t}$	1.872	1.964	1.991	2.036	
Scheme 2	$E_1^{N,\Delta t}$	6.1280e-01	1.4884e-01	3.6471e-02	9.0623e-03	2.2620e-03
	$R_1^{N,\Delta t}$	2.028	2.042	2.029	2.002	
	$E_2^{N,\Delta t}$	5.9452e-01	1.5039e-01	3.7589e-02	9.3952e-03	2.3471e-03
	$R_2^{N,\Delta t}$	1.983	2.000	2.000	2.000	
Euler Scheme	$E_1^{N,\Delta t}$	6.8887e-01	1.8494e-01	4.6807e-02	1.1732e-02	2.8928e-03
	$R_1^{N,\Delta t}$	1.897	1.982	1.996	2.019	
	$E_2^{N,\Delta t}$	7.0525e-01	1.7203e-01	4.2581e-02	1.0608e-02	2.6388e-03
	$R_2^{N,\Delta t}$	2.035	2.014	2.004	2.007	

TABLE 4.2: Present algorithm with the Euler Scheme: Iteration counts taking $\varepsilon_1 = 10^{-9}$ in Example 4.1.

$\varepsilon_2 = 10^{-n}$	$N=2^5$ $M=4$	$N=2^6$ $M=4^2$	$N=2^7$ $M=4^3$	$N=2^8$ $M=4^4$	$N=2^9$ $M=4^5$
$n=1$	4	5	6	6	7
2	3	3	5	5	6
3	3	3	4	5	5
4	3	3	4	5	4
5	3	3	3	4	4
6	3	3	3	3	3
7	2	2	2	2	3
8	2	2	2	2	2
9	2	2	2	2	2
10	1	1	1	1	1

For Scheme 1, Scheme 2, and the Euler Scheme, the uniform errors and uniform convergence rates for the solution components are presented in Table 4.1 for Example 4.1. The numerical results presented in Table 4.1 are almost similar for all the three schemes. Tables 4.2 and 4.3 display the number of iterations to achieve the stopping criterion. The number of iterations for the proposed algorithm with Scheme 1 and Scheme 2 are same, whereas these counts are slightly different for the classical Euler

TABLE 4.3: Present algorithm with Scheme 1 or Scheme 2 : Iteration counts taking $\varepsilon_1 = 10^{-9}$ in Example 4.1.

$\varepsilon_2 = 10^{-n}$	$N = 2^5$ $M = 4$	$N = 2^6$ $M = 4^2$	$N = 2^7$ $M = 4^3$	$N = 2^8$ $M = 4^4$	$N = 2^9$ $M = 4^5$
$n = 1$	4	5	5	6	6
2	3	3	4	4	5
3	3	3	3	4	4
4	3	3	4	3	4
5	3	3	3	4	4
6	3	3	3	3	4
7	2	2	2	2	3
8	2	2	2	2	2
9	2	2	2	2	2
10	1	1	1	1	1

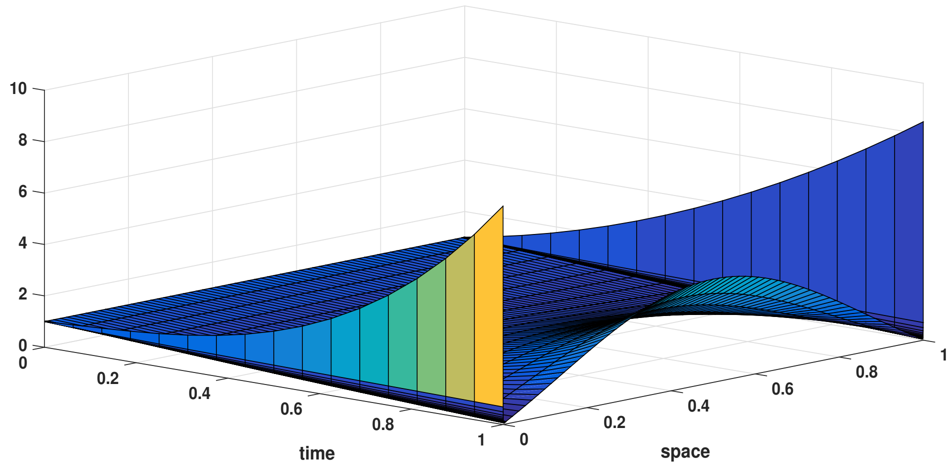

FIGURE 4.1: Component 1 with Scheme 1 for Example 4.1 with $\varepsilon_1 = 10^{-7}, \varepsilon_2 = 10^{-5}$ and $N = 64, M = 16$.

Scheme.

Example 4.2. Consider problem (4.1) with the following data

$$f_1(x, t, \boldsymbol{u}) = 4u_1 - u_2 + \cos(u_2) - 2t(x - x^2 + \sin(\pi x)) - 4,$$

$$f_2(x, t, \boldsymbol{u}) = -u_1 - \sin(u_1) + 7u_2 + \sin^2(u_2) - xt - 1,$$

$$\boldsymbol{\varphi}_0 = (10t \sin t, 10 \cos t(1 - \exp(-t)))^T, \ \boldsymbol{\varphi}_1 = (10t \sin t, 10 \cos t(1 - \exp(-t)))^T,$$

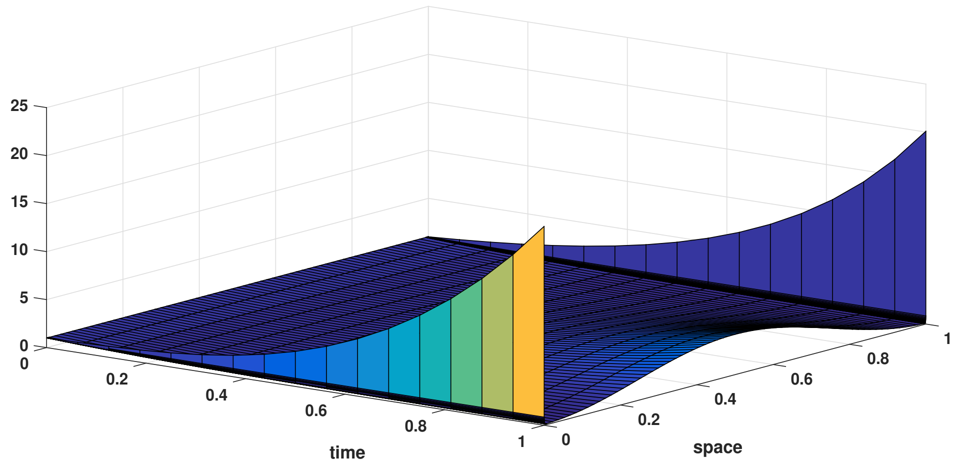

Figure 4.2: Component 2 with Scheme 1 for Example 4.1 with $\varepsilon_1 = 10^{-7}, \varepsilon_2 = 10^{-5}$ and $N = 64, M = 16$.

$\boldsymbol{\phi}(x) = (0,0)^T$.

Table 4.4: Uniform errors $\mathbf{E}^{N,\Delta t}$ and uniform convergence rates $\mathbf{R}^{N,\Delta t}$ for Example 4.2.

Schemes		$N=2^5$ $M=1/4$	$N=2^6$ $M=4^2$	$N=2^7$ $M=4^3$	$N=2^8$ $M=4^4$	$N=2^9$ $M=4^5$
Scheme 1	$E_1^{N,\Delta t}$	1.2734e-02	3.9623e-03	1.0423e-03	2.6382e-04	6.6120e-05
	$R_1^{N,\Delta t}$	1.684	1.927	1.982	1.996	
	$E_2^{N,\Delta t}$	3.5016e-02	9.5811e-03	2.5464e-03	7.0867e-04	1.8222e-04
	$R_2^{N,\Delta t}$	1.869	1.912	1.845	1.959	
Scheme 2	$E_1^{N,\Delta t}$	1.1026e-02	3.3801e-03	9.1004e-04	2.3167e-04	5.7932e-05
	$R_1^{N,\Delta t}$	1.706	1.893	1.973	1.999	
	$E_2^{N,\Delta t}$	2.0197e-02	7.8078e-03	2.4604e-03	6.0747e-04	1.5148e-04
	$R_2^{N,\Delta t}$	1.371	1.666	2.018	2.001	
Euler Scheme	$E_1^{N,\Delta t}$	1.4421e-02	4.8217e-03	1.3339e-03	3.4251e-04	8.6492e-05
	$R_1^{N,\Delta t}$	1.581	1.854	1.959	1.985	
	$E_2^{N,\Delta t}$	1.9394e-02	7.9896e-03	2.6077e-03	6.9169e-04	1.7736e-04
	$R_2^{N,\Delta t}$	1.279	1.615	1.915	1.963	

The uniform convergence rates $\mathbf{R}^{N,\Delta t}$ and uniform errors $\mathbf{E}^{N,\Delta t}$ are computed in the same manner as before. The uniform errors and uniform convergence rates for the Scheme 1, Scheme 2 and the Euler scheme for the solution components are presented

TABLE 4.5: Present algorithm with the Euler Scheme: Iteration counts taking $\varepsilon_1 = 10^{-9}$ in Example 4.2.

$\varepsilon_2 = 10^{-n}$	$N = 2^5$ $M = 4$	$N = 2^6$ $M = 4^2$	$N = 2^7$ $M = 4^3$	$N = 2^8$ $M = 4^4$	$N = 2^9$ $M = 4^5$
$n = 1$	5	5	6	6	7
2	4	4	5	5	6
3	3	4	4	5	5
4	3	4	4	4	4
5	3	3	4	4	3
6	3	3	3	3	3
7	2	2	2	2	2
8	2	2	2	2	2
9	2	2	2	2	2
10	1	1	1	1	1

TABLE 4.6: Present algorithm with Scheme 1 or Scheme 2 : Iteration counts taking $\varepsilon_1 = 10^{-9}$ in Example 4.2.

$\varepsilon_2 = 10^{-n}$	$N = 2^5$ $M = 4$	$N = 2^6$ $M = 4^2$	$N = 2^7$ $M = 4^3$	$N = 2^8$ $M = 4^4$	$N = 2^9$ $M = 4^5$
$n = 1$	4	5	6	6	6
2	3	4	4	5	5
3	3	4	4	4	5
4	3	3	4	4	4
5	3	3	3	4	3
6	3	3	3	3	3
7	2	2	2	2	3
8	2	2	2	2	2
9	2	2	2	2	2
10	1	1	1	1	1

in Table 4.4 for Example 4.2. Further, Tables 4.5 and 4.6 show how many iterations are necessary to satisfy the stopping constraint. Here, we also observe the same behavior of the numerical results as in Example 4.1.

To show the efficiency of the algorithm with Schemes 1 and 2, we compare the computational cost required by the the algorithm with Schemes 1, 2 and the Euler Scheme in Tables 4.7 and 4.8 for Examples 4.1 and 4.2 respectively. These results are calculated for fixed values $\varepsilon_1 = 10^{-7}$ and $\varepsilon_2 = 10^{-5}$ and different values of

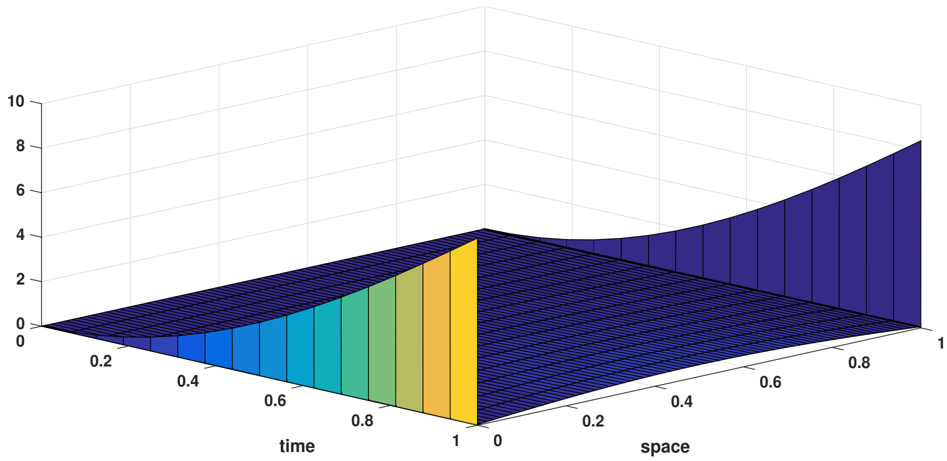

FIGURE 4.3: Component 1 with Scheme 2 for Example 4.2 with $\varepsilon_1 = 10^{-7}, \varepsilon_2 = 10^{-5}$ and $N = 64, M = 16$.

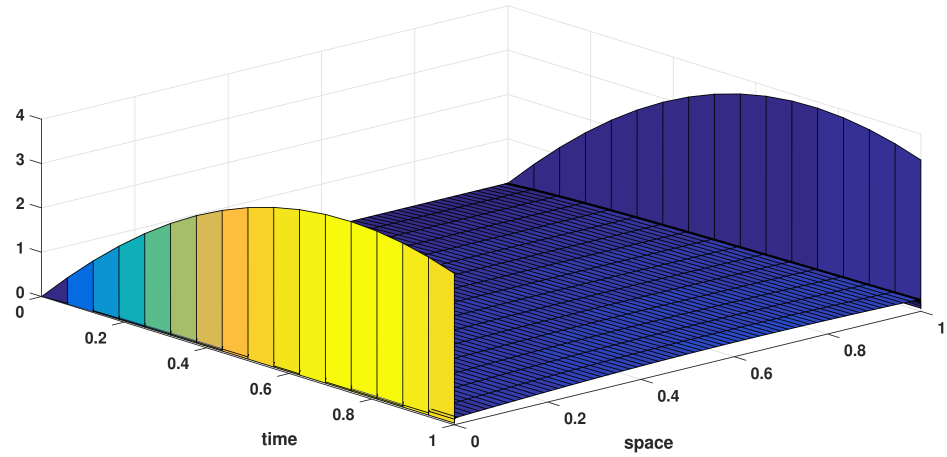

FIGURE 4.4: Component 2 with Scheme 2 for Example 4.2 with $\varepsilon_1 = 10^{-7}, \varepsilon_2 = 10^{-5}$ and $N = 64, M = 16$.

TABLE 4.7: The used CPU time in seconds for Example 4.1 with $\varepsilon_1 = 10^{-7}$, $\varepsilon_2 = 10^{-5}$.

Algorithm↓	$N = 2^5$ $M = 4^2$	$N = 2^6$ $M = 4^3$	$N = 2^7$ $M = 4^4$	$N = 2^8$ $M = 4^5$	$N = 2^9$ $M = 4^6$
Scheme 1	0.434	2.610	26.878	271.484	6854.164
Scheme 2	0.569	2.824	28.986	291.506	7086.662
Euler Scheme	1.064	4.095	57.706	933.209	46471.232

TABLE 4.8: The used CPU time in seconds for Example 4.2 with $\varepsilon_1 = 10^{-7}$, $\varepsilon_2 = 10^{-5}$.

Algorithm↓	$N = 2^5$ $M = 4^2$	$N = 2^6$ $M = 4^3$	$N = 2^7$ $M = 4^4$	$N = 2^8$ $M = 4^5$	$N = 2^9$ $M = 4^6$
Scheme 1	0.895	2.198	18.481	209.241	5821.146
Scheme 2	0.783	2.856	20.185	249.665	6056.191
Euler Scheme	1.377	3.765	52.840	1066.341	42924.251

discretization parameters N and Δt. These results clearly show that the algorithm with Schemes 1 and 2 is computationally efficient than the Euler Scheme.

4.5 Conclusions

We have proposed a domain decomposition algorithm to solve the semilinear coupled system of singularly perturbed parabolic problems. On each subdomain, a classical central difference scheme in space along with the splitting of components technique in time, are utilized. We have shown that the proposed algorithm is parameter uniform, with the accuracy of almost second order in space variable and one in time variable. To support the theoretical findings and show the efficiency of the proposed algorithm, we have included two test problems.

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