

Chapter 5

Copure $D2$ Modules

In 1976, Nicholson [47] introduced the concept of direct-projective modules. He defined an R -module N is direct-projective if for any direct summand K of N with projection $p : N \rightarrow K$ and any epimorphism $g : N \rightarrow K$, there exists an $f \in \text{End}_R(N)$ such that $g \circ f = p$. In 2003, Nicholson and Yusif [48] proved that the class of direct-projective modules is equivalent to the class of $D2$ modules. An R -module N is called $D2$ if the quotient module N/K is isomorphic to a direct summand of N , then K is also a direct summand of N . In 1984, V.A. Hiremath [24] introduced the concept of copurity which is dual to the concept of purity by employing co-finitely related modules. In 2022, Maurya et al.[41] introduced copure $C2$ as the generalization of $C2$ modules. So motivated by the above theory of copure $C2$ modules. In this chapter, we introduced the new concept of copure $D2$ modules as the dual of copure $C2$ modules and explored the properties of direct summands and direct sums of copure $D2$ modules. We also give new characterizations of semi-simple rings with respect to copure $D2$ modules and find their connections with copure $D2$ modules, copure-rickart modules, endoregular modules, etc.

5.1 Copure D2 Modules

We first define copure D2 modules, with some of their examples, and study their various properties.

Definition 5.1.1. *An R -module N is called a copure D2 module (or copure-direct-projective module), if for any copure submodule B of N such that $N/B \cong A$ and $A \leq^{\oplus} N$, it follows that B is a direct summand of N .*

In general, every direct summand of an R -module N is pure and copure submodule [24]. Consequently, it follows that a D2 module is also a copure D2 module. Next, we provide some examples of copure D2 modules out of which the first two show copure D2 modules is a strict generalization of D2 modules.

Example 5.1.2. *We discuss the following examples of copure D2 modules:-*

1. *Given a right R -module N , every direct summand of N is also a copure submodule of N . Consequently, every D2 module is a copure D2 module, but the converse is not necessarily true. As an example, consider the \mathbb{Z} -module $\mathbb{Z} \oplus \mathbb{Z}_n$, where $n \in \mathbb{N}$. This module is copure D2 but not a D2 module since both \mathbb{Z} and \mathbb{Z}_n have no copure submodules over \mathbb{Z} .*
2. *We can similarly construct another example $\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ over the ring \mathbb{Z} , this module is a copure D2 but not D2.*
3. *Every copure simple module is also a copure D2 module. A module N is called copure simple when its only copure submodules are the trivial submodules.*

Remark 5.1.3. *An R -module N is called as copure D2 if the copure exact sequence $0 \rightarrow \text{Ker}(f) \rightarrow N \rightarrow N/\text{Ker}(f) \rightarrow 0$ splits for every $f \in \text{End}_R(N)$ such that $\text{Img}(f) \leq^{\oplus} N$.*

Theorem 5.1.4. *The following conditions are equivalent for an R -module N :*

1. N is a copure D2 module;
2. If $\text{Img}(f)$ is a direct summand of N with $\text{Ker}(f)$ is a copure submodule of N for every $f \in \text{End}_R(N)$, then $\text{Ker}(f)$ is also a direct summand of N ;
3. Let K be a direct summand of N and $f : N \rightarrow K$ be an epimorphism such that $\text{Ker}(f)$ is copure in N . Then there exists $g \in \text{End}_R(N)$ such that the following diagram commutes:

$$\begin{array}{ccc} & & N \\ & \nearrow g & \downarrow \pi \\ N & \xrightarrow{f} & K \end{array}$$

equivalently, $f \circ g = \pi$, where π is a projection map.

Proof. (1) \Rightarrow (2) Using the fact, $N/\text{Ker}(f) \cong \text{Img}(f)$ (By Fundamental theorem of Module Homomorphism) and $\text{Ker}(f)$ is copure submodule of N , hence $\text{Ker}(f) \leq^\oplus N$ as N is a copure D2 module.

(2) \Rightarrow (1) Consider $N/B \cong K \leq^\oplus N$ with B copure submodule of N and let $N/B \cong eN$ for some idempotent $e \in \text{End}_R(N)$. Let π be a projection map from N to K and f be an epimorphism from N to K . We set some s as $s = e \circ f^{-1} \circ \pi$, then $\text{Ker}(s) = B$ and $\text{Img}(s) = eN$. Since $\text{Ker}(s)$ is a direct summand in N , it follows that B is a direct summand of N .

(1) \Rightarrow (3) It is obvious.

(3) \Rightarrow (1) Let B be any copure submodule of N such that $N/B \cong K \leq^\oplus N$. Consider $h : N \rightarrow N/B$, since $\text{Ker}(h) \cong B$ and copure in N . By hypothesis, we

have some $g \in \text{End}_R(N)$ such that $h \circ g = \pi$. Hence, g is the required splitting of h , and $\text{Ker}(h)$ is a direct summand of N . \square

Next, we discuss the Morita equivalence with respect to copure $D2$ modules. Two rings R and S are Morita equivalent if their category of modules $\text{Mod-}R$ and $\text{Mod-}S$ are equivalent.

Proposition 5.1.5. *Let R and S be rings for which there is a Morita equivalence $\phi : \text{Mod-}R \rightarrow \text{Mod-}S$ be their Morita equivalence. If a module N is copure $D2$, then its image under ϕ , denoted as $\phi(N)$, is also copure $D2$, and vice versa.*

Proof. Let $K \leq N$ be a copure submodule then $\phi(K)$ is a copure submodule of $\phi(N)$, since copurity is a Morita equivalent property. Hence, the proof is clear. \square

Proposition 5.1.6. *Let N be a copure $D2$ R -module such that $N = N_1 \oplus N_2$ and $f : N_1 \rightarrow N_2$ be an R -homomorphism satisfying the conditions that the $\text{Img}(f)$ is a direct summand of N_2 and the $\text{Ker}(f)$ is a copure submodule of N_1 . Then $\text{Ker}(f)$ is also a direct summand of N_1 .*

Proof. Let $f : N_1 \rightarrow N_2$ be the module homomorphism, $\pi : N \rightarrow N_1$ be the canonical projection. We have $f \circ \pi : N \rightarrow N_2$. Moreover, $\text{Img}(f \circ \pi) = \text{Img}(f)$. Therefore, $N/\text{Ker}(f \circ \pi) \cong \text{Img}(f \circ \pi) = \text{Img}(f)$. Since $\text{Img}(f) \leq^\oplus N_2$, it is a direct summand of the module N . Hence, by the definition of a copure $D2$ module, we have that $\text{Ker}(f \circ \pi) = N_2 \oplus \text{Ker}(f)$ is copure in $N_2 \oplus N_1$. Since $\text{Ker}(f)$ is copure in N_1 , it follows that $\text{Ker}(f \circ \pi)$ is copure in N . Therefore, we have $\text{Ker}(f \circ \pi) \leq^\oplus N$. Since $\text{Ker}(f)$ is a submodule of N_1 . Hence, $\text{Ker}(f) \leq^\oplus N_1$.

\square

Corollary 5.1.7. *Let N be a copure D2 R -module where $N = N_1 \oplus N_2$ with an R -epimorphism $f : N_1 \rightarrow N_2$ such that $\text{Ker}(f)$ is copure in N_1 . Then $\text{Ker}(f)$ is a direct summand of N_1 .*

Definition 5.1.8. *A module N is called a copure dual Rickart module if for every $f \in \text{End}_R(N)$, $\text{Img}(f)$ is a copure submodule of N and direct summand of N .*

Proposition 5.1.9. *Suppose that an R -module $N = N_1 \oplus N_2$ is a copure D2 module, where N_1 is copure simple and copure dual Rickart module. Then, either $\text{Hom}_R(N_1, N_2) = 0$ or for every nonzero homomorphism $f : N_1 \rightarrow N_2$, if $\text{Ker}(f)$ is a copure submodule of N_1 , then f is a monomorphism.*

Proof. We consider $\text{Hom}_R(N_1, N_2) \neq 0$ as other case is trivial. If a non-zero R -homomorphism $f : N_1 \rightarrow N_2$ and N_1 is copure simple, this implies $\text{Ker}(f) \neq 0$ is not copure submodule, implies $\text{Ker}(f) \neq 0$ is not direct summand of N_1 . Hence, $\text{Ker}(f)$ is trivial and $f \in \text{Hom}_R(N_1, N_2)$ is a monomorphism. \square

Proposition 5.1.10. *Let N be an R -module such that for every $f \in \text{End}_R(N)$, the image $\text{Img}(f)$ is a direct summand of N and $N/\text{Ker}(f)$ is a copure injective submodule of N . Then, N is a copure D2 module if and only if N is a D2 module.*

Proof. Since $N/\text{Ker}(f)$ is copure injective, we can conclude that the exact sequence is copure and $0 \rightarrow N/\text{Ker}(f) \rightarrow N$ splits. Moreover, since $N/\text{Ker}(f) \cong \text{Img}(f)$, which is a direct summand of N . Therefore, we got our assertion. \square

5.2 Direct summands and Direct Sums of Copure $D2$ modules

In this section, we study the properties of direct summands and direct sums of copure $D2$ modules. The following proposition verifies that a direct summand of a copure $D2$ module is a copure $D2$ module.

Proposition 5.2.1. *If N is a copure $D2$ module with $K \leq^\oplus N$, then K is also a copure $D2$ module.*

Proof. Let K be a direct summand of N and A be a copure submodule of K such that $K/A \cong B \leq^\oplus K$ for some submodule B of K , we are required to prove $A \leq^\oplus K$. Since N is copure $D2$, so there exists some T , a direct summand of N such that $N/A \cong T \leq^\oplus N$. Since A is copure in K this implies A is copure in N , therefore A will be a direct summand of N as N is a copure $D2$ module. Since A is a copure submodule of K . Hence $A \leq^\oplus K$. \square

Now, it is worth noting that the direct sum of two copure $D2$ modules may not necessarily be a copure $D2$ module. Consider \mathbb{Z} -modules $N_1 = \prod_{n=1}^{\infty} \mathbb{Z}_2$ and $N_2 = \prod_{n=1}^{\infty} \mathbb{Z}_2 / \bigoplus_{n=1}^{\infty} \mathbb{Z}_2$ which are both copure $D2$ but $N_1 \oplus N_2$ is not a copure $D2$ module because $f \in \text{Hom}_R(N_1, N_2)$ and $\text{Ker}(f)$ is copure submodule of N_1 but not a direct summand of N_1 .

Definition 5.2.2. *Let N and K be copure $D2$ R -modules. N is called relatively copure $D2$ with respect to K , if there exists an R -homomorphism $f \in \text{Hom}_R(N, K)$ such that $\text{Img}(f)$ is a direct summand of K and $\text{Ker}(f)$ is a copure submodule N , then $\text{Ker}(f)$ is a direct summand of N .*

Next, in our discussion, we will examine the direct sum of a copure $D2$ module in relation to the copure intersection property. A module is said to possess the copure intersection property when the intersection of any two copure submodules remains copure. Suppose N be an R -module such that it satisfies the copure intersection property, then for any decomposition of module N into the direct sums of submodules K and T say $N = K \oplus T$, and for every $f \in \text{Hom}_R(T, K)$, it follows that $\text{Ker}(f)$ is a copure submodule of T .

Proposition 5.2.3. *Let $N = K \oplus T$ be a copure $D2$ module with the copure intersection property, and T is a dual Rickart module. Then K is relatively copure $D2$ with respect to T .*

Proof. It is obvious. □

In particular, every direct sum of copure $D2$ modules is copure $D2$ if the modules are relatively copure $D2$ to each other.

Theorem 5.2.4. *Let $N = \bigoplus_{i \in \mathbb{N}} N_i$. Then every N_i is relatively copure $D2$ module with respect to others if and only if N is a copure $D2$ module.*

Proof. If N is copure $D2$ module then $\bigoplus_{i \in \mathbb{N}} N_i$ is copure $D2$ and it follows from Proposition 5.1.6 and Definition 5.2.2, each N_i is relative copure $D2$. For the converse part, let some copure submodule P of N such that $N/P \cong K \leq^{\oplus} N$, then N will be a direct summand of some N_k . Moreover, P is a copure submodule of some N_t , since N_t is relatively copure $D2$ to N_k . Hence, P is a direct summand of module N . As $k, t \in \mathbb{N}$ are arbitrary, N is a copure $D2$ module. □

5.3 Characterization of rings using copure $D2$ modules

This section deals with the new characterizations of copure $D2$ modules over some rings. First, we consider the endomorphism ring of copure $D2$ modules and develop an equivalent condition for copure $D2$ and endoregular modules. Since every endoregular module is copure $D2$, the converse of this statement may not be true. For example, assume \mathbb{Z}_{p^n} as \mathbb{Z} -module where p is any prime number and $n \in \mathbb{N}$, which is copure $D2$ module but not an endoregular module.

Definition 5.3.1. *A module N is known as a copure Rickart module if, for every $f \in \text{End}_R(N)$, the kernel $\text{Ker}(f)$ is a copure submodule of the module N and direct summand of N .*

Theorem 5.3.2. *For a copure $D2$ R -module N , the following statements are equivalent :*

1. N is copure Rickart and dual Rickart module;
2. $N \oplus N$ has SSP and copure intersection property;
3. N is an Endoregular module.

Proof. (1) \Rightarrow (3) To prove that N is endoregular module, we need to show $\text{Ker}(f)$ and $\text{Img}(f)$ are direct summands for all $f \in \text{End}_R(N)$. $\text{Img}(f)$ is a direct summand of N since N is dual Rickart, also given N is a copure Rickart module, hence $\text{Ker}(f)$ is copure for all $f \in \text{End}_R(N)$. So we get $\text{Ker}(f)$ is a direct summand of N . Hence, N is the Endoregular module.

(2) \Rightarrow (3) $\text{Img}(g) \leq^\oplus N$ since $N \oplus N$ has SSP for every $g \in \text{End}_R(N)$ and due to copure intersection property, $\text{Ker}(f)$ is copure for every $f \in \text{End}_R(N)$. This

implies $N/\text{Ker}(f) \leq \text{Img}(f) \leq^{\oplus} N \implies \text{Ker}(f) \leq^{\oplus} N$. Hence, $\text{Ker}(f)$ is a direct summand of N , as N is a copure $D2$ module.

(3) \implies (1) and (3) \implies (2) It is clear from the definition of Endoregular module. \square

Recall a ring R is called a copure semisimple ring if every copure submodule of an R -module N is a direct summand of N . In the following proposition, we find when a copure Rickart module is a copure $D2$.

Proposition 5.3.3. *Every copure Rickart module is copure $D2$ module over copure semi-simple ring.*

Proof. Suppose N is a copure Rickart module over a copure semi-simple ring R . Let P be a copure submodule of N such that $N/P \cong N \leq^{\oplus} N$. As R is a copure semi-simple ring, P is a direct summand of N this implies N is a copure $D2$ module. \square

Proposition 5.3.4. *Let R be a ring over which every R -module is a fully copure module. Then the following statements are equivalent:*

1. R is a semi-simple ring;
2. All R -modules are relatively copure $D2$ to any R -module.

Proof. (1) \implies (2) It is clear.

(2) \implies (1) Suppose I is an ideal of a ring R . We have to show that I is a direct summand of R . From (2), if every R -module is relatively copure $D2$ to any R -module, then the R -module R is relatively copure $D2$ to R/I as an R -module. Then by definition for R -homomorphism $f : R \rightarrow R/I$, $\text{Ker}(f) = I$ is copure in R and $\text{Ker}(f)$ is a direct summand of R . Since I is an arbitrary ideal of R . Therefore, R is a semi-simple ring. \square

Next, we characterize copure D2 modules with respect to copure semisimple rings.

Proposition 5.3.5. *The following statements are equivalent:-*

1. *R is a copure semisimple ring;*
2. *Every finitely generated R -module is a copure D2;*
3. *Every 2-generated module is a copure D2.*

Proof. (1) \Rightarrow (2) \Rightarrow (3) It is clear.

(3) \Rightarrow (1) We are required to prove that every copure ideal, say I of R is a direct summand of R . Let $R \oplus R/I$ be a copure D2 module. So, from Theorem 5.1.4, there exists $f : R \rightarrow R/I$ then $\text{Ker}(f) \cong I$, which is copure in R . Hence I is direct summand of R . Therefore R is a copure semisimple ring.

□

Next, we characterize the concept of copure D2 in respect of copure semisimple ring.

Definition 5.3.6. *Let N and N' be R -modules. Then N is called N' copure D2, if for a module homomorphism $g : N \rightarrow N'/P$ where P is a copure submodule of N' and N'/P is isomorphic to a direct summand of N , $\exists h \in \text{Hom}_R(N, N')$ such that $\pi \circ h = g$, where π is natural epimorphism from N' to N'/P .*

$$\begin{array}{ccc}
 & & N \\
 & \swarrow h & \downarrow g \\
 N' & \xrightarrow{\pi} & N'/P
 \end{array}$$

This implies the above diagram commutes.

Proposition 5.3.7. *For every copure submodule P of N , if N/P is isomorphic to a direct summand of R -module N and N/P is N -copure D2, then R -module N is copure semisimple.*

Proof:- Since N/P is N copure D2 it follows that the identity mapping from N/P to N/P has a lifting $g : N/P \rightarrow N$. Hence short exact sequence $0 \rightarrow P \rightarrow N \rightarrow N/P$ splits, which implies $P \leq^{\oplus} N$.

Proposition 5.3.8. *Let $N = N' \oplus N''$ be a copure D2 and N' is a copure simple module. Then N' is N'' -copure D2 module.*

Proof. This follows from the above definition. □

Recall a ring is called copure hereditary if every factor module of a copure injective module is copure injective.

Proposition 5.3.9. *Let N be a copure injective module over a copure hereditary ring R . Then R -module N is a copure D2 module.*

Proof. Since R -module N is copure injective over the copure hereditary ring, so every quotient module of copure injective is copure injective. For copure submodule P of N then N/P is copure injective hence, $P \leq^{\oplus} N$. Therefore R -module N is copure D2. □

Lemma 5.3.10. *Let T be a projective module and $T \oplus K$ be a copure D2 module where K is an R -module. If there exists an epimorphism $f : T \rightarrow K$ such that $\text{Ker}(f)$ is copure, then K is a projective module.*

Proof. Since f is an epimorphism with $\text{Ker}(f)$ as copure in T , then Corollary 5.1.7 implies $\text{Ker}(f)$ is a direct summand of T . Hence, epimorphism f splits and K is projective. □

In the next proposition, we characterize von Neumann regular rings in terms of copure $D2$ modules.

In [20], The concept of pure $D2$ modules is introduced as a generalization of $D2$ modules. If any pure submodule B of N satisfies $N/B \cong A$ and $A \leq^{\oplus} N$, then B is a direct summand of N , is called as pure $D2$ module. In this section, copure $D2$ modules are introduced as another generalization of $D2$ modules, and since these two ideas are generalizations of $D2$ modules. The next proposition shows under what conditions these two concepts are equivalent:

Proposition 5.3.11. *Over a Dedekind domain R , an R -module N is pure $D2$ if and only if it is copure $D2$.*

Proof. The concepts of copurity and purity coincide over the Dedekind domain (Remark 22, [24]). So, copure $D2$ is the same as pure $D2$ over the Dedekind domain. \square

Recall that a ring R is called classical if it is commutative and every co-finitely generated R -module is linearly compact. [55]

Proposition 5.3.12. *For a classical ring R , the following statements are equivalent:*

1. R is a copure semi-simple ring;
2. R is a PDS ring;
3. Every R -module is copure $D2$;
4. Every R -module is pure $D2$.

Proof. (1) \Leftrightarrow (2) It follows from [Proposition 19(ii), [25],].

(2) \Leftrightarrow (4) R is a PDS ring if and only if every pure submodule is a direct summand of an R module if and only if every R -module is pure $D2$.

(1) \Leftrightarrow (3) R is a copure semi-simple ring if and only if every copure submodule is a direct summand if and only if every R -module is copure $D2$.

□