Chapter 2

An Infeasible Interior-point Method to Generate the Nondominated Sets for Multiobjective Optimization Problems

2.1 Introduction

As described in Section 1.5, the cone method [47] is one of the most efficient scalarization techniques that can generate convex and nonconvex part of the Pareto surface. Also, this method does not require the prior information about the location of nondominated set. This method provides a set of single objective optimization subproblems whose nature are probably nonconvex [47]. In this chapter, an infeasible IPM is implemented to solve the single objective optimization subproblems.

2.2 Motivation

Interior-point methods require strictly feasible points as starting points. In theory, this requirement does not seem to be particularly restrictive, but it can be costly in computation. To overcome this deficiency, most existing practical algorithms allow positive but infeasible starting points and seek feasibility and optimality simultaneously. Algorithms of this type is known as infeasible interior-point methods (IIPMs). This type of methods firstly introduced by Megiddo [85] for solving LP problems. The IIPM is widely considered the most efficient method. Interior-point algorithms are based on the line search techniques. Generally, IPMs follow the following steps to solve the optimization problems:

(i) convert the inequality constraints into equality using the slack or surplus variables,

(ii) omit the nonnegative variables by keeping them within the barrier terms,

(iii) move equality constraints to the objective with the Lagrange transformation to obtain an unconstrained optimization problem and write first order optimality conditions for it, and

(iv) apply Newton method to solve the first order optimality conditions (i.e., to solve a system of nonlinear equations).

Since IIPMs do not require the iterates to be feasible, they have to concern with two conflicting objectives at each iteration: the reduction of infeasibility and the reduction of objective function. Both goals must be considered when deciding whether the new iterate will be accepted or rejected. Most algorithms combine optimality and feasibility into one merit function to make this choice.

When an MOP is transformed into single objective optimization subproblems, finding the initial feasible point will not be possible for each subproblem. Therefore, an

IIPM [86] will be the better choice to solve the single objective optimization subproblems.

2.3 Contributions

In this chapter, an IIPM is proposed to generate the nondominated set of nonlinear multiobjective optimization problems with the help of the direction-based cone method. We derive the proposed method for both convex and nonconvex problems. In order to solve the parametric optimization problems of the cone method, the IIPM starts with an initial iterate outside the feasible region, and then gradually reduces the primal and dual infeasibility measures and the objective function value across the iterations with the help of a merit function. The convergence results of the proposed method are provided. We provide the performance of the proposed method on a variety of convex and nonconvex multiobjective test problems.

The main contributions of this chapter are as follows:

(i) we develop a new decomposition based approach for solving MOPs (convex and nonconvex) which uses an IIPM to solve each single objective optimization subproblem.

(ii) Initially, we develop an algorithm to deal with the convex case of the single objective optimization subproblem. We also provide the convergence analysis of this method and an estimate of the number of iterations to reach an ϵ-precise solution. Then, the algorithm extends to solve the nonconvex MOPs.

(iii) We provide the performance of the proposed methods on a variety of convex and nonconvex MOPs test problems. Performance comparison between the proposed method and popular existing solvers is provided with respect to two performance measures and the corresponding relative efficiency measures.

In this chapter, we will consider $\bar{\mathcal{J}} = 0$, hence, MOP (1.1) will convert into the following MOP:

$$\left.\begin{aligned} &\text{minimize} \quad F(x) = (f_1(x), f_2(x), \ldots, f_{\mathcal{P}}(x))^\top, \ \mathcal{P} \geq 2 \\ &\text{subject to} \quad h_j(x) \geq 0, \ j = 1, 2, \ldots, \mathcal{J}, \end{aligned}\right\} \tag{2.1}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ and $h_j : \mathbb{R}^n \to \mathbb{R}$ are taken twice continuously differentiable functions for all $i \in \mathcal{I}$ and $j = 1, 2, \ldots, \mathcal{J}$ throughout this thesis. Hence, the feasible set in decision space will be $\mathcal{X} = \{x \in \mathbb{R}^n : h_j(x) \geq 0, \ j = 1, 2, \ldots, \mathcal{J}\}$ for MOP (2.1).

2.4 Cone method

To obtain a (weakly) non-dominated point of $F(\mathcal{X})$, the *cone method* [47] suggests to solve the following minimization problem corresponding to a particular $\hat{\beta} \in \mathbb{S}^{\mathcal{P}-1}_{\geqq} = \mathbb{S}^{\mathcal{P}-1} \cap \mathbb{R}^{\mathcal{P}}_{\geqq}$ (where $\mathbb{S}^{\mathcal{P}-1}$ represents the unit sphere in $\mathbb{R}^{\mathcal{P}}$):

$$\left.\begin{aligned} &\text{minimize} \quad t \\ &\text{subject to} \quad t\hat{\beta} \geqq F(x), \\ &\qquad\qquad\quad h_j(x) \geq 0, \ j = 1, 2, \ldots, \mathcal{J}, \\ &\qquad\qquad\quad t \geq 0. \end{aligned}\right\} \tag{2.2}$$

In fact, $\mathcal{Y}_{wN} = \bigcup_{\hat{\beta} \, \in \, \mathbb{S}^{\mathcal{P}-1}_{\geqq}} \left\{ t_{\hat{\beta}} \hat{\beta} : t_{\hat{\beta}} \text{ is `min } t\text{' of (2.2)} \right\}$. From the complete weakly non-dominated set $\mathcal{Y}_{wN}$, the process to filter out the set of all nondominated points is detailed in [47].

We note that the problem (2.2) is a single-objective parametric problem, with parameter $\hat{\beta}$, corresponding to the MOP (2.1). To generate the complete nondominated set, one needs to solve the problem (2.2) for all $\hat{\beta}$'s in $\mathbb{S}^{\mathcal{P}-1}_{\geqq}$. For obtaining a uniformly spreaded nondominated points, the work in [47] suggests to take the following expression of $\hat{\beta}$:

$$\left(\cos\theta_1, \cos\theta_2\sin\theta_1, \cos\theta_3\sin\theta_2\sin\theta_1, \ldots, \cos\theta_{\mathcal{P}-1}\prod_{i=1}^{\mathcal{P}-2}\sin\theta_i, \prod_{i=1}^{\mathcal{P}-1}\sin\theta_i\right), \tag{2.3}$$

where $0 \leq \theta_i \leq \frac{\pi}{2}, i = 1, 2, \ldots, \mathcal{P} - 1$. In the proposed algorithm, we follow the suggested way in [47] on the number of grid points for θ_i's to obtain a discrete subset of nondominated points.

Note that the MOP (2.1) is convex if f_i and $-h_j$ are convex functions for all $i \in \mathcal{I}$ and $j = 1, 2, \ldots, \mathcal{J}$; otherwise, the MOP (2.1) is nonconvex. For each $\hat{\beta}$, the parametric subproblem (2.2) of the cone method is convex (respectively, nonconvex) if the problem (2.1) is convex (respectively, nonconvex). The case of convex problems (Section 2.6) and noncovex problems (Section 2.7) are considered separately. Initially, we build an Algorithm 1 to deal with the convex case of the parametric problem (2.2), and later we extend it to solve the nonconvex case (see Algorithm 3). In fact, Algorithm 3 eventually merges the algorithms for convex and nonconvex case in one single algorithm.

The next section formulates the Newton scheme for the IPM to solve the parametric problem (2.2).

2.5 Analysis of (2.2) based on the infeasible interior-point method

This section investigates a parametric scalar optimization problem that is equivalent to the problem (2.2). In the sequel, we formulate a log-barrier problem corresponding to the problem (2.2). Afterwards, to solve the formulated log-barrier problem, Karush-Kuhn-Tucker (KKT) conditions are derived. In order to find an approximate solution of KKT conditions, IIPM utilizes Newton's method to find the direction along which to proceed. Thus, an explicit expression of the search direction to the Newton system is also derived.

We note that by denoting $\mathrm{x} = (x_1, x_2, \ldots, x_n, t)^\top$, $c = (0, 0, \ldots, 0, 1)^\top$, $\hat{\beta} = (\beta_1, \beta_2, \ldots, \beta_{\mathcal{P}})^\top$, $f_i(\mathrm{x}) = f_i(x)$, $i \in \mathcal{I}$ and $F(\mathrm{x}) = F(x)$, the parametric problem (2.2) can be written in the following form:

$$\left.\begin{array}{ll} \text{minimize} & c^\top \mathrm{x} \\ \text{subject to} & \hat{\beta} c^\top \mathrm{x} - F(\mathrm{x}) - v = 0, \\ & h(\mathrm{x}) - w = 0, \\ & v \geqq 0, \quad w \geqq 0, \end{array}\right\} \tag{2.4}$$

where $h(\mathrm{x}) = (h_1(\mathrm{x}), h_2(\mathrm{x}), \ldots, h_{\mathcal{J}}(\mathrm{x}), h_{\mathcal{J}+1}(\mathrm{x}))^\top$ with $h_{\mathcal{J}+1}(\mathrm{x}) = c^\top \mathrm{x}$, $v = (v_1, v_2, \ldots, v_{\mathcal{P}})^\top$; $w = \left(w_1, w_2, \ldots, w_{\mathcal{J}}, w_{\mathcal{J}+1}\right)^\top$ being vectors of slack variables. We eliminate the inequality constraints in (2.4) by placing them inside a barrier term as follows:

$$\left.\begin{array}{ll} \text{minimize} & b_\mu (\mathrm{x}, v, w) \\ \text{subject to} & \hat{\beta} c^\top \mathrm{x} - F(\mathrm{x}) - v = 0, \\ & h(\mathrm{x}) - w = 0, \end{array}\right\} \tag{2.5}$$

where

$$b_\mu (\mathrm{x}, v, w) = c^\top \mathrm{x} - \mu \left(\sum_{i=1}^{\mathcal{P}} \log(v_i) + \sum_{j=1}^{\mathcal{J}+1} \log(w_j) \right)$$

and $\mu > 0$ is the barrier parameter. In the rest of this chapter, we use the following notations:

$$p = (\mathrm{x}, v, w) \quad \text{and} \quad \Omega = (p, y, z) = (\mathrm{x}, v, w, y, z).$$

The Lagrangian for the problem (2.5) is given by

$$L_{\hat{\beta}}(\Omega, \mu) = b_\mu (\mathrm{x}, v, w) - y^\top \left(\hat{\beta} c^\top \mathrm{x} - F(\mathrm{x}) - v \right) - z^\top \left(h(\mathrm{x}) - w \right),$$

where $y \in \mathbb{R}^{\mathcal{P}}$ and $z \in \mathbb{R}^{\mathcal{J}+1}$ are Lagrangian multipliers.

The first-order KKT conditions for a minimum of (2.5) are

$$\left.\begin{aligned} &\nabla_{\mathrm{x}} L_{\hat{\beta}}(\Omega,\mu) \equiv c - \left(\nabla_{\mathrm{x}}\left(\hat{\beta} c^{\top}\mathrm{x} - F(\mathrm{x})\right)\right)^{\top} y - (\nabla_{\mathrm{x}} h(\mathrm{x}))^{\top} z = 0,\\ &\nabla_{v} L_{\hat{\beta}}(\Omega,\mu) \equiv -\mu V^{-1} e + y = 0, \quad y \geqq 0,\\ &\nabla_{w} L_{\hat{\beta}}(\Omega,\mu) \equiv -\mu W^{-1} e + z = 0, \quad z \geqq 0,\\ &\nabla_{y} L_{\hat{\beta}}(\Omega,\mu) \equiv F(\mathrm{x}) - \hat{\beta} c^{\top}\mathrm{x} + v = 0,\\ &\nabla_{z} L_{\hat{\beta}}(\Omega,\mu) \equiv -h(\mathrm{x}) + w = 0, \end{aligned}\right\} \tag{2.6}$$

where $V = \mathrm{diag}(v_1, v_2, \ldots, v_{\mathcal{P}}), W = \mathrm{diag}(w_1, w_2, \ldots, w_{\mathcal{J}+1})$, e is a vector of all 1's of dimension $\mathcal{P}$ or $(\mathcal{J}+1)$ according to the context, and $\nabla_{\mathrm{x}}\left(\hat{\beta} c^{\top}\mathrm{x} - F(\mathrm{x})\right)$ and $\nabla_{\mathrm{x}} h(\mathrm{x})$ are the Jacobian matrices of the functions $\hat{\beta} c^{\top}\mathrm{x} - F(\mathrm{x})$ and $h(\mathrm{x})$, respectively. We modify (2.6) by multiplying the second and third equations by V and W, respectively. Accordingly, we get the following standard primal-dual system:

$$\left.\begin{aligned} c - \left(\nabla_{\mathrm{x}}\left(\hat{\beta} c^{\top}\mathrm{x} - F(\mathrm{x})\right)\right)^{\top} y - (\nabla_{\mathrm{x}} h(\mathrm{x}))^{\top} z = 0,\\ -\mu e + VYe = 0, \quad y \geqq 0,\\ -\mu e + WZe = 0, \quad z \geqq 0,\\ -F(\mathrm{x}) + \hat{\beta} c^{\top}\mathrm{x} - v = 0,\\ h(\mathrm{x}) - w = 0, \end{aligned}\right\} \tag{2.7}$$

where $Y = \mathrm{diag}(y_1, y_2, \ldots, y_{\mathcal{P}})$ and $Z = \mathrm{diag}(z_1, z_2, \ldots, z_{\mathcal{J}+1})$. To find a solution to the primal-dual system (2.7), we apply Newton's method. For a simplified appearance of the expressions below, we introduce the following notations:

$$A_{\hat{\beta}}(\mathrm{x}) = \nabla_{\mathrm{x}}\left(\hat{\beta} c^{\top}\mathrm{x} - F(\mathrm{x})\right), \; B(\mathrm{x}) = \nabla_{\mathrm{x}} h(\mathrm{x})$$

and

$$H(\mathrm{x}, y, z) = \sum_{i=1}^{\mathcal{P}} y_i \nabla^2 f_i(\mathrm{x}) - \sum_{j=1}^{\mathcal{J}+1} z_j \nabla^2 h_j(\mathrm{x}), \quad y \geqq 0, \quad z \geqq 0. \tag{2.8}$$

For a given barrier parameter $\mu > 0$, the Newton direction $(\Delta\mathrm{x}, \Delta v, \Delta w, \Delta y, \Delta z)$ at a point (x, v, w, y, z) is obtained by solving the following Newton system for (2.7):

$$\begin{bmatrix} H(\mathrm{x},y,z) & 0 & 0 & -(A_{\hat\beta}(\mathrm{x}))^\top & -(B(\mathrm{x}))^\top \\ 0 & Y & 0 & V & 0 \\ 0 & 0 & Z & 0 & W \\ A_{\hat\beta}(\mathrm{x}) & -I & 0 & 0 & 0 \\ B(\mathrm{x}) & 0 & -I & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\mathrm{x} \\ \Delta v \\ \Delta w \\ \Delta y \\ \Delta z \end{bmatrix} = - \begin{bmatrix} c - (A_{\hat\beta}(\mathrm{x}))^\top y - (B(\mathrm{x}))^\top z \\ -\mu e + VYe \\ -\mu e + WZe \\ \hat\beta c^\top \mathrm{x} - F(\mathrm{x}) - v \\ h(\mathrm{x}) - w \end{bmatrix}. \tag{2.9}$$

The matrix on the left of (2.9) is not symmetric. However, it can be easily symmetrized by multiplying the first equation by -1, the second equation by $-V^{-1}$ and the third equation by $-W^{-1}$. Accordingly, we get

$$\begin{bmatrix} -H(\mathrm{x},y,z) & 0 & 0 & (A_{\hat\beta}(\mathrm{x}))^\top & (B(\mathrm{x}))^\top \\ 0 & -V^{-1}Y & 0 & -I & 0 \\ 0 & 0 & -W^{-1}Z & 0 & -I \\ A_{\hat\beta}(\mathrm{x}) & -I & 0 & 0 & 0 \\ B(\mathrm{x}) & 0 & -I & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\mathrm{x} \\ \Delta v \\ \Delta w \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} \sigma_{\hat\beta} \\ -\gamma_1 \\ -\gamma_2 \\ \varrho_{\hat\beta} \\ \rho \end{bmatrix}, \tag{2.10}$$

where

$$\left.\begin{aligned} \sigma_{\hat\beta}(\mathrm{x},y,z) &= c - (A_{\hat\beta}(\mathrm{x}))^\top y - (B(\mathrm{x}))^\top z, \\ \gamma_1(v,y) &= \mu V^{-1}e - y, \\ \gamma_2(w,z) &= \mu W^{-1}e - z, \\ \varrho_{\hat\beta}(\mathrm{x},v) &= F(\mathrm{x}) + v - \hat\beta c^\top \mathrm{x}, \\ \text{and}\ \ \rho(\mathrm{x},w) &= w - h(\mathrm{x}). \end{aligned}\right\} \tag{2.11}$$

The reason behind making the system symmetric is that we can use the Cholesky factorization [87] to solve the system. Note that the notations $\sigma_{\hat\beta}$, $\varrho_{\hat\beta}$, ρ, γ_1 and γ_2 depend on x, y, z, v and w. If $\rho_{\hat\beta}$ and ρ do not vanish at a point, then the point is *primal infeasible*. Therefore, $\varrho_{\hat\beta}$ and ρ together denote *primal infeasibility*. In contrast, if $\rho_{\hat\beta}$ and ρ vanish at a point, then the point is primal feasible. Similarly, If $\sigma_{\hat\beta}$ does not vanish at a point, then the point is dual infeasible. Therefore, $\sigma_{\hat\beta}$ denotes the *dual* *infeasibility.*

We note that second and third equations of (2.10) can be used to eliminate Δv and Δw without producing any off-diagonal fill-in in the remaining system with the help of the following equations:

$$\left.\begin{aligned} \Delta v &= VY^{-1}(\gamma_1 - \Delta y) \\ \Delta w &= WZ^{-1}(\gamma_2 - \Delta z). \end{aligned}\right\} \tag{2.12}$$

Accordingly, from (2.10), the resulting *reduced KKT system* is given by

$$\begin{bmatrix} -H(\mathrm{x}, y, z) & (A_{\hat{\beta}}(\mathrm{x}))^\top & (B(\mathrm{x}))^\top \\ A_{\hat{\beta}}(\mathrm{x}) & VY^{-1} & 0 \\ B(\mathrm{x}) & 0 & WZ^{-1} \end{bmatrix} \begin{bmatrix} \Delta \mathrm{x} \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} \sigma_{\hat{\beta}} \\ \varrho_{\hat{\beta}} + VY^{-1}\gamma_1 \\ \rho + WZ^{-1}\gamma_2 \end{bmatrix}. \tag{2.13}$$

To find a solution to (2.13), the algorithm that we propose below (Algorithm 1) starts from a given initial point $\left(\mathrm{x}^{(0)}, v^{(0)}, w^{(0)}, y^{(0)}, z^{(0)}\right)$; then, at the k-th iteration, it determines a search direction $\left(\Delta \mathrm{x}^{(k)}, \Delta v^{(k)}, \Delta w^{(k)}, \Delta y^{(k)}, \Delta z^{(k)}\right)$ by solving (2.13) at $\left(\mathrm{x}^{(k)}, v^{(k)}, w^{(k)}, y^{(k)}, z^{(k)}\right)$; lastly, it chooses a step length $\alpha^{(k)}$ and then finds the next iterate by

$$\left.\begin{aligned} \mathrm{x}^{(k+1)} &= \mathrm{x}^{(k)} + \alpha^{(k)}\Delta \mathrm{x}^{(k)} \\ v^{(k+1)} &= v^{(k)} + \alpha^{(k)}\Delta v^{(k)} \\ w^{(k+1)} &= w^{(k)} + \alpha^{(k)}\Delta w^{(k)} \\ y^{(k+1)} &= y^{(k)} + \alpha^{(k)}\Delta y^{(k)} \\ z^{(k+1)} &= z^{(k)} + \alpha^{(k)}\Delta z^{(k)}, \end{aligned}\right\} \tag{2.14}$$

where $\alpha^{(k)}$ is the step length that is detailed in Section 2.6.3.

In order to compute a search direction, we need to solve an system (2.13). The following theorem gives the explicit form of the search direction.

Theorem 2.1 *Let the point* $\Omega = (\mathrm{x}, y, z, v, w)$ *be such that* $y > 0,\ z > 0,\ v > 0$ *and* $w > 0$*. We denote* $N_{\hat{\beta}}(\Omega) = H(\mathrm{x}, y, z) + (A_{\hat{\beta}}(\mathrm{x}))^\top V^{-1} Y A_{\hat{\beta}}(\mathrm{x}) + (B(\mathrm{x}))^\top W^{-1} Z B(\mathrm{x})$*. If at a point* Ω*,* $N_{\hat{\beta}}$ *is nonsingular, then the system (2.10) has a unique solution. In* *particular,*

$$\left.\begin{aligned}
\Delta \mathrm{x} \;=\;& -N_{\hat{\beta}}^{-1}c + N_{\hat{\beta}}^{-1}(A_{\hat{\beta}}(\mathrm{x}))^{\top}V^{-1}Y\varrho_{\hat{\beta}} + N_{\hat{\beta}}^{-1}(A_{\hat{\beta}}(\mathrm{x}))^{\top}\mu V^{-1}e + \mu N_{\hat{\beta}}^{-1}(B(\mathrm{x}))^{\top}W^{-1}e\\
&+ N_{\hat{\beta}}^{-1}(B(\mathrm{x}))^{\top}W^{-1}Z\rho,\\
\Delta v \;=\;& -A_{\hat{\beta}}(\mathrm{x})N_{\hat{\beta}}^{-1}c - \left(I - A_{\hat{\beta}}(\mathrm{x})N_{\hat{\beta}}^{-1}(A_{\hat{\beta}}(\mathrm{x}))^{\top}V^{-1}Y\right)\varrho_{\hat{\beta}} + \mu A_{\hat{\beta}}(\mathrm{x})N_{\hat{\beta}}^{-1}(A_{\hat{\beta}}(\mathrm{x}))^{\top}V^{-1}e,\\
&+ A_{\hat{\beta}}(\mathrm{x})N_{\hat{\beta}}^{-1}(B(\mathrm{x}))^{\top}W^{-1}Z\rho + \mu A_{\hat{\beta}}(\mathrm{x})N_{\hat{\beta}}^{-1}(B(\mathrm{x}))^{\top}W^{-1}e \quad \textit{and}\\
\Delta w \;=\;& -B(\mathrm{x})N_{\hat{\beta}}^{-1}c - \left(I - B(\mathrm{x})N_{\hat{\beta}}^{-1}(B(\mathrm{x}))^{\top}W^{-1}Z\right)\rho + B(\mathrm{x})N^{-1}(A_{\hat{\beta}}(\mathrm{x}))^{\top}V^{-1}Y\varrho_{\hat{\beta}}\\
&+ \mu B(\mathrm{x})N_{\hat{\beta}}^{-1}(A_{\hat{\beta}}(\mathrm{x}))^{\top}V^{-1}e + \mu B(\mathrm{x})N_{\hat{\beta}}^{-1}(B(\mathrm{x}))^{\top}W^{-1}e.
\end{aligned}\right\} \quad (2.15)$$

Proof: By solving the second and the third equations of (2.13) for Δy and Δz, we get $\Delta y = V^{-1}Y\varrho_{\hat{\beta}} + \gamma_1 - V^{-1}YA_{\hat{\beta}}(\mathrm{x})\Delta\mathrm{x}$ and $\Delta z = W^{-1}Z\rho + \gamma_2 - W^{-1}ZB(\mathrm{x})\Delta\mathrm{x}$. Eliminating Δy and Δz from the first block of the system (2.13), we get

$$\Delta\mathrm{x} = -N_{\hat{\beta}}^{-1}c + \left(\mu N_{\hat{\beta}}^{-1}(A_{\hat{\beta}}(\mathrm{x}))^{\top}V^{-1}Y\varrho_{\hat{\beta}}\right) + \left(\mu N_{\hat{\beta}}^{-1}(B(\mathrm{x}))^{\top}W^{-1}e + N_{\hat{\beta}}^{-1}(B(\mathrm{x}))^{\top}W^{-1}Z\rho\right).$$

We notice that the square matrix of order $(n+\mathcal{J}+\mathcal{P}+1)\times(n+\mathcal{J}+\mathcal{P}+1)$ on the left side of the system (2.13) is quasi-definite and therefore nonsingular in nature (see [87]). Hence, using $\Delta\mathrm{x}$, we can compute Δy and Δz, and finally Δv and Δw uniquely as follows:

$$\begin{aligned}
\Delta v \;=\;& -A_{\hat{\beta}}(\mathrm{x})N_{\hat{\beta}}^{-1}c - \left(I - A_{\hat{\beta}}(\mathrm{x})N_{\hat{\beta}}^{-1}(A_{\hat{\beta}}(\mathrm{x}))^{\top}V^{-1}Y\right)\varrho_{\hat{\beta}} + \mu A_{\hat{\beta}}(\mathrm{x})N_{\hat{\beta}}^{-1}(A_{\hat{\beta}}(\mathrm{x}))^{\top}V^{-1}e\\
&+ A_{\hat{\beta}}(\mathrm{x})N_{\hat{\beta}}^{-1}(B(\mathrm{x}))^{\top}W^{-1}Z\rho + \mu A_{\hat{\beta}}(\mathrm{x})N_{\hat{\beta}}^{-1}(B(\mathrm{x}))^{\top}W^{-1}e\\
\text{and}\quad \Delta w \;=\;& -B(\mathrm{x})N_{\hat{\beta}}^{-1}c - \left(I - B(\mathrm{x})N_{\hat{\beta}}^{-1}(B(\mathrm{x}))^{\top}W^{-1}Z\right)\rho + B(\mathrm{x})N_{\hat{\beta}}^{-1}(A_{\hat{\beta}}(\mathrm{x}))^{\top}V^{-1}Y\varrho_{\hat{\beta}}\\
&+ \mu B(\mathrm{x})N_{\hat{\beta}}^{-1}(A_{\hat{\beta}}(\mathrm{x}))^{\top}V^{-1}e + \mu B(\mathrm{x})N_{\hat{\beta}}^{-1}(B(\mathrm{x}))^{\top}W^{-1}e.
\end{aligned}$$

□

In Section 2.6, we describe an algorithm to solve convex multiobjective optimization problems with the help of the reduced KKT system (2.13), the sequence of iterations (2.14) and Theorem 2.1.

2.6 Algorithm for convex multiobjective optimization

In this section, we consider the MOP (2.1) where each $f_i,\ i \in \mathcal{I}$ is convex and each $h_j,\ j = 1, 2, \ldots, \mathcal{J}$ is concave. Hence, the MOP under consideration in this section is a convex problem. Evidently, the single objective parametric problem (2.2) is convex and the Hessian matrix $H(\mathrm{x}, y, z)$ is positive semi-definite. Then, the matrix $N_{\hat{\beta}}(\mathrm{x}, y, z, v, w) = H(\mathrm{x}, y, z) + (A_{\hat{\beta}}(\mathrm{x}))^\top V^{-1} Y A_{\hat{\beta}}(\mathrm{x}) + (B(\mathrm{x}))^\top W^{-1} Z B(\mathrm{x})$ is positive semi-definite. Since the algorithm that we propose (Algorithm 1) solves the system (2.13) with the help of Cholesky factorization, it is important that the algorithm preserves the positive semi-definiteness of the Hessian matrix $H(\mathrm{x}, y, z)$ across the iterations. If the Hessian matrix is not positive definite, the algorithm suitably perturbs the Hessian (see Section 2.7). Therefore, Section 2.6 assumes that the Hessian matrix $H(\mathrm{x}, y, z)$ is positive definite, and hence the matrix $N_{\hat{\beta}}(\mathrm{x}, y, z, v, w)$ is positive definite. Consequently, the search directions can be determined by Theorem 2.1.

Once an iterative point becomes feasible, in order to retain the feasibility of the next iterates, we need to maintain nonnegativity of the slack variables v and w and of the dual variables y and z. Hence, the step length for movement at each iteration needs to be chosen carefully. Usually, merit functions [88–90] are used to determine suitable step lengths along the search directions.

2.6.1 Merit functions

Feasible interior-point algorithms [55, 91] are known to maintain feasibility of every iterate. Thus, in feasible IPMs, one focuses only on finding suitable descent directions. In infeasible IPMs, however, the iterations start from outside the feasible region. Therefore, if we attempt to apply an infeasible IPM, in addition to finding a descent direction, we need to reduce infeasibility at every iteration. Progress of both of the objective function and infeasibility can be done by monitoring a suitable merit function reduction at every iteration. These progresses are obtained by suitably truncating the step length along the search directions (2.13) so that there is substantial reduction in the merit function.

Some variants [30, 90] of merit functions for solving the system (2.7) are as follows:

$$\phi_1(\mathrm{x}, v, w, \eta, \mu) = b_\mu(\mathrm{x}, v, w) + \eta\left(\|\varrho_{\hat{\beta}}(\mathrm{x}, v)\|_1 + \|\rho(\mathrm{x}, w)\|_1\right),$$

$$\phi_2(\mathrm{x}, v, w, y, z) = \|\sigma_{\hat{\beta}}\|_2^2 + \|YVe\|_2^2 + \|WZe\|_2^2 + \|\varrho_{\hat{\beta}}(\mathrm{x}, v)\|_2^2 + \|\rho(\mathrm{x}, w)\|_2^2,$$

$$\psi_{\eta,\mu}(\mathrm{x}, v, w) = b_\mu(\mathrm{x}, v, w) + \frac{\eta}{2}\left(\|\varrho_{\hat{\beta}}(\mathrm{x}, v)\|_2^2 + \|\rho(\mathrm{x}, w)\|_2^2\right), \quad \text{etc.,}$$

where $\mu > 0,\ \eta > 0$.

The merit function $\phi_1(\mathrm{x}, v, w, \eta, \mu)$ is exact but nondifferentiable due to the l_1 norm. The word 'exact' refers to the existence of a positive scalar η_0 such that for every $\eta \geq \eta_0$, a local minimum point of the problem (2.5) is also a local minimum point of $\phi_1(\mathrm{x}, v, w, \eta, \mu)$.

The l_2 merit function $\phi_2(\mathrm{x}, v, w, y, z)$ was introduced by El Bakry et al. [90]. They showed that the algorithm is convergent whenever the Jacobian of the system (2.10) is nonsingular. The proposed algorithm in [90] was tested on the Hock and Schittkowski test problems by Shanno and Simantiraki [92]. They found that the algorithm fails on some problems.

In this chapter, we use the differentiable l_2 merit function $\psi_{\eta,\mu}$ (see [30]). A mild disadvantage of the merit function $\psi_{\eta,\mu}$ is that η needs to be large enough to guarantee the convergence to a feasible point. The following theorem verifies that if the problem (2.1) is such that the Hessian matrix $H(\mathrm{x}, y, z)$ is positive definite, the search direction given by Theorem 2.1 is a descent direction for $\psi_{\eta,\mu}(\mathrm{x}, v, w)$ provided η is large enough.

Theorem 2.2 *Consider the barrier problem (2.5) and the point $\Omega = (\mathrm{x}, v, w, y, z)$ is such that $v > 0,\ w > 0,\ y > 0$ and $z > 0$. Suppose that $N_{\hat{\beta}}$ is positive definite at Ω. Then, for any $\mu > 0$, the following results hold:*

1. *If the point Ω is primal feasible, i.e., $\varrho_{\hat{\beta}} = \rho = 0$, then either $(\Delta\mathrm{x}, \Delta v, \Delta w)$ is a*

descent direction for the barrier function $b_\mu(\mathrm{x}, v, w)$ or the point Ω satisfies the KKT conditions (2.7).

2. *If the point Ω is not primal feasible, then there exists an $\eta_{\min} \geq 0$ such that for each $\eta > \eta_{\min}$, $(\Delta \mathrm{x}, \Delta v, \Delta w)$ is a descent direction for the merit function $\psi_{\eta,\mu}(\mathrm{x}, v, w)$.*

Proof: 1. From the expression of $b_\mu(\mathrm{x}, v, w)$, we obtain $\nabla_{\mathrm{x}} b_\mu = c,\ \nabla_v b_\mu = -\mu V^{-1} e$ and $\nabla_w b_\mu = -\mu W^{-1} e$. Denote $y = \mu V^{-1} e,\ z = \mu W^{-1} e$ and $\sigma_{\hat{\beta}} = c - (A_{\hat{\beta}}(\mathrm{x}))^\top y - (B(\mathrm{x}))^\top z$. By the expression of $\Delta \mathrm{x},\ \Delta v$ and Δw from Theorem 2.1, we obtain

$$\begin{bmatrix} \nabla_{\mathrm{x}} b_\mu & \nabla_v b_\mu & \nabla_w b_\mu \end{bmatrix} \begin{bmatrix} \Delta \mathrm{x} & \Delta v & \Delta w \end{bmatrix}^\top$$

$$= -\sigma_{\hat{\beta}}^\top N_{\hat{\beta}}^{-1} \sigma_{\hat{\beta}} + \sigma_{\hat{\beta}}^\top N_{\hat{\beta}}^{-1} (A_{\hat{\beta}}(\mathrm{x}))^\top V^{-1} Y \varrho_{\hat{\beta}} + y^\top \varrho_{\hat{\beta}} + z^\top \rho + \sigma_{\hat{\beta}}^\top N_{\hat{\beta}}^{-1} (B(\mathrm{x}))^\top W^{-1} Z \rho.$$

Since the given point (x, v, w, y, z) is primal feasible, $\varrho_{\hat{\beta}} = \rho = 0$. Hence, we get

$$\begin{bmatrix} \nabla_{\mathrm{x}} b_\mu \\ \nabla_v b_\mu \\ \nabla_w b_\mu \end{bmatrix}^\top \begin{bmatrix} \Delta \mathrm{x} \\ \Delta v \\ \Delta w \end{bmatrix} = -\sigma_{\hat{\beta}}^\top N_{\hat{\beta}}^{-1} \sigma_{\hat{\beta}}. \tag{2.16}$$

As $N_{\hat{\beta}}$ is positive definite, $N_{\hat{\beta}}^{-1}$ is positive definite. Thus, $\sigma_{\hat{\beta}}^\top N_{\hat{\beta}}^{-1} \sigma_{\hat{\beta}} \geq 0$.

- *Case* 1. If $\sigma_{\hat{\beta}}^\top N_{\hat{\beta}}^{-1} \sigma_{\hat{\beta}} > 0$, then (2.16) shows that $(\Delta \mathrm{x}, \Delta v, \Delta w)$ is a descent direction for the barrier function.
- *Case* 2. If $\sigma_{\hat{\beta}}^\top N_{\hat{\beta}}^{-1} \sigma_{\hat{\beta}} = 0$, then $\sigma_{\hat{\beta}} = 0$. In this case, $\varrho_{\hat{\beta}} = \rho = 0$ and $\sigma_{\hat{\beta}} = 0$ together satisfy the KKT system (2.7).

2. We note that $\nabla_{\mathrm{x}} \left(\|\varrho_{\hat{\beta}}(\mathrm{x}, v)\|_2^2 + \|\rho(\mathrm{x}, w)\|_2^2 \right) = -2 \left(\varrho_{\hat{\beta}}(\mathrm{x}, v)^\top A_{\hat{\beta}}(\mathrm{x}) + \rho(\mathrm{x}, w)^\top B(\mathrm{x}) \right)$.

Therefore,

$$\begin{aligned}
&\begin{bmatrix} \nabla_{\mathrm{x}} \left(\|\varrho_{\hat{\beta}}(\mathrm{x}, v)\|_2^2 + \|\rho(\mathrm{x}, w)\|_2^2 \right) \\ \nabla_v \left(\|\varrho_{\hat{\beta}}(\mathrm{x}, v)\|_2^2 \right) \\ \nabla_w \left(\|\rho(\mathrm{x}, w)\|_2^2 \right) \end{bmatrix}^\top \begin{bmatrix} \Delta \mathrm{x} \\ \Delta v \\ \Delta w \end{bmatrix} \\
= & -2\varrho_{\hat{\beta}}^\top (A_{\hat{\beta}}(\mathrm{x})\Delta \mathrm{x} - \Delta v) - 2\rho^\top (B(\mathrm{x})\Delta \mathrm{x} - \Delta w) \\
= & -2\left(\|\varrho_{\hat{\beta}}\|_2^2 + \|\rho\|_2^2 \right).
\end{aligned}$$

Further,

$$\begin{aligned}
\begin{bmatrix} \nabla_{\mathrm{x}} \psi_{\eta,\mu} \\ \nabla_v \psi_{\eta,\mu} \\ \nabla_w \psi_{\eta,\mu} \end{bmatrix}^\top \begin{bmatrix} \Delta \mathrm{x} \\ \Delta v \\ \Delta w \end{bmatrix} &= \begin{bmatrix} \nabla_{\mathrm{x}} b_\mu \\ \nabla_v b_\mu \\ \nabla_w b_\mu \end{bmatrix}^\top \begin{bmatrix} \Delta \mathrm{x} \\ \Delta v \\ \Delta w \end{bmatrix} + \begin{bmatrix} -\eta \varrho_{\hat{\beta}}^\top A_{\hat{\beta}}(\mathrm{x}) - \eta \rho^\top B(\mathrm{x}) \\ \eta \varrho_{\hat{\beta}} \\ \eta \rho \end{bmatrix}^\top \begin{bmatrix} \Delta \mathrm{x} \\ \Delta v \\ \Delta w \end{bmatrix} \\
&= -\sigma_{\hat{\beta}}^\top N_{\hat{\beta}}^{-1} \sigma_{\hat{\beta}} + \sigma_{\hat{\beta}}^\top N_{\hat{\beta}}^{-1} (A(\mathrm{x}))^\top V^{-1} Y \varrho_{\hat{\beta}} + y^\top \varrho_{\hat{\beta}} + z^\top \rho \\
&\quad + \sigma_{\hat{\beta}}^\top N_{\hat{\beta}}^{-1} (B(\mathrm{x}))^\top W^{-1} Z \rho - \eta \left(\|\varrho_{\hat{\beta}}\|_2^2 + \|\rho\|_2^2 \right).
\end{aligned}$$

We denote

$$\Gamma_{\hat{\beta}}(\Omega) = \sigma_{\hat{\beta}}^\top N_{\hat{\beta}}^{-1} (A(\mathrm{x}))^\top V^{-1} Y \varrho_{\hat{\beta}} + y^\top \varrho_{\hat{\beta}} + z^\top \rho + \sigma_{\hat{\beta}}^\top N_{\hat{\beta}}^{-1} (B(\mathrm{x}))^\top W^{-1} Z \rho - \sigma_{\hat{\beta}}^\top N_{\hat{\beta}}^{-1} \sigma_{\hat{\beta}}. \quad (2.17)$$

Then,

$$\begin{bmatrix} \nabla_{\mathrm{x}} \psi_{\eta,\mu} \\ \nabla_v \psi_{\eta,\mu} \\ \nabla_w \psi_{\eta,\mu} \end{bmatrix}^\top \begin{bmatrix} \Delta \mathrm{x} \\ \Delta v \\ \Delta w \end{bmatrix} = \Gamma_{\hat{\beta}}(\Omega) - \eta \left(\|\varrho_{\hat{\beta}}\|_2^2 + \|\rho\|_2^2 \right).$$

Since the given point (x, v, w, y, z) is not a feasible point, i.e., either or both of $\|\varrho_{\hat{\beta}}\|_2 \neq 0$ and $\|\rho\|_2 \neq 0$, the following two cases arise:

- *Case* 1. In this case, we consider $\Gamma_{\hat{\beta}}(\Omega) \leq 0$. This implies

$$\begin{bmatrix} \nabla_{\mathrm{x}} \psi_{\eta,\mu} \\ \nabla_{v} \psi_{\eta,\mu} \\ \nabla_{w} \psi_{\eta,\mu} \end{bmatrix}^{\top} \begin{bmatrix} \Delta \mathrm{x} \\ \Delta v \\ \Delta w \end{bmatrix} < 0.$$

  Hence, $(\Delta \mathrm{x}, \Delta v, \Delta w)$ is a descent direction of $\psi_{\eta,\mu}$ for any $\eta > 0$. In this case, the result follows with $\eta_{\min} = 0$.

- *Case* 2. Let $\Gamma_{\hat{\beta}}(\Omega) > 0$. In this case, by choosing an η so that

$$\eta > \frac{\Gamma_{\hat{\beta}}(\Omega)}{\left(\|\varrho_{\hat{\beta}}\|_2^2 + \|\rho\|_2^2\right)} = \eta_{\min},$$

  we see that $(\Delta \mathrm{x}, \Delta v, \Delta w)$ is a descent direction for $\psi_{\eta,\mu}$. Hence, the result follows.

□

Note 2.1 *Theorem 2.2 suggests to keep* η *as zero for the merit function* $\psi_{\eta,\mu}$ *as long as either the current iterate is feasible or the direction* $(\Delta \mathrm{x}, \Delta v, \Delta w)$ *is descent. In case the current iterate is neither feasible nor* $(\Delta \mathrm{x}, \Delta v, \Delta w)$ *is descent, choose* η *not smaller than*

$$\eta_{\min} = \frac{\Gamma_{\hat{\beta}}(\Omega)}{\left(\|\varrho_{\hat{\beta}}\|_2^2 + \|\rho\|_2^2\right)}, \tag{2.18}$$

where $\Gamma_{\hat{\beta}}(\Omega)$ *is given by* (2.17)*; in the proposed algorithm, we typically choose* $\eta = 10\eta_{\min}$.

So far, we have not discussed the choice of the barrier parameter μ, step length, and strategy for choosing the initial point. Next, we detail these concerns in the subsections 2.6.2, 2.6.3 and 2.6.4.

2.6.2 Choice of the barrier parameter μ

The selection of the barrier parameter has an important role in IPMs, as the optimality is achieved whenever the barrier parameter decreases to zero (see [65]). Feasible interior point methods [65] are generally based on the strategy of central path. Therefore, whenever the primal-dual feasible point (x, v, w, y, z) lies on the central path, it has to satisfy the second and third equations of (2.7), i.e., $v^\top y = \mathcal{P}\mu$ and $w^\top z = (\mathcal{J}+1)\mu$. Hence, the exact value of μ at any iteration is given by $\mu = \frac{v^\top y + w^\top z}{\mathcal{P}+\mathcal{J}+1}$. In the proposed algorithm, as the iterations start from outside the feasible region, an estimate of the barrier parameter μ is chosen as (see [93]):

$$\mu = r\frac{v^\top y + w^\top z}{\mathcal{P}+\mathcal{J}+1} \text{ for some } r \in (0,1). \tag{2.19}$$

This choice of μ shows the convergence (Theorem 2.4) of the iterative sequence generated by the proposed Algorithm 1. Computational evidence, as demonstrated in Section 2.8, also shows that this choice of barrier parameter (2.19) works well for the merit function $\psi_{\eta,\mu}$.

2.6.3 Choice of the step length

The proposed algorithm updates the iteration point at the end of each iteration by (2.14). When choosing the step length at every iteration, attention must be given so that the slack variables v_i and w_j, and the dual variables y_i and z_j, stay positive for all $i \in \mathcal{I}$ and $j = 1, 2, \ldots, \mathcal{J}+1$ across the iterations, otherwise the barrier function (see (2.5)) will not be well defined. For this positivity, we choose the step length α at every iteration by the following standard ratio formula (see [93]):

$$\alpha \;=\; \min\left\{\delta\left(\max_{i,j}\left\{-\frac{\Delta v_i}{v_i}, -\frac{\Delta y_i}{y_i}, -\frac{\Delta w_j}{w_j}, -\frac{\Delta z_j}{z_j}\right\}\right)^{-1},\ 1\right\}, \tag{2.20}$$

where $0 < \delta \leq 1$. The results in [86] demonstrate that $\delta = 0.95$ works well in practice. Therefore, we also select this value of δ in the proposed algorithm.

The choice of step length α by (2.20) is generally used to maintain positivity of the nonnegative variables for linear and quadratic programming [93]. However, for general nonlinear programming, it might be the case that a slight decrease in the objective function contributes to a large rise of infeasibility. Therefore, at each iteration, the interior-point method chooses $\bar{\alpha} \in (0, \alpha]$ such that the merit function follows the following Armijo rule:

$$\psi_{\eta,\mu}(p + \bar{\alpha}\Delta p) - \psi_{\eta,\mu}(p) \leq \kappa\bar{\alpha} \left(\nabla_p \psi_{\eta,\mu}(p)\right)^\top \Delta p, \tag{2.21}$$

where $\kappa \in (0, 1)$, $p = (\mathrm{x}, v, w)$ and $\Delta p = (\Delta \mathrm{x}, \Delta v, \Delta w)$. If the point p is such that the Hessian matrix $H(\mathrm{x}, y, z)$ is positive definite, then the matrix $N_{\hat{\beta}}(\Omega)$ is positive definite, and hence the direction Δp calculated by Theorem 2.1 is descent for the merit function $\psi_{\eta,\mu}(p)$ (see Theorem 2.2). Consequently, the right side of (2.21) is negative, which confirms that the merit function $\psi_{\eta,\mu}(p)$ reduces at the point $p + \bar{\alpha}\Delta p$.

Note 2.2 *Note that as Δp is a descent direction for the merit function $\psi_{\eta,\mu}$, there exists an $\alpha^* > 0$ that satisfies the condition (2.21) (see Section 2.5 in [33]). In finding an estimate of the largest α^* (which we denote as $\bar{\alpha}$) that satisfies the condition (2.21), a common way is to choose the largest value in $\{\zeta^2\alpha, \zeta^3\alpha, \ldots\}$ such that the condition (2.21) holds, where $\zeta \in (0, 1)$. Notice that there exist $l \in \{2, 3, \ldots\}$ such that $\bar{\alpha}_l = \zeta^l \alpha \leq \alpha^*$ that will satisfy the condition (2.21).*

2.6.4 Choice of the initial point

To solve the convex multiobjective optimization problem (2.1), the method that we propose first converts the MOP into a set of parametric scalar optimization problems (2.2). Subsequently, the method chooses the parameter $\hat{\beta}$ by (2.3) and then initializes the algorithm by selecting an appropriate starting point. Interior-point methods not only require a suitable initialization of x but also of the slack variables $v_i,\ i \in \mathcal{I}$, and $w_j,\ j = 1, 2, \ldots, \mathcal{J}+1$. For a given $\mathrm{x}^{(0)}$, an initialization of x, the slack variable values are given by (see (2.7))

$$v^{(0)} = \hat{\beta} c^\top \mathrm{x}^{(0)} - F(\mathrm{x}^{(0)}) \ \text{ and } \ w^{(0)} = h(\mathrm{x}^{(0)}). \tag{2.22}$$

With the computed values of $v^{(0)}$ and $w^{(0)}$, two difficulties may occur:

1. if $\mathrm{x}^{(0)}$ is not feasible, then at least one component of each of the vectors $v^{(0)}$ and $w^{(0)}$ becomes negative, which makes the barrier function in (2.5) undefined, and
2. if $\mathrm{x}^{(0)}$ is feasible, then it may be possible that one or more components of $v^{(0)}$ and $w^{(0)}$ are very close to zero. This impedes progress towards the solution.

To resolve both the issues, one way of initializing $v^{(0)}$ and $w^{(0)}$ (see [86]) is

$$\left.\begin{aligned} v_j^{(0)} &= \begin{cases} \xi_1 & \text{if } \beta_i c^\top \mathrm{x}^{(0)} - f_i(\mathrm{x}^{(0)}) = 0, \quad i = 1, 2, \ldots, \mathcal{P} \\ |\beta_i c^\top \mathrm{x}^{(0)} - f_i(\mathrm{x}^{(0)})| & \text{otherwise} \end{cases} \\ w_i^{(0)} &= \begin{cases} \xi_2 & \text{if } h_j(\mathrm{x}^{(0)}) = 0, \quad j = 1, 2, \ldots, \mathcal{P}+1, \\ |h_j(\mathrm{x}^{(0)})| & \text{otherwise,} \end{cases} \end{aligned}\right\} \tag{2.23}$$

where $\xi_1 > 0$ and $\xi_2 > 0$. This method of initializing v_i's and w_j's holds the variables v and w at least as large as ξ_1 and ξ_2, and it eliminates both of the above-mentioned difficulties. In practice, we use $\xi_1 = 1$ and $\xi_2 = 1$.

Recall that the solution of the multiobjective problem (2.1) will be achieved by solving the parametric single objective optimization problem (2.2). The interior-point method reformulates the problem (2.2) as a parametric barrier problem (2.5). Algorithm 1 solves the parametric barrier problem (2.5), via solving (2.10), and saves the objective vectors of the Pareto optimal solutions in 'Nondominated Set' for each $\hat{\beta} \in \mathbb{S}_{\geqq}^{\mathcal{P}-1}$.

The set 'Nondominated Set' in the Algorithm 1 gives a discrete approximation of the complete nondominated set of the problem (2.1). By increasing the number of $\hat{\beta}$, the entire $\mathbb{S}_{\geqq}^{\mathcal{P}-1}$ can be covered, and hence Algorithm 1 is capable of capturing the complete nondominated set of the problem (2.1). This is due to the fact that for each nondominated point, there is a $\hat{\beta} \in \mathbb{S}_{\geqq}^{\mathcal{P}-1}$ (see Theorem 2 in [47]).

As we aim to find a solution of (2.5), the solution point is required to satisfy primal feasibility, dual feasibility and complementary slackness. We, thus, use the following merit function

$$\nu(\Omega) = \max\{\|\sigma_{\hat{\beta}}\|_1, \|\varrho_{\hat{\beta}}\|_1, \|\rho\|_1, \|VYe\|_1, \|WZe\|_1\} \tag{2.24}$$

to measure overall progress of the iterative points. For a given $\epsilon > 0$, we define the approximated KKT point as $\Omega = (\mathrm{x}, v, w, y, z)$ such that $v > 0,\ w > 0,\ y > 0,\ z > 0$ and $\nu(\Omega) \leq \epsilon$. Note that $\nu(\Omega) = 0$ if and only if Ω is a KKT point. Thus, Algorithm 1 attempts to reduce merit function (2.24) until it is less than a precision parameter ϵ. In addition, Algorithm 1 also reduces the merit function

$$\psi_{\eta,\mu}(p) = c^\top \mathrm{x} - \mu \left(\sum_{i=1}^{k} \log(v_i) + \sum_{j=1}^{m+1} \log(w_j) \right) + \frac{\eta}{2} \left(\|\varrho_{\hat{\beta}}(\mathrm{x}, v)\|_2^2 + \|\rho(\mathrm{x}, w)\|_2^2 \right)$$

with adaptive μ and η to find a minimum of the problem (2.5).

Algorithm 1 is applicable to MOPs for which the Hessian matrix $H(\mathrm{x}, y, z)$ is positive definite at every iteration. For example, if all the objective functions f_i's are twice continuously differentiable and convex, and the constraint functions h_j's are twice continuously differentiable and concave, then $H(\mathrm{x}, y, z)$ is positive definite. In case we are not sure about the positive definiteness of $H(\mathrm{x}, y, z)$, Algorithm 1 is not applicable for such a problem, and for such a case, we need to apply Algorithm 3. In fact, Algorithm 3 is applicable irrespective of the positive definiteness of $H(\mathrm{x}, y, z)$. If, however, we somehow know apriori that $H(\mathrm{x}, y, z)$ is positive definite, we must prefer to apply Algorithm 1 over Algorithm 3 since in Algorithm 3 there are some extra computations due to the line numbers 8 to 14.

Algorithm 1 IIPM for the MOPs when $H(\mathrm{x}, y, z)$ is positive definite

Aim: To generate a discrete approximation of the nondominated set $\mathcal{D}$ of the MOP (2.1)
1: **Inputs**
Provide $f_1, f_2, \ldots, f_{\mathcal{P}}$ which are twice continuously differentiable objective functions of (2.1)
Provide $h_1, h_2, \ldots, h_{\mathcal{J}}$ which are twice continuously differentiable constraints functions of (2.1)
Provide N, the number of different $\hat{\beta}$'s corresponding to which the subproblem (2.5) is to be solved
2: **Initialization**
Set nondominated set $\mathcal{D} \leftarrow \emptyset$
To solve (2.5), for a given $\hat{\beta}$, provide an initial point $\mathrm{x}^{(0)}$
Choose any positive ξ_1 and ξ_2 for (2.23)
Initialize $v^{(0)}$, $w^{(0)}$ by (2.23) and $y^{(0)}, z^{(0)}$ by any positive values
Choose any positive r, δ and $\kappa \in (0, 1)$ for (2.19), (2.20) and (2.21), respectively
Give a value of the precision parameter $\epsilon > 0$ for the optimum solutions to (2.5)
Set $k \leftarrow 0$
Set $\mu^{(k)} \leftarrow r\frac{(v^{(k)})^\top y^{(k)} + (w^{(k)})^\top z^{(k)}}{\mathcal{J}+\mathcal{P}+1}$
3: **Main Steps to solve** (2.5) **for** N **different** $\hat{\beta}$**'s**
4: **for** $i = 1 : 1 : N$ **do**
(Lines 5 to 36 solves (2.5) for a given $\hat{\beta}$ and find $\mathrm{x}^{(k)}$ as a solution to (2.5). Line 38 keeps stacking up the nondominated points of the MOP (2.1) in the set $\mathcal{D}$)
5: Choose a direction $\hat{\beta}$ by using (2.3)
6: Set $\nu(\Omega^{(k)}) \leftarrow \max\{\|\sigma_{\hat{\beta}}^{(k)}\|_1, \|\varrho_{\hat{\beta}}^{(k)}\|_1, \|\rho^{(k)}\|_1, \|V^{(k)}Y^{(k)}e\|_1, \|W^{(k)}Z^{(k)}e\|_1\}$ (according to (2.24))
7: **while** $\nu(\Omega^{(k)}) \geq \epsilon$ **do**
8: Solve the system (2.13) for $(\Delta\mathrm{x}^{(k)}, \Delta v^{(k)}, \Delta w^{(k)})$ with the help of Cholesky factorization
9: Choose step length α by the formula (2.20)
10: Set $p^{(k+1)} \leftarrow p^{(k)} + \alpha\Delta p^{(k)}$, $y^{(k+1)} \leftarrow y^{(k)} + \alpha\Delta y^{(k)}$, $z^{(k+1)} \leftarrow z^{(k)} + \alpha\Delta z^{(k)}$
11: Calculate $\Gamma_{\hat{\beta}}(\Omega^{(k)})$ by the expression (2.17)
12: **if** $\varrho_{\hat{\beta}}(\mathrm{x}^{(k)}, v^{(k)}) = 0$, $\rho(\mathrm{x}^{(k)}, w^{(k)}) = 0$ and $\psi_{\eta,\mu}(\Omega^{(k+1)}) \leq \psi_{\eta,\mu}(\Omega^{(k)})$ **then**
13: Update $\mu^{(k+1)} \leftarrow r\frac{(v^{(k+1)})^\top y^{(k+1)} + (w^{(k+1)})^\top z^{(k+1)}}{\mathcal{J}+\mathcal{P}+1}$
14: Update $\nu(\Omega^{(k+1)})$ according to (2.24)
15: Set $k \leftarrow k + 1$
16: **else**
17: **if** $\Gamma_{\hat{\beta}}(\Omega^{(k)}) < 0$ **then**
18: Set $\eta \leftarrow 0$
19: Backtrack $\alpha^{(k)} \in [0, \alpha]$ until the condition (2.21) holds
20: Update $\Omega^{(k+1)} \leftarrow \Omega^{(k)} + \alpha^{(k)}\Delta\Omega^{(k)}$
21: Update $\mu^{(k+1)} \leftarrow r\frac{(v^{(k+1)})^\top y^{(k+1)} + (w^{(k+1)})^\top z^{(k+1)}}{\mathcal{J}+\mathcal{P}+1}$
22: Update $\nu(\Omega^{(k+1)})$ according to (2.24)
23: Set $k \leftarrow k + 1$
24: **else**
25: Calculate $\eta_{\min}$ by equation (2.18)
26: Set $\eta = 10\eta_{\min}$
27: Backtrack $\alpha^{(k)} \in [0, \alpha]$ until the condition (2.21) holds
28: Update $\Omega^{(k+1)} \leftarrow \Omega^{(k)} + \alpha^{(k)}\Delta\Omega^{(k)}$
29: **if** $\psi_{\hat{\beta}}(\Omega^{(k+1)}) \leq \psi_{\hat{\beta}}(\Omega^{(k)})$ **then**
30: Update $\mu^{(k+1)} \leftarrow r\frac{(v^{(k+1)})^\top y^{(k+1)} + (w^{(k+1)})^\top z^{(k+1)}}{\mathcal{J}+\mathcal{P}+1}$
31: Update $\nu(\Omega^{(k+1)})$ according to (2.24)
32: Set $k \leftarrow k + 1$
33: **end if**
34: **end if**
35: **end if**
36: **end while**
37: Calculate $F(\mathrm{x}^{(k)}) = \hat{\beta}c^T\mathrm{x}^{(k)} - v^{(k)}$
38: Update the nondominated set $\mathcal{D} \leftarrow \mathcal{D} \bigcup \{F(\mathrm{x}^{(k)})\}$
39: **end for**
40: **return** the set $\mathcal{D}$ (a discrete approximation of the whole nondominated set)

2.6.5 Well-definedness of Algorithm 1

The well-definedness of Algorithm 1 depends on line numbers 8 and 9. For Algorithm 1, the Hessian matrix $H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})$ is positive definite at any k-th iteration. Hence, the system (2.13) is consistent and Cholesky factorization (see [87]) is applicable to obtain $(\Delta \mathrm{x}^{(k)}, \Delta y^{(k)}, \Delta z^{(k)})$. Thus, line 8 is well-defined. For the line 9, we initially take the values of $v^{(0)}$, $w^{(0)}$, $y^{(0)}$ and $w^{(0)}$ positive and thus nonzero. Then the rule (2.20) is applicable to obtain the step length α at the initial iteration. Rule (2.20) also keeps the values of $v^{(k)}$, $w^{(k)}$, $y^{(k)}$ and $w^{(k)}$ positive at any k-th iteration (see Section 2.6.3). This implies that line number 9 is also well-defined.

2.6.6 Convergence analysis

In this subsection, we examine the convergence of the proposed algorithm. Algorithm 1 generates the sequence (2.14) by computing the direction and step length at every iteration and minimizes the merit function $\psi_{\eta,\mu}(p)$. Apart from the reduction of $\psi_{\eta,\mu}(p)$, it is essential to calculate the reduction of $\nu(\Omega)$ because we use $\nu(\Omega) < \epsilon$ as the stopping criterion of Algorithm 1. The following questions arise concerning the merit function $\nu(\Omega)$:

(i) Does the Algorithm 1 reduce the values of primal infeasibilities $(\rho, \varrho_{\hat{\beta}})$, dual infeasibility $(\sigma_{\hat{\beta}})$ and complementarity (γ_1, γ_2) at each iteration?

(ii) Does the sequence (2.14) generated by the Algorithm 1 converge to an optimal point of the problem (2.5)?

(iii) How many iterations does the algorithm need to reach an ϵ-precise solution for a given $\epsilon > 0$?

Answers to these three questions are addressed in Subsection 2.6.7. Answer to (i) is provided by Theorem 2.3. For the answer to (ii) and (iii), see Theorem 2.4.

2.6.7 Measures of progress

The progress of the interior-point method towards a solution of the KKT system (2.7) can be observed by estimating the following three criteria at consecutive iterates:

(i) primal infeasibility $(\|\varrho_{\hat{\beta}}\|_1, \|\rho\|_1)$,

(ii) dual infeasibility $(\|\sigma_{\hat{\beta}}\|_1)$ and

(iii) complementarity $(\|VYe\|_1, \|WZe\|_1)$.

In the derivation below, we use the combined complementarity by $\gamma = v^\top y + w^\top z$. The following theorem calculates the reduction of criteria (i), (ii) and (iii) in one iteration.

Theorem 2.3 *Consider the barrier problem* (2.5)*. At the k-th iterative point* $\Omega^{(k)} = \left(\mathrm{x}^{(k)}, v^{(k)}, w^{(k)}, y^{(k)}, z^{(k)}\right)$*, let*

(i) the Hessian matrix $H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})$ *be positive definite,*

(ii) the direction $\Delta\Omega^{(k)} = (\Delta\mathrm{x}^{(k)}, \Delta v^{(k)}, \Delta w^{(k)}, \Delta y^{(k)}, \Delta z^{(k)})$ *in Algorithm 1 be given by solving the system* (2.13)*,*

(iii) $\alpha^{(k)}$ *be given by* (2.20)*,*

(iv) for some $\zeta \in (0, 1)$*, the largest value in* $\left\{\zeta^2\alpha^{(k)}, \zeta^3\alpha^{(k)}, \zeta^4\alpha^{(k)}, \ldots\right\}$ *that satisfies the condition* (2.21) *be* $\bar{\alpha}^{(k)}$*, and*

(v) the next iterative point $\Omega^{(k+1)} = \left(\mathrm{x}^{(k+1)}, v^{(k+1)}, w^{(k+1)}, y^{(k+1)}, z^{(k+1)}\right)$ *of* $\Omega^{(k)}$ *of the proposed Algorithm 1 be given by* $\Omega^{(k+1)} = \Omega^{(k)} + \bar{\alpha}^{(k)}\Delta\Omega^{(k)}$*.*

Suppose the primal infeasibilities, dual infeasibility and complementarity at $\Omega^{(k)}$ *and* $\Omega^{(k+1)}$ *are* $(\varrho_{\hat{\beta}}^{(k)}, \rho^{(k)}, \sigma_{\hat{\beta}}^{(k)}, \gamma^{(k)})$ *and* $(\varrho_{\hat{\beta}}^{(k+1)}, \rho^{(k+1)}, \sigma_{\hat{\beta}}^{(k+1)}, \gamma^{(k+1)})$*, respectively. If there exists* $M > 0$ *such that* $\Delta\Omega^{(k)}$ *is bounded by* M*, i.e.,* $\|\Delta\Omega^{(k)}\|_\infty \leq M$*, then*

$$\|\rho^{(k+1)}\|_1 \leq (1 - \bar{\alpha}^{(k)})\|\rho^{(k)}\|_1$$

$$\|\sigma_{\hat{\beta}}^{(k+1)}\|_1 \leq (1 - \bar{\alpha}^{(k)})\|\sigma_{\hat{\beta}}^{(k)}\|_1 \text{ and}$$

$$\gamma^{(k+1)} \leq \gamma^{(k)}(1 - \bar{\alpha}^{(k)}(1 - r)) + M(\|\varrho_{\hat{\beta}}^{(k)}\|_1 + \|\rho^{(k)}\|_1 + \|\sigma_{\hat{\beta}}^{(k)}\|_1).$$

Proof: We note that

$$
\begin{aligned}
\varrho_{\hat{\beta}}^{(k+1)} &= F(\mathrm{x}^{(k+1)}) + v^{(k+1)} - \hat{\beta} c^\top \mathrm{x}^{(k+1)} \\
&= F(\mathrm{x}^{(k)} + \bar{\alpha}^{(k)} \Delta \mathrm{x}^{(k)}) + v^{(k)} + \bar{\alpha}^{(k)} \Delta v^{(k)} - \hat{\beta} c^\top (\mathrm{x}^{(k)} + \bar{\alpha}_l^{(k)} \Delta \mathrm{x}^{(k)}) \\
&\geq F(\mathrm{x}^{(k)}) + \bar{\alpha}^{(k)} \nabla F(\mathrm{x}^{(k)}) \Delta \mathrm{x}^{(k)} + v^{(k)} + \bar{\alpha}^{(k)} \Delta v^{(k)} - \hat{\beta} c^\top \mathrm{x}^{(k)} - \bar{\alpha}^{(k)} \hat{\beta} c^\top \Delta \mathrm{x}^{(k)} \quad \text{(since } F \text{ is a convex function)} \\
&= \left(F(\mathrm{x}^{(k)}) + v^{(k)} - \hat{\beta} c^\top \mathrm{x}^{(k)} \right) - \bar{\alpha}^{(k)} \left(\nabla \left(\hat{\beta} c^\top \mathrm{x}^{(k)} - F(\mathrm{x}^{(k)}) \right) \Delta \mathrm{x}^{(k)} - \Delta v^{(k)} \right). \qquad (2.25)
\end{aligned}
$$

As the quantities $F(\mathrm{x}^{(k)}) + v^{(k)} - \hat{\beta} c^\top \mathrm{x}^{(k)}$ and $\nabla_{\mathrm{x}} \left(\hat{\beta} c^\top \mathrm{x}^{(k)} - F(\mathrm{x}^{(k)}) \right)$ are the primal infeasibility $\varrho_{\hat{\beta}}^{(k)}$ (see the second last expression of (2.11)) and $A_{\hat{\beta}}(\mathrm{x}^{(k)})$, respectively, we get from (2.25) that $\varrho_{\hat{\beta}}^{(k+1)} \geq \varrho_{\hat{\beta}}^{(k)} - \bar{\alpha}^{(k)} \left(A_{\hat{\beta}}(\mathrm{x}^{(k)}) \Delta \mathrm{x}^{(k)} - \Delta v^{(k)} \right)$.

The quantity $A_{\hat{\beta}}(\mathrm{x}^{(k)}) \Delta \mathrm{x}^{(k)} - \Delta v^{(k)}$ is equal to $\varrho_{\hat{\beta}}^{(k)}$ (see the second last equation of system (2.10)). Hence,

$$
\varrho_{\hat{\beta}}^{(k+1)} \geq \left(1 - \bar{\alpha}^{(k)}\right) \varrho_{\hat{\beta}}^{(k)}. \qquad (2.26)
$$

As $\hat{\beta} c^\top \mathrm{x}^{(k)} \geq F(\mathrm{x}^{(k)})$ (see (2.4)), we have $\varrho_{\hat{\beta}}^{(k)} \leq 0$ for all $k = 0, 1, 2 \ldots$. Hence, the inequality (2.26) yields

$$
\|\varrho_{\hat{\beta}}^{(k+1)}\|_1 \leq (1 - \bar{\alpha}^{(k)}) \|\varrho_{\hat{\beta}}^{(k)}\|_1. \qquad (2.27)
$$

Next, notice that

$$
\begin{aligned}
\rho^{(k+1)} = w^{(k+1)} - h(\mathrm{x}^{(k+1)}) &= w^{(k)} + \bar{\alpha}^{(k)} \Delta w^{(k)} - h(\mathrm{x}^{(k)} + \bar{\alpha}^{(k)} \Delta \mathrm{x}^{(k)}) \\
&\geq (1 - \bar{\alpha}^{(k)}) \rho^{(k)} \quad (\text{as } -h \text{ is a convex function}). \qquad (2.28)
\end{aligned}
$$

As $h(\mathrm{x}^{(l)}) - w^{(l)} \geq 0$ (see (2.4)), we have $\rho^{(l)} \leq 0$ for all $l = 0, 1, 2 \ldots$. Hence, the inequality (2.28) becomes

$$
\|\rho^{(k+1)}\|_1 \leq (1 - \bar{\alpha}^{(k)}) \|\rho^{(k)}\|_1. \qquad (2.29)
$$

Similarly, a reduction in dual infeasibility is determined by

$$\begin{aligned}
\sigma_{\hat{\beta}}^{(k+1)} &= c - (A_{\hat{\beta}}(\mathrm{x}^{(k+1)}))^\top y^{(k+1)} - (B(\mathrm{x}^{(k+1)}))^\top z^{(k+1)} \\
&= c - A_{\hat{\beta}}(\mathrm{x}^{(k)} + \bar{\alpha}^{(k)}\Delta\mathrm{x}^{(k)})^\top (y^{(k)} + \bar{\alpha}^{(k)}\Delta y^{(k)}) - B(\mathrm{x}^{(k)} + \bar{\alpha}^{(k)}\Delta\mathrm{x}^{(k)})^\top (z^{(k)} + \bar{\alpha}^{(k)}\Delta z^{(k)}) \\
&\leq c - (A_{\hat{\beta}}(\mathrm{x}^{(k)}))^\top y^{(k)} + \bar{\alpha}^{(k)} \left(\sum_{i=1}^{s} y_i^{(k)} \nabla^2 f_i(\mathrm{x}^{(k)})\right) \Delta\mathrm{x}^{(k)} - \bar{\alpha}^{(k)} (A_{\hat{\beta}}(\mathrm{x}^{(k)}))^\top \Delta y^{(k)} \\
&\quad - (B(\mathrm{x}^{(k)}))^\top z^{(k)} - \bar{\alpha}^{(k)} \left(\sum_{j=1}^{m} z_j^{(k)} \nabla^2 h_j(\mathrm{x}^{(k)})\right) \Delta\mathrm{x}^{(k)} - \bar{\alpha}^{(k)} (B(\mathrm{x}^{(k)}))^\top \Delta z^{(k)} \quad (\because F \text{ and } -h \text{ are convex}) \\
&= \bar{\alpha}^{(k)} \left(H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})\Delta\mathrm{x}^{(k)} - (A_{\hat{\beta}}(\mathrm{x}^{(k)}))^\top \Delta y^{(k)} - (B(\mathrm{x}^{(k)}))^\top \Delta z^{(k)}\right) \\
&\quad + \left(c - (A_{\hat{\beta}}(\mathrm{x}^{(k)}))^\top y^{(k)} - (B(\mathrm{x}^{(k)}))^\top z^{(k)}\right). \qquad (2.30)
\end{aligned}$$

Since $\sigma_{\hat{\beta}}^{(k)} = -H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})\Delta\mathrm{x}^{(k)} + (A_{\hat{\beta}}(\mathrm{x}^{(k)}))^\top \Delta y^{(k)} + (B(\mathrm{x}^{(k)}))^\top \Delta z^{(k)}$ (from the first component of (2.10)) and $\sigma_{\hat{\beta}}^{(k)} = c - (A_{\hat{\beta}}(\mathrm{x}^{(k)}))^\top y^{(k)} - (B(\mathrm{x}^{(k)}))^\top z^{(k)}$, therefore (2.30) yields $\sigma_{\hat{\beta}}^{(k+1)} \leq (1 - \bar{\alpha}^{(k)})\sigma_{\hat{\beta}}^{(k)}$, and hence

$$\|\sigma_{\hat{\beta}}^{(k+1)}\|_1 \leq (1 - \bar{\alpha}^{(k)})\|\sigma_{\hat{\beta}}^{(k)}\|_1. \qquad (2.31)$$

For the complemantarity values,

$$\begin{aligned}
\gamma^{(k+1)} &= (v^{(k+1)})^\top y^{(k+1)} + (w^{(k+1)})^\top z^{(k+1)} \\
&= (v^{(k)})^\top y^{(k)} + (w^{(k)})^\top z^{(k)} + \bar{\alpha}^{(k)} \left((v^{(k)})^\top \Delta y^{(k)} + (\Delta v^{(k)})^\top y^{(k)} + (w^{(k)})^\top \Delta z^{(k)} + (\Delta w^{(k)})^\top z^{(k)}\right) \\
&\quad + (\bar{\alpha}^{(k)})^2 \left((\Delta v^{(k)})^\top \Delta y^{(k)} + (\Delta w^{(k)})^\top \Delta z^{(k)}\right).
\end{aligned}$$

We note that

$$(v^{(k)})^\top \Delta y^{(k)} + (\Delta v^{(k)})^\top y^{(k)} = e^\top (V^{(k)}\Delta y^{(k)} + Y^{(k)}\Delta v^{(k)}) = e^\top(\mu e - V^{(k)}Y^{(k)}e) = \mu s - (v^{(k)})^\top y^{(k)}.$$

Similarly, $(w^{(k)})^\top \Delta z^{(k)} + (\Delta w^{(k)})^\top z^{(k)} = \mu(m+1) - (w^{(k)})^\top z^{(k)}$ and

$$\begin{aligned}
(\Delta v^{(k)})^\top \Delta y^{(k)} + (\Delta w^{(k)})^\top \Delta z^{(k)} &= (\Delta\mathrm{x}^{(k)})^\top \sigma_{\hat{\beta}}^{(k)} + (\Delta\mathrm{x}^{(k)})^\top H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})\Delta\mathrm{x}^{(k)} \\
&\quad - (\varrho_{\hat{\beta}}^{(k)})^\top \Delta y^{(k)} - (\rho^{(k)})^\top \Delta z^{(k)}.
\end{aligned}$$

Therefore,

$$\gamma^{(k+1)} = \gamma^{(k)}(1 - \alpha^{(k)}(1-r)) + (\bar{\alpha}^{(k)})^2 \left((\Delta\mathrm{x}^{(k)})^\top \sigma_{\hat{\beta}}^{(k)} + (\Delta\mathrm{x}^{(k)})^\top H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})\Delta\mathrm{x}^{(k)} - (\varrho_{\hat{\beta}}^{(k)})^\top \Delta y^{(k)} - (\rho^{(k)})^\top \Delta z^{(k)}\right).$$

By Hölder inequality, we have

$$|(\Delta \mathrm{x}^{(k)})^\top \sigma_{\hat{\beta}}^{(k)}| \leq \|\sigma_{\hat{\beta}}^{(k)}\|_1 \|\Delta \mathrm{x}^{(k)}\|_\infty, \qquad |(\Delta \mathrm{x}^{(k)})^\top H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)}) \Delta \mathrm{x}^{(k)}| \leq \lambda_{\max} \|\Delta \mathrm{x}^{(k)}\|_\infty^2,$$

$$|(\Delta y^{(k)})^\top \varrho_{\hat{\beta}}^{(k)}| \leq \|\varrho_{\hat{\beta}}^{(k)}\|_1 \|\Delta y^{(k)}\|_\infty \quad \text{and} \quad |(\Delta z^{(k)})^\top \rho^{(k)}| \leq \|\rho^{(k)}\|_1 \|\Delta z^{(k)}\|_\infty,$$

where $\lambda_{\max}$ is the maximum eigenvalue of the matrix $H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})$.

Hence,

$$\begin{aligned} \gamma^{(k+1)} \leq \gamma^{(k)}(1 - \bar{\alpha}^{(k)}(1-r)) + \Big(& \|\varrho_{\hat{\beta}}^{(k)}\|_1 \|(\bar{\alpha}^{(k)})^2 \Delta y^{(k)}\|_\infty + \|\rho^{(k)}\|_1 \|(\bar{\alpha}^{(k)})^2 \Delta z^{(k)}\|_\infty \\ & + \|\sigma_{\hat{\beta}}^{(k)}\|_1 \|(\bar{\alpha}^{(k)})^2 \Delta \mathrm{x}^{(k)}\|_\infty + \lambda_{\max} \|(\bar{\alpha}^{(k)})^2 \Delta \mathrm{x}^{(k)}\|_\infty \Big) \end{aligned} \tag{2.32}$$

because $0 < \bar{\alpha}^{(k)} \leq 1$ (see (2.20)).

As we have considered that $\Delta\Omega^{(k)}$ is bounded by M, i.e., $\|\Delta\Omega^{(k)}\|_\infty \leq M$, thus

$$\|\Delta \mathrm{x}^{(k)}\|_\infty \leq M, \quad \|\Delta v^{(k)}\|_\infty \leq M, \quad \|\Delta w^{(k)}\|_\infty \leq M, \quad \|\Delta y^{(k)}\|_\infty \leq M, \quad \text{and} \quad \|\Delta z^{(k)}\|_\infty \leq M.$$

Hence, the inequality (2.32) can be written as follows:

$$\gamma^{(k+1)} \leq \gamma^{(k)}(1 - \bar{\alpha}^{(k)}(1-r)) + M \left(\|\varrho_{\hat{\beta}}^{(k)}\|_1 + \|\rho^{(k)}\|_1 + \|\sigma_{\hat{\beta}}^{(k)}\|_1 + \lambda_{\max} \right). \tag{2.33}$$

□

Note 2.3 *The dual problem corresponding to the primal problem (2.4) is*

$$\left.\begin{aligned} &\textit{maximize} \quad c^\top \mathrm{x} - y^\top (\hat{\beta} c^\top \mathrm{x} - F(\mathrm{x})) - z^\top h(\mathrm{x}) + ((A_{\hat{\beta}}(\mathrm{x}))^\top y + (B(\mathrm{x}))^\top z - c)^\top \mathrm{x} \\ &\textit{subject to} \quad (A_{\hat{\beta}}(\mathrm{x}))^\top y + (B(\mathrm{x}))^\top z = c, \\ &\qquad\qquad\quad y, z \geq 0. \end{aligned}\right\} \tag{2.34}$$

The difference between the values of the dual objective function and the primal ob-

jective function is

$$\begin{aligned} G &= y^\top(\hat{\beta}c^\top \mathrm{x} - F(\mathrm{x})) + z^\top h(\mathrm{x}) - ((A_{\hat{\beta}}(\mathrm{x}))^\top y + (B(\mathrm{x}))^\top z - c)^\top \mathrm{x} \\ &= \gamma - y^\top \varrho_{\hat{\beta}} - z^\top \rho + \sigma_{\hat{\beta}}^\top \mathrm{x}. \end{aligned}$$

At any point (x, v, w, y, z), *the value of* G *can be estimated by*

$$\begin{aligned} |G| &\leq \gamma + |\varrho_{\hat{\beta}}^\top y| + |\rho^\top z| + |\sigma_{\hat{\beta}}^\top \mathrm{x}| \\ &\leq \gamma + \|\varrho_{\hat{\beta}}\|_1 \|y\|_\infty + \|\rho\|_1 \|z\|_\infty + \|\sigma_{\hat{\beta}}\|_1 \|\mathrm{x}\|_\infty. \end{aligned} \tag{2.35}$$

The last inequality holds by the Hölder inequality. For a precision parameter $\epsilon > 0$, *if* $\|\varrho_{\hat{\beta}}\|_1 < \epsilon$, $\|\rho\|_1 < \epsilon$, $\|\sigma_{\hat{\beta}}\|_1 < \epsilon$ *and* $\gamma < \epsilon$, *then the gap* G *becomes very close to zero, and hence the current solution is approximately optimal for* (2.5).

Next, we calculate the reduction in primal infeasibilities, dual infeasibility and complementarity. In addition, the convergence of Algorithm 1 is addressed in Theorem 2.4.

Theorem 2.4 *Consider the barrier problem* (2.5). *Assume that the sequence* $\big(\mathrm{x}^{(k)}, v^{(k)}, w^{(k)}, y^{(k)}, z^{(k)}\big)$ *generated by Algorithm 1 is bounded and Hessian matrix* $H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})$ *is positive definite at every* k*-th iteration. Let the quantities* $\varrho_{\hat{\beta}}^{(k)}$, $\rho^{(k)}$, $\sigma_{\hat{\beta}}^{(k)}$ *and* $\gamma^{(k)}$ *denote infeasibility and complementary values at the* k*-th iterate. Then, the following three results hold.*

(i) Suppose $\tau > 0$ *and* $M > 0$ *are such that for all* k,

$$\alpha^{(k)} \geq \tau, \quad \|\mathrm{x}^{(k)}\|_\infty \leq M, \quad \|v^{(k)}\|_\infty \leq M, \quad \|w^{(k)}\|_\infty \leq M, \quad \|y^{(k)}\|_\infty \leq M \quad \text{and} \quad \|z^{(k)}\|_\infty \leq M.$$

Then,

$$\|\varrho_{\hat{\beta}}^{(k)}\|_1 \leq (1-\tau)^k \|\varrho_{\hat{\beta}}^{(0)}\|_1, \qquad \|\rho^{(k)}\|_1 \leq (1-\tau)^k \|\rho^{(0)}\|_1, \qquad \|\sigma_{\hat{\beta}}^{(k)}\|_1 \leq (1-\tau)^k \|\sigma_{\hat{\beta}}^{(0)}\|_1 \tag{2.36}$$

and there exists an $\tilde{M} > 0$ *such that* $\gamma^{(k)} \leq (1-\bar{\tau})^k \tilde{M}$, *where* $\bar{\tau} = \tau(1-r)$, *(ii) For $k > \bar{K}$, infeasibilities and complementarity become less than a precision value $\epsilon > 0$, where*

$$\bar{K} = \max\left\{ \frac{\log\left(\frac{\epsilon}{\|\varrho_{\hat{\beta}}^{(0)}\|_1}\right)}{\log(1-\tau)}, \frac{\log\left(\frac{\epsilon}{\|\rho^{(0)}\|_1}\right)}{\log(1-\tau)}, \frac{\log\left(\frac{\epsilon}{\|\sigma_{\hat{\beta}}^{(0)}\|_1}\right)}{\log(1-\tau)}, \frac{\log\left(\frac{\epsilon}{\bar{M}}\right)}{\log(1-\bar{\tau})} \right\}.$$

(iii) If $\sum_{k=1}^{\infty} \alpha^{(k)} = \infty$, then

$$\lim_{k\to\infty} \|\varrho_{\hat{\beta}}^{(k)}\|_1 = 0, \quad \lim_{k\to\infty} \|\rho^{(k)}\|_1 = 0, \quad \lim_{k\to\infty} \|\sigma_{\hat{\beta}}^{(k)}\|_1 = 0 \quad \textit{and} \quad \lim_{k\to\infty} \gamma^{(k)} = 0.$$

Proof: (i) For the k-th iteration, inequality (2.27) and $\alpha^{(k)} \geq \tau$ lead to

$$\|\varrho_{\hat{\beta}}^{(k)}\|_1 \leq (1-\tau)\|\varrho_{\hat{\beta}}^{(k-1)}\|_1 \leq (1-\tau)^2\|\varrho_{\hat{\beta}}^{(k-2)}\|_1 \leq \cdots \leq (1-\tau)^k\|\varrho_{\hat{\beta}}^{(0)}\|_1.$$

Similarly, from (2.29) and (2.31), we obtain

$$\|\rho^{(k)}\|_1 \leq (1-\tau)^k\|\rho^{(0)}\|_1 \quad \text{and} \quad |\sigma_{\hat{\beta}}^{(k)}\|_1 \leq (1-\tau)^k\|\sigma_{\hat{\beta}}^{(0)}\|_1.$$

An estimate of the complementarity decrease at the k-th iteration is calculated by (2.33):

$$\begin{aligned}\gamma^{(k)} &\leq \gamma^{(k-1)}(1-\tau(1-r)) + M(1-\tau)^{k-1}(\|\varrho_{\hat{\beta}}^{(0)}\|_1 + \|\rho^{(0)}\|_1 + \|\sigma_{\hat{\beta}}^{(0)}\|_1 + \lambda_{\max}) \\ &= (1-\bar{\tau})\gamma^{(k-1)} + \bar{M}(1-\tau)^{k-1},\end{aligned}$$

where $\bar{\tau} = \tau(1-r)$ and $\bar{M} = M(\|\varrho_{\hat{\beta}}^{(0)}\|_1 + \|\rho^{(0)}\|_1 + \|\sigma_{\hat{\beta}}^{(0)}\|_1 + \lambda_{\max})$. Similarly,

$$\begin{aligned}\gamma^{(k)} &\leq (1-\bar{\tau})\left[(1-\bar{\tau})\gamma^{(k-2)} + \bar{M}(1-\tau)^{k-2}\right] + \bar{M}(1-\tau)^{k-1} \\ &= (1-\bar{\tau})^2\gamma^{(k-2)} + \bar{M}(1-\tau)^{k-1}\left[\frac{1-\bar{\tau}}{1-\tau} + 1\right].\end{aligned}$$

Proceeding in a similar way, we see that

$$\begin{aligned}\gamma^{(k)} &\leq (1-\bar{\tau})^2\left[(1-\bar{\tau})\gamma^{(k-3)} + \bar{M}(1-\tau)^{k-3}\right] + \bar{M}(1-\tau)^{k-1}\left[\frac{1-\bar{\tau}}{1-\tau}+1\right]\\ &= (1-\bar{\tau})^3\gamma^{(k-3)} + \bar{M}(1-\tau)^{k-1}\left[\left(\frac{1-\bar{\tau}}{1-\tau}\right)^2 + \frac{1-\bar{\tau}}{1-\tau}+1\right]\\ &\quad\dots\\ &\leq (1-\bar{\tau})^k\gamma^{(0)} + \bar{M}(1-\tau)^{k-1}\left[\left(\frac{1-\bar{\tau}}{1-\tau}\right)^{k-1} + \ldots + \frac{1-\bar{\tau}}{1-\tau}+1\right]\\ &= (1-\bar{\tau})^k\gamma^{(0)} + \bar{M}\frac{(1-\bar{\tau})^k-(1-\tau)^k}{\tau-\bar{\tau}}.\end{aligned}$$

Since $\bar{\tau} = \tau(1-r)$, we can write $\frac{(1-\bar{\tau})^k-(1-\tau)^k}{\tau-\bar{\tau}} \leq \frac{(1-\bar{\tau})^k}{r\tau}$. Hence, $\gamma^{(k)} \leq (1-\bar{\tau})^k\left(\gamma^{(0)} + \frac{\bar{M}}{r\tau}\right)$.

Denoting $\tilde{M} = \left(\gamma^{(0)} + \frac{\bar{M}}{r\tau}\right)$, we get $\gamma^{(k)} \leq (1-\bar{\tau})^k\tilde{M}$.

(ii) We note from (2.36) that $\|\varrho_{\hat{\beta}}^{(k)}\|_1 \leq (1-\tau)^k\|\varrho_{\hat{\beta}}^{(0)}\|_1 < \epsilon$ holds if

$$\begin{aligned} &(1-\tau)^k < \frac{\epsilon}{\|\varrho_{\hat{\beta}}^{(0)}\|_1}\\ \text{i.e., if}\quad &k\log(1-\tau) < \log\left(\frac{\epsilon}{\|\varrho_{\hat{\beta}}^{(0)}\|_1}\right)\\ \text{i.e., if}\quad &k > \frac{\log\left(\frac{\epsilon}{\|\varrho_{\hat{\beta}}^{(0)}\|_1}\right)}{\log(1-\tau)}\quad \text{since}\quad 0<\tau<1.\end{aligned}$$

Similarly, from other inequalities in (2.36), $\|\rho^{(k)}\| < \epsilon$, $\|\sigma_{\hat{\beta}}^{(k)}\|_1 < \epsilon$ and $\gamma^{(k)} < \epsilon$ hold if

$$k > \frac{\log\left(\frac{\epsilon}{\|\rho^{(0)}\|_1}\right)}{\log(1-\tau)},\quad k > \frac{\log\left(\frac{\epsilon}{\|\sigma_{\hat{\beta}}^{(0)}\|_1}\right)}{\log(1-\tau)}\quad \text{and}\quad k > \frac{\log\left(\frac{\epsilon}{\tilde{M}}\right)}{\log(1-\bar{\tau})},\ \text{respectively.}$$

Hence, the result follows.

(*iii*) We first prove that $\prod_{k=1}^{\infty}(1-\alpha^{(k)}) = 0$ if $\sum_{k=1}^{\infty}\alpha^{(k)} = \infty$.

For $k \in \mathbb{N}$, we denote

$$p_k = (1-\alpha^{(1)})(1-\alpha^{(2)})\cdots(1-\alpha^{(k)}) \text{ and } s_k = \alpha^{(1)} + \alpha^{(2)} + \cdots + \alpha^{(k)}.$$

Using the fact that $1 - x \le e^{-x}$ for every $x \in \mathbb{R}$, we have

$$p_k \le e^{-\alpha^{(1)}} e^{-\alpha^{(2)}} \cdots e^{-\alpha^{(k)}} = e^{-s_k}.$$

Hence, $\prod_{k=1}^{\infty}(1-\alpha^{(k)}) \le e^{-\sum_{k=1}^{\infty}\alpha^{(k)}} = 0$. Since $0 \le \alpha^{(k)} \le 1$ for all k (see (2.20)),

$$\prod_{k=1}^{\infty}(1-\alpha^{(k)}) = 0. \tag{2.37}$$

Now, we proceed with the main part of the theorem. By 2.4, we can write that

$$\|\varrho_{\hat{\beta}}^{(k)}\|_1 \le (1-\alpha^{(1)})(1-\alpha^{(2)})\cdots(1-\alpha^{(k)})\|\varrho_{\hat{\beta}}^{(0)}\|_1.$$

As k approaches to infinity, we obtain

$$0 \le \lim_{k\to\infty}\|\varrho_{\hat{\beta}}^{(k)}\|_1 \le \prod_{k=1}^{\infty}(1-\alpha^{(k)})\|\varrho_{\hat{\beta}}^{(0)}\|_1 = 0.$$

Thus, $\lim_{k\to\infty}\|\varrho_{\hat{\beta}}^{(k)}\|_1 = 0$. Similarly, the remaining limits can be shown.

□

Section 2.7 extends Algorithm 1 for nonconvex multiobjective optimization problems.

2.7 Algorithm for nonconvex multiobjective optimization problems

So far, we discussed that the nondominated points of the multiobjective optimization problem (2.1) are obtained by solving the single objective parametric problem (2.2). Algorithm 1 has the ability to handle those single objective parametric problems in which the Hessian matrix $H(\mathrm{x}, y, z)$ (calculated by (2.8)) is positive definite at every iteration. However, in case of nonconvex problems, $H(\mathrm{x}, y, z)$ and hence $N_{\hat{\beta}}(\mathrm{x}, y, z, v, w)$ may fail to be positive definite. In such a case, we lose the descent property given in Theorem 2.2. To maintain the descent property, the following substitution is taken in place of the Hessian matrix $H(\mathrm{x}, y, z)$

$$\tilde{H}(\mathrm{x}, y, z) = H(\mathrm{x}, y, z) + \lambda I, \tag{2.38}$$

where I is the identity matrix of order $(n+1) \times (n+1)$ and the choice of λ (see Section 2.7.1) is such that $\tilde{H}(\mathrm{x}, y, z)$ is positive definite.

The primal and dual directions are determined by solving the following system

$$\begin{bmatrix} -\tilde{H}(\mathrm{x}, y, z) & (A_{\hat{\beta}}(\mathrm{x}))^\top & B((\mathrm{x}))^\top \\ A_{\hat{\beta}}(\mathrm{x}) & VY^{-1} & 0 \\ B(\mathrm{x}) & 0 & WZ^{-1} \end{bmatrix} \begin{bmatrix} \Delta \mathrm{x} \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} \sigma_{\hat{\beta}} \\ \varrho_{\hat{\beta}} + VY^{-1}\gamma_1 \\ \rho + WZ^{-1}\gamma_2 \end{bmatrix}. \tag{2.39}$$

The solution of the system (2.39) provides the explicit formulas (2.40) for the primal-dual directions

$$\left.\begin{aligned} \Delta \mathrm{x} &= \tilde{N}_{\hat{\beta}}^{-1}\left(-\sigma_{\hat{\beta}} + \left(A_{\hat{\beta}}(\mathrm{x})\right)^\top (\gamma_1 + V^{-1}Y\varrho_{\hat{\beta}}) + (B(\mathrm{x}))^\top(\gamma_2 + W^{-1}Z\rho)\right), \\ \Delta v &= -\varrho_{\hat{\beta}} + A_{\hat{\beta}}(\mathrm{x})\Delta \mathrm{x}, \\ \Delta w &= -\rho + B(\mathrm{x})\Delta \mathrm{x}, \\ \Delta y &= \gamma_1 + V^{-1}Y\left(\varrho_{\hat{\beta}} - A_{\hat{\beta}}(\mathrm{x})\Delta \mathrm{x}\right), \\ \Delta z &= \gamma_2 + W^{-1}Z\left(\rho - B(\mathrm{x})\Delta \mathrm{x}\right), \end{aligned}\right\} \tag{2.40}$$

where $\tilde{N}_{\hat{\beta}} = \tilde{N}_{\hat{\beta}}(\Omega) = \tilde{H}(\mathrm{x}, y, z) + (A_{\hat{\beta}}(\mathrm{x}))^\top V^{-1} Y A_{\hat{\beta}}(\mathrm{x}) + (B(x))^\top W^{-1} Z B(\mathrm{x})$. Note that primal-dual directions (2.40) are determined in a similar way to Theorem 2.2.

Undoubtedly, the following question arises naturally: what is the effect of perturbation (2.38) on primal infeasibility, dual infeasibility and complementarity? Theorem 2.5 addresses the reduction in primal and dual infeasibility due to the perturbation. In this result, we assume that the search directions are computed with $\tilde{H}(\mathrm{x}, y, z)$ instead of $H(\mathrm{x}, y, z)$.

Theorem 2.5 *Consider the following perturbations in* x, v, w, y *and* z*:*

$$\bar{\mathrm{x}} = \mathrm{x} + \varepsilon \Delta \mathrm{x}, \quad \bar{v} = v + \varepsilon \Delta v, \quad \bar{w} = w + \varepsilon \Delta w, \quad \bar{y} = y + \varepsilon \Delta y, \quad \bar{z} = z + \varepsilon \Delta z,$$

where ε *is calculated according to Subsection 2.6.3. If* $\bar{\varrho}_{\hat{\beta}} = \varrho_{\hat{\beta}}(\bar{\mathrm{x}}, \bar{v}), \quad \bar{\rho} = \rho(\bar{\mathrm{x}}, \bar{w})$ *and* $\bar{\sigma}_{\hat{\beta}} = \sigma_{\hat{\beta}}(\bar{\mathrm{x}}, \bar{y}, \bar{z})$*, then*

$$\begin{aligned}
\bar{\varrho}_{\hat{\beta}} &= (1-\varepsilon)\varrho_{\hat{\beta}} + o(\varepsilon^2), \\
\bar{\rho} &= (1-\varepsilon)\rho + o(\varepsilon^2), \\
\bar{\sigma}_{\hat{\beta}} &= (1-\varepsilon)\sigma_{\hat{\beta}} - \varepsilon\lambda\Delta\mathrm{x} + o(\varepsilon^2), \\
\bar{v}^\top \bar{y} &= (1-\varepsilon(1-\delta_1))v^\top y + o(\varepsilon^2), \ \textit{and} \\
\bar{w}^\top \bar{z} &= (1-\varepsilon(1-\delta_2))w^\top y + o(\varepsilon^2),
\end{aligned}$$

where $\delta_1 = \frac{\mathcal{P}\mu}{v^\top y}$ *and* $\delta_2 = \frac{(\mathcal{J}+1)\mu}{w^\top z}$.

Proof: We have

$$\begin{aligned}
\bar{\varrho}_{\hat{\beta}} &= F(\bar{\mathrm{x}}) + \bar{v} - \hat{\beta}c^\top \bar{\mathrm{x}} \\
&= F(\mathrm{x} + \varepsilon\Delta\mathrm{x}) + v + \varepsilon\Delta v - \hat{\beta}c^\top(\mathrm{x} + \varepsilon\Delta\mathrm{x}) \\
&= F(\mathrm{x}) + \varepsilon\nabla F(\mathrm{x})\Delta\mathrm{x} + o(\varepsilon^2) + v + \varepsilon\Delta v - \hat{\beta}c^\top\mathrm{x} - \varepsilon\hat{\beta}c^\top\Delta\mathrm{x} \\
&= \left(F(\mathrm{x}) + v - \hat{\beta}c^\top\mathrm{x}\right) - \varepsilon\left(\left(\hat{\beta}c^\top - \nabla F(\mathrm{x})\right)\Delta\mathrm{x} - \Delta v\right) + o(\varepsilon^2)
\end{aligned}$$

$$
\begin{aligned}
&= \varrho_{\hat{\beta}} - \varepsilon\left(A_{\hat{\beta}}(\mathrm{x})\Delta\mathrm{x} - \Delta v\right) + o(\varepsilon^2) \\
&= (1-\varepsilon)\varrho_{\hat{\beta}} + o(\varepsilon^2),
\end{aligned}
\tag{2.41}
$$

where the last equality holds from the third block of equation (2.9). Similarly, $\bar{\rho}$ can be determined. Next, we have

$$
\begin{aligned}
\bar{\sigma}_{\hat{\beta}} &= c - (A_{\hat{\beta}}(\bar{\mathrm{x}}))^\top \bar{y} - (B(\bar{\mathrm{x}}))^\top \bar{z} \\
&= c - (A_{\hat{\beta}}(\mathrm{x}+\varepsilon\Delta\mathrm{x}))^\top (y+\varepsilon\Delta y) - (B(\mathrm{x}+\varepsilon\Delta\mathrm{x}))^\top (z+\varepsilon\Delta z) \\
&= c - (A_{\hat{\beta}}(\mathrm{x}))^\top y + \varepsilon\left(\sum_{i=1}^{k} y_i \nabla^2 f_i(\mathrm{x})\right)\Delta\mathrm{x} - \varepsilon(A_{\hat{\beta}}(\mathrm{x}))^\top \Delta y - (B(\mathrm{x}))^\top z \\
&\quad - \varepsilon\left(\sum_{j=1}^{m} z_j \nabla^2 h_j(\mathrm{x})\right)\Delta\mathrm{x} - \varepsilon(B(\mathrm{x}))^\top \Delta z + o(\varepsilon^2) \\
&= c + \varepsilon\left(\tilde{H}(\mathrm{x},y,z)\Delta\mathrm{x} - (A_{\hat{\beta}}(\mathrm{x}))^\top \Delta y - (B(\mathrm{x}))^\top \Delta z\right) - (A_{\hat{\beta}}(\mathrm{x}))^\top y - (B(\mathrm{x}))^\top z + o(\varepsilon^2) \\
&= (1-\varepsilon)\sigma_{\hat{\beta}} - \varepsilon\lambda\Delta\mathrm{x} + o(\varepsilon^2),
\end{aligned}
\tag{2.42}
$$

where the last equality holds from the first block of equation (2.9) in which $H(\mathrm{x},y,z)$ is replaced by $\tilde{H}(\mathrm{x},y,z)$. Again, we see that

$$
\begin{aligned}
\bar{v}^\top \bar{y} &= (v+\varepsilon\Delta v)^\top (y+\varepsilon\Delta y) \\
&= v^\top y + \varepsilon(v^\top \Delta y + \Delta v^\top y) + o(\varepsilon^2) \\
&= v^\top y + \varepsilon e^\top (V\Delta y + Y\Delta v) + o(\varepsilon^2) \\
&= v^\top y + \varepsilon e^\top (\mu e - VYe) + o(\varepsilon^2) \\
&= v^\top y + \varepsilon(s\mu - v^\top y) + o(\varepsilon^2) \\
&= v^\top y - \varepsilon(1-\delta_1)v^\top y + o(\varepsilon^2) \\
&= (1-\varepsilon(1-\delta_1))v^\top y + o(\varepsilon^2).
\end{aligned}
\tag{2.43}
$$

Similarly, we can compute the reduction in $\bar{w}^\top \bar{z}$ by

$$
\bar{w}^\top \bar{z} \;=\; (1-\varepsilon(1-\delta_2))w^\top z + o(\varepsilon^2). \tag{2.44}
$$

□

Note that Theorem 2.5 shows dual infeasibility $\sigma_{\hat{\beta}}$ may fail to reduce whenever $\lambda > 0$, even if an $\varepsilon \in (0,1)$ is taken along the search direction. This behaviour can affect the overall performance of the algorithm. However, when $o(\lambda) = o(\varepsilon)$, then $o(\varepsilon\lambda\Delta \mathrm{x}) = o(\varepsilon^2)$ and hence the convergence of $\bar{\sigma}_{\hat{\beta}}$ to 0 is guaranteed (by (2.42) and $\varepsilon \in (0,1)$). Therefore, a method is presented (see Algorithm 2) to determine λ so that $N_{\hat{\beta}}(\mathrm{x}, y, z, v, w)$ remains positive definite as well as $o(\lambda) = o(\varepsilon)$ (line number 6 of Algorithm 2). Also, note that the formulas of the primal-dual directions (2.40) involve the perturbation form of Hessian (2.38), where λ is taken as described in the subsection 2.7.1.

2.7.1 Determination of λ

We initially start with an $LDL^\top$ factorization of the following symmetric matrix of the reduced KKT system (2.13):

$$S = \begin{bmatrix} -H(\mathrm{x}, y, z) & (A_{\hat{\beta}}(\mathrm{x}))^\top & (B(\mathrm{x}))^\top \\ A_{\hat{\beta}}(\mathrm{x}) & VY^{-1} & 0 \\ B(\mathrm{x}) & 0 & WZ^{-1} \end{bmatrix}.$$

The matrix S is quasidefinite whenever $H(\mathrm{x}, y, z)$ is positive definite [87], and therefore there is a guarantee of LDL^T factorization. When $H(\mathrm{x}, y, z)$ is not positive definite then the LDL^T factorization will not be applicable. Therefore, we choose the value of λ so that the matrix $\tilde{H}(\mathrm{x}, y, z) = H(\mathrm{x}, y, z) + \lambda I$ becomes positive definite. Thereafter, we replace the matrix $H(\mathrm{x}, y, z)$ by $\tilde{H}(\mathrm{x}, y, z)$ in the matrix S and solve the system (2.39) with the help of a variant of LDL^T factorization, i.e., Cholesky factorization. The value of λ is computed by Algorithm 2.

Note 2.4 *In this note, we show mathematically that the value of λ that is computed by Algorithm 2 is such that the dual infeasiblity $\sigma_{\hat{\beta}}$ reduces at every iteration. From Note*

Algorithm 2 A method to determine a value of λ so that $\tilde{H}(\text{x}, y, z)$ becomes positive definite

Aim: To determine a value of λ to be used in (2.38) to make $\tilde{H}(\text{x}, y, z)$ positive definite matrix

1: **Inputs**
Provide $n+1$, the dimension of the vector x
Provide the matrix $H(\text{x}, y, z)$ (see (2.8)) of dimension $(n+1)\times(n+1)$
(In the below, I is the identity matrix of order $(n+1)\times(n+1)$)

2: **Initialization**
Choose any $\zeta \in (0,1)$

3: **Main Steps** (Steps 4 to 7) to find a λ such that $\tilde{H}(\text{x}, y, z)$ is positive definite

4: Calculate the minimum eigenvalue ($\lambda_{\min}$) of the symmetric matrix $H(\text{x}, y, z)$

5: Set $H(\text{x}, y, z) \leftarrow H(\text{x}, y, z) + |\lambda_{\min}| I$

6: Choose λ as the smallest value in $\left\{\zeta^2|\lambda_{\min}|, \zeta^3|\lambda_{\min}|, \zeta^4|\lambda_{\min}|, \ldots\right\}$ such that

$$H(\text{x}, y, z) + \lambda I \ \text{ is positive definite}$$

7: **return** the value λ $\left(\text{for which } \tilde{H}(\text{x}, y, z) = H(\text{x}, y, z) + \lambda I \ \text{ is positive definite}\right)$

2.2, we have

$$\begin{aligned} &\varepsilon = \bar{\alpha}_l = \zeta^l \alpha \ \text{for some } l \in \{2,3,4,\ldots\} \ \text{ and } \zeta \in (0,1) \\ or\ , \ &\frac{\varepsilon}{\zeta} = \zeta^{l-1}\alpha \to 0 \ \text{ as } \zeta \to 0. \end{aligned} \tag{2.45}$$

Similarly, from line 6 of Algorithm 2, we have

$$\begin{aligned} &\lambda = \zeta^{\ell}|\lambda_{\min}| \ \ \text{for some } \ell \in \{2,3,4,\ldots\} \ \text{ and } \zeta \in (0,1) \\ or\ , \ &\frac{\lambda}{\zeta} = \zeta^{\ell-1}|\lambda_{\min}| \to 0 \ \text{ as } \zeta \to 0. \end{aligned} \tag{2.46}$$

From (2.45) and (2.46), it is clear that $\varepsilon = o(\zeta)$ and $\lambda = o(\zeta)$. Now equation (2.42) can be written as follows:

$$\bar{\sigma}_{\hat{\beta}} = (1-\bar{\alpha}_l)\sigma_{\hat{\beta}} + o(\zeta^2).$$

As $\lim\limits_{l\to\infty} \bar{\alpha}_l = 0$, so there exists a natural number $\hat{K}$ such that for every $l \geq \hat{K}$, the following inequality holds:

$$\|\bar{\sigma}_{\hat{\beta}}\|_1 \leq (1-\bar{\alpha}_l)\|\sigma_{\hat{\beta}}\|_1. \tag{2.47}$$

The inequality (2.47) *confirms that the value of* λ *calculated by Algorithm 2 reduces the dual infeasibility* ($\sigma_{\hat{\beta}}$) *at each iteration.*

In the case of convex MOP, if the Hessian matrix $H(\mathrm{x}, y, z)$ is positive definite then the Newton's direction (2.15) is a descent direction for the merit function $\psi_{\eta,\mu}(\Omega)$ (see Theorem 2.2). However, if the problem is nonconvex and the Hessian matrix $H(\mathrm{x}, y, z)$ fails to be positive definite at any point of iteration, then we perturb the Hessian matrix and calculate the reduction in primal and dual infeasibilities (Theorem 2.5). Theorem 2.5 does not guarantee a reduction in dual infeasibility. The following Algorithm 3 has the ability to handle the Hessian matrix $H(\mathrm{x}, y, z)$ whether it is positive definite or not.

2.7.2 Well-definedness of Algorithm 3

Algorithm 3 not only solves convex MOPs but also nonconvex MOPs. Algorithm 1 works only when the Hessian matrix $H(\mathrm{x}, y, z)$ is positive definite at every iteration. Algorithm 3 handles both the cases, whether the Hessian matrix $H(\mathrm{x}, y, z)$ is positive definite or not. The well-definedness of Algorithm 3 depends on lines number 10, 14 and 17. The line 10 is well-defined as the Hessian matrix $H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})$ is positive definite at any k-th iteration, and hence the system (2.13) is solvable by Cholesky factorization. At the k-th iteration, when $H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})$ is not positive definite, Algorithm 3 computes the value of $\lambda_{\min}$ by Algorithm 2 so that the matrix $\tilde{H}(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})$ becomes positive definite. Now Cholesky factorization is applicable to solve the system (2.39). Hence, the line number 14 is well-defined. The reason of well-definedness of line number 17 is identical to line number 9 in Algorithm 1.

As an illustration, we consider the following problem

$$\left.\begin{array}{ll} \text{minimize}_{x\in\mathbb{R}^2} & (f_1(x), \quad f_2(x))^\top \\ \text{subject to} & (x_1+1)^2 + x_2^2 \leq 4, \ (x_1+2)^2 + (x_2+2)^2 \leq 4, \\ & x_1 \in [-5, 2], \ x_2 \in [-5, 3], \end{array}\right\} \tag{2.48}$$

Algorithm 3 IIPM for MOPs when $H(\mathrm{x}, y, z)$ is not necessarily positive definite

Aim: To generate a discrete approximation of the nondominated set $\mathcal{D}$ of the MOP (2.1)
1: **Inputs**
Provide $f_1, f_2, \ldots, f_{\mathcal{P}}$ which are twice continuously differentiable objective functions of (2.1)
Provide $h_1, h_2, \ldots, h_{\mathcal{J}}$ which are twice continuously differentiable constraints functions of (2.1)
Provide N, the number of different $\hat{\beta}$'s corresponding to which the subproblem (2.5) is to be solved
2: **Initialization**
Set nondominated set $\mathcal{D} \leftarrow \emptyset$
To solve (2.5) for a given $\hat{\beta}$, provide an initial point $\mathrm{x}^{(0)}$
Choose any positive ξ_1 and ξ_2 for (2.23)
Initialize $v^{(0)}$, $w^{(0)}$ by (2.23) and $y^{(0)}$, $z^{(0)}$ by any positive values
Give the values of parameters r, δ and $0 < \kappa < 1$ according to (2.19), (2.20) and (2.21), respectively
Give a value of the precision parameter $\epsilon > 0$
Set $k \leftarrow 0$
Set $\mu^{(k)} \leftarrow r\frac{(v^{(k)})^\top y^{(k)} + (w^{(k)})^\top z^{(k)}}{\mathcal{J}+\mathcal{P}+1}$
3: **Main Steps to solve** (2.5) **for** N **different** $\hat{\beta}$**'s**
4: **for** $i = 1 : 1 : N$ **do**
(Lines 5 to 42 solves (2.5) for a $\hat{\beta}$ and find $\mathrm{x}^{(k)}$ as a solution to (2.5). Line 44 keeps stacking up the nondominated points of the MOP (2.1) in the set $\mathcal{D}$)
5: Choose a direction $\hat{\beta}$ by using (2.3)
6: Set $\nu(\Omega^{(k)}) \leftarrow \max\{\|\sigma_{\hat{\beta}}^{(k)}\|_1, \|\varrho_{\hat{\beta}}^{(k)}\|_1, \|\rho^{(k)}\|_1, \|V^{(k)}Y^{(k)}e\|_1, \|W^{(k)}Z^{(k)}e\|_1\}$ (by (2.24))
7: **while** $\nu(\Omega^{(k)}) \geq \epsilon$ **do**
8: Calculate the Hessian matrix $H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})$ by using (2.8)
9: **if** $H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})$ is positive definite **then**
10: Solve the system (2.13) by Cholesky factorization
11: **else**
12: Compute the value of perturbation parameter $\lambda_{\min}$ for $H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)})$ by Algorithm 2
13: Set $\tilde{H}(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)}) \leftarrow H(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)}) + \lambda_{\min} I$
14: Solve the system (2.39) by Cholesky factorization
15: **end if**
16: Find primal-dual directions $\Delta\Omega^{(k)}$
17: Choose step length α by formula (2.20)
18: Set $p^{(k+1)} \leftarrow p^{(k)} + \alpha\Delta p^{(k)},\ y^{(k+1)} \leftarrow y^{(k)} + \alpha\Delta y^{(k)},\ z^{(k+1)} \leftarrow z^{(k)} + \alpha\Delta z^{(k)}$
19: Calculate $\Gamma_{\hat{\beta}}(\Omega^{(k)})$ by the expression (2.17)
20: **if** $\varrho_{\hat{\beta}}(\mathrm{x}^{(k)}, v^{(k)}) = 0,\ \rho(\mathrm{x}^{(k)}, w^{(k)}) = 0$ and $\psi_{\eta,\mu}(\Omega^{(k+1)}) \leq \psi_{\eta,\mu}(\Omega^{(k)})$ **then**
21: Update $\mu^{(k+1)} \leftarrow r\frac{(v^{(k+1)})^\top y^{(k+1)} + (w^{(k+1)})^\top z^{(k+1)}}{\mathcal{J}+\mathcal{P}+1}$
22: Update $\nu(\Omega^{(k+1)})$ according to (2.24)
23: Set $k \leftarrow k + 1$
24: **else**
25: **if** $\Gamma_{\hat{\beta}}(\Omega^{(k)}) < 0$ **then**
26: Set $\eta = 0$ and backtrack $\alpha^{(k)} \in [0, \alpha]$ until the condition (2.21) holds
27: Update $\Omega^{(k+1)} \leftarrow \Omega^{(k)} + \alpha^{(k)}\Delta\Omega^{(k)}$
28: Update $\mu^{(k+1)} \leftarrow r\frac{(v^{(k+1)})^\top y^{(k+1)} + (w^{(k+1)})^\top z^{(k+1)}}{\mathcal{J}+\mathcal{P}+1}$
29: Update $\nu(\Omega^{(k+1)})$ according to (2.24)
30: Set $k \leftarrow k + 1$
31: **else**
32: Calculate $\eta_{\min}$ by equation (2.18)
33: Set $\eta = 10\eta_{\min}$ and backtrack $\alpha^{(k)} \in [0, \alpha]$ until the condition (2.21) holds
34: Update $\Omega^{(k+1)} \leftarrow \Omega^{(k)} + \alpha^{(k)}\Delta\Omega^{(k)}$
35: **if** $\psi_{\hat{\beta}}(\Omega^{(k+1)}) \leq \psi_{\hat{\beta}}(\Omega^{(k)})$ **then**
36: Update $\mu^{(k+1)} \leftarrow r\frac{(v^{(k+1)})^\top y^{(k+1)} + (w^{(k+1)})^\top z^{(k+1)}}{\mathcal{J}+\mathcal{P}+1}$
37: Update $\nu(\Omega^{(k+1)})$ according to (2.24)
38: Set $k \leftarrow k + 1$
39: **end if**
40: **end if**
41: **end if**
42: **end while**
43: Calculate $F(\mathrm{x}^{(k)}) = \hat{\beta}c^T\mathrm{x}^{(k)} - v^{(k)}$
44: Update nondominated set $\mathcal{D} \leftarrow \mathcal{D} \bigcup \{F(\mathrm{x}^{(k)})\}$
45: **end for**
46: **return** the set $\mathcal{D}$ (a discrete approximation of the nondominated set)

where $f_1(x) = (x_1 + 3)^2 + (x_2 - 2)^2$ and $f_2(x) = x_1^2 + (x_2 + 3)^2$. The location of the nondominated set of this problem is not known a priori. The cone formulation of the problem (2.48) is as follows:

$$\left.\begin{aligned}
&\text{minimize} \quad t \\
&\text{subject to} \quad t\cos\theta \geq f_1(x),\ t\sin\theta \geq f_2(x), \\
&\qquad (x_1+1)^2 + x_2^2 \leq 4,\ (x_1+2)^2 + (x_2+2)^2 \leq 4, \\
&\qquad x_1 \in [-5,2],\ x_2 \in [-5,3],
\end{aligned}\right\} \tag{2.49}$$

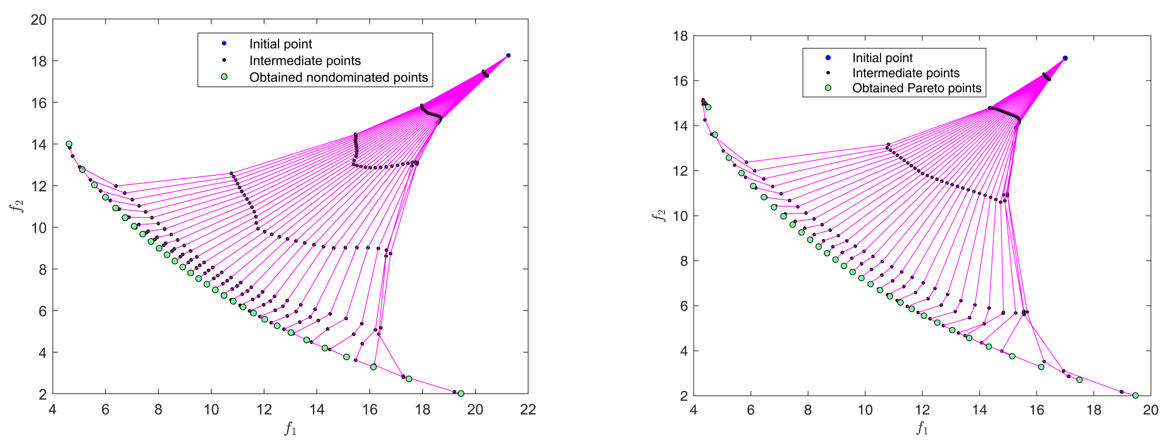
where $\theta \in [0, \pi/2]$. The convergence of Algorithm 3 towards the nondominated points of the problem (2.48) is shown in Figure 2.1(a). To obtain the nondominated points of the problem (2.48), Algorithm 3 solves the formulated problem (2.49) for different values of $\theta \in [0, \pi/2]$. It starts with an initial point $(x_1^{(0)}, x_2^{(0)}, t^{(0)}) = (1.5, 1, 15)$ and a value $\theta \in [0, \pi/2]$, and then gradually moves towards the efficient point. After finding one efficient point, Algorithm 3 changes the value of $\theta \in [0, \pi/2]$ and then converges to another efficient point. We have shown all the iterations in the objective space (see Figure 2.1(a)). The blue point $(21.25, 18.25)$ in Figure 2.1(a) is the starting point in the objective space and green points are the generated nondominated points corresponding to different values of $\theta \in [0, \pi/2]$. Note that the initial point remains unchanged throughout the entire process (see Figure 2.1(a)).

One can also check the performance of the proposed algorithm by taking different strategies of starting points. Here we discuss three strategies of choosing starting points and compare them with respect to the number of iterations and time. For every strategy, we take the problem (2.48) as a test example and solve its cone formulation (2.49) for 30 values of $\theta \in [0, \pi/2]$ by Algorithm 3 with accuracy value $\epsilon = 10^{-4}$.

Strategy 2.1 *In the first strategy, we solve the problem (2.48) by taking different initial points and check the performance of Algorithm 3 based on the total number of iterations and time. We take four different initial points and calculate the total number of iterations and time corresponding to every initial point (see Table 2.1). For every* *initial point, convergence of the iterations are depicted in Figures 2.1 and 2.2. The data depicted in Table 2.1 shows that Algorithm 3 performs better in terms of iterations and time if the image of initial points are taken closer to the nondominated frontier (see Figures 2.1 and 2.2), i.e., this strategy will speed up the generation of nondominated points if the chosen initial point is closer to the true frontier.*

Table 2.1: Strategy 2.1

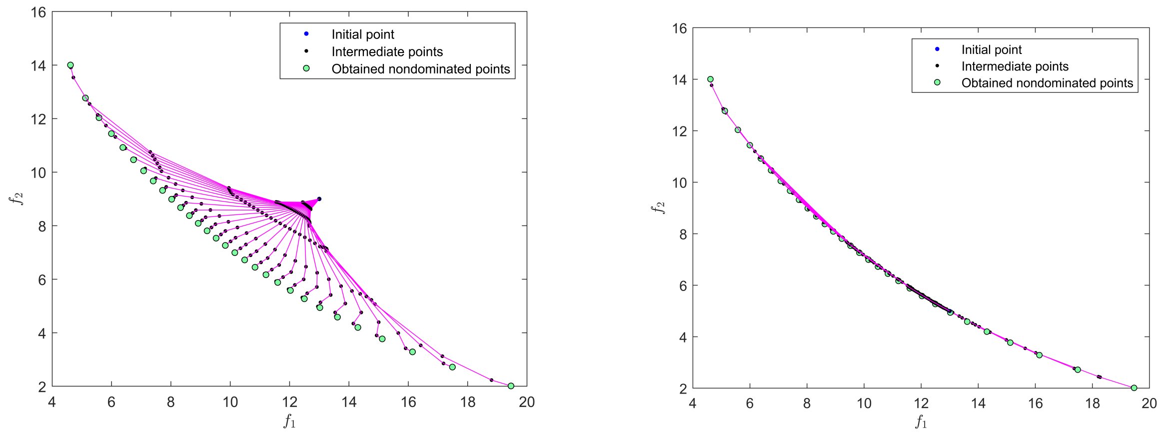
Initial point $(\mathrm{x}^{(0)})$	$\left(f_1(\mathrm{x}^{(0)}), f_2(\mathrm{x}^{(0)})\right)$	Total iterations	Total time taken (in seconds)
$(1.5, 1, 15)$	$(21.25, 18.25)$	333	488.83
$(1, 1, 15)$	$(17, 17)$	327	441.55
$(0, 0, 15)$	$(13, 9)$	300	401.62
$(-1, -1, 15)$	$(13, 5)$	254	356.65



(a) (b)

Figure 2.1: Obtained nondominated points by taking initial points $(1.5, 1, 15)$ and $(1, 1, 15)$, respectively

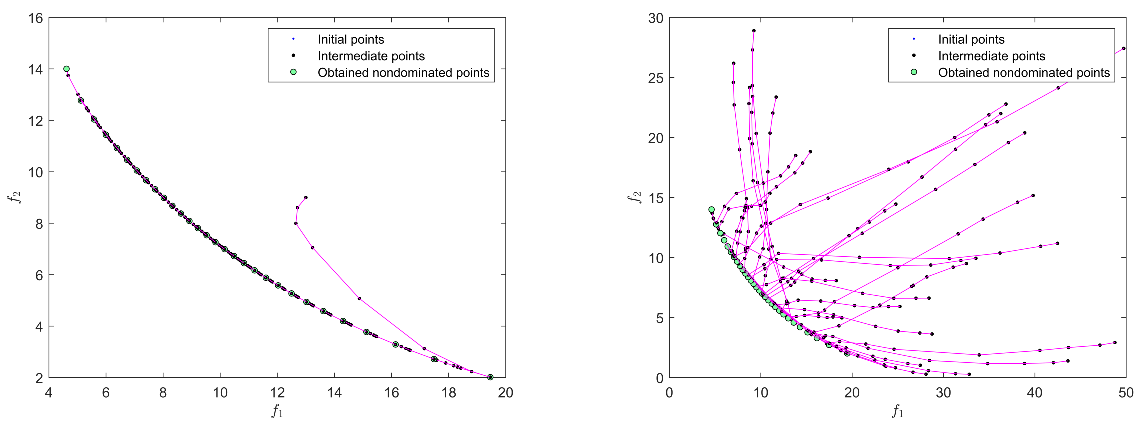
Strategy 2.2 In this strategy, we chose the initial point as the last generated nondominated point. Under this strategy, we measure the performance of Algorithm 3 to solve the problem (2.49) by taking $(0, 0, 15)$ as an initial point for first $\hat{\beta}$. Thereafter, for the next $\hat{\beta}$, the initial point is taken as the generated nondominated point by taking $(0, 0, 15)$ as the initial point. For the next $\hat{\beta}$, the initial point is taken as the last nondominated point (see Figure 2.3(a)). It is found that in this strategy Algorithm 3 took



(a) (b)

Figure 2.2: Obtained nondominated points by taking initial points $(0, 0, 15)$ and $(-1, -1, 15)$, respectively

251 iterations and 352.36 seconds to generate 30 nondominated points of the problem (2.48).



(a) (b)

Figure 2.3: Obtained nondominated points by taking initial points according to Strategy 2.2 and Strategy 2.3, respectively

Strategy 2.3 *In this strategy, we employ random initial point. Under this strategy, we solve the problem* (2.49) *by taking random initial points from the set* $[-5, 5] \times [-5, 5] \times [10, 20]$ *for every* $\hat{\beta}$. *To generate 30 nondominated points (see Figure 2.3(b)), Algorithm*

3 took 307 iterations and 529.90 seconds.

From these three strategies, we experienced that Strategy 2.2 is expected to take less iterations and time than the other two strategies to generate the nondominated set.

Next, we provide a list of λ-values taken up by Algorithm 3 (in line 12) for a nonconvex problem. We consider a typical tri-objective optimization problem (comet problem) which is nonconvex in nature and known to have difficulty in generating its nondominated set by the state-of-the-art solvers. The mathematical representation of the comet problem is as follows:

$$\left.\begin{array}{ll} \text{minimize}_{x\in\mathbb{R}^3} & (f_1(x), \quad f_2(x), \quad f_3(x))^\top \\ \text{subject to} & 1 \leq x_1 \leq 3.5, \ -2 \leq x_2 \leq 2, \ 0 \leq x_3 \leq 1, \end{array}\right\} \tag{2.50}$$

where $f_1(x) = (1+x_3)(x_1^3x_2^2-10x_1-4x_2)$, $f_2(x) = (1+x_3)(x_1^3x_2^2-10x_1+4x_2)$ and $f_3(x) = 3(1+x_3)x_2^2$.

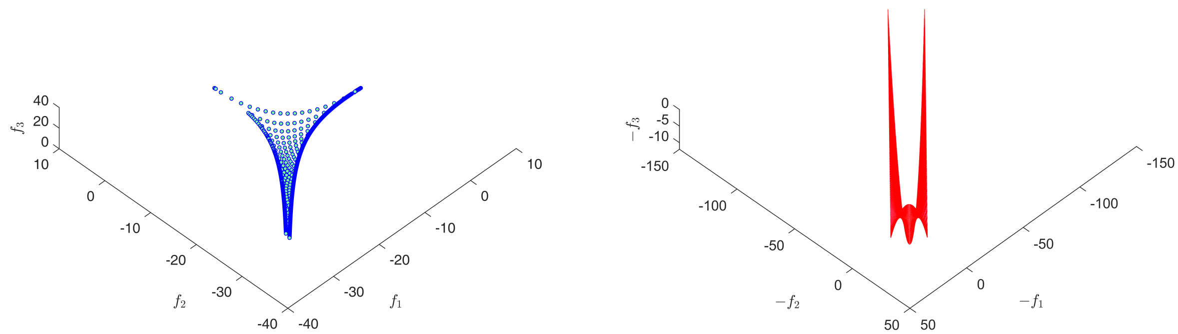
The nondominated surface of the problem (2.50) looks like a comet. One end of the nondominated set of the comet problem is a wide-spreaded region which continuously reduces to a thinner region on the other end. Capturing the nondominated solutions simultaneously from both of the wide-spreaded and thin areas is a challenging task (see Figure 2.4(b)). The generated nondominated set of the comet problem by Algorithm 3 is depicted in Figure 2.4(a). As the comet problem is a nonconvex problem, the Hessian matrix is not positive definite across the iterations of Algorithm 3. However, Algorithm 3 takes a positive definite approximation of the Hessian by adding λI with it (see the line 12). In Table 2.2, we provide the value of λ that is computed by Algorithm 2 with the precision parameter $\epsilon = 10^{-5}$ for an exemplary direction $\hat{\beta} = (0.7071, 0.5721, 0.4156)$, and the value of $\nu(\Omega)$ (see (2.24)). From Table 2.2 we see that as the number of iterations progresses, the value of $\nu(\Omega)$ reduces and gets less than 10^{-5}.

Table 2.2: Description of the λ values of the comet problem for $\hat{\beta} = (0.7071, 0.5721, 0.4156)$

Number of iterations (k)	Definiteness of $H\left(\mathrm{x}^{(k)}, y^{(k)}, z^{(k)}\right)$	Value of λ (by Algorithm 2)	$\nu(\Omega^{(k+1)})$
0	No	0.7578	5997.96
1	No	0.9389	5997.33
2	No	0.1497	62.04
3	No	0.0641	0.9666
4	No	0.2725	0.9324
5	No	0.3585	0.8797
6	No	0.3434	0.0022
7	No	0.0224	0.000006

2.8 Computational tests

In this section, a MATLAB implementation of Algorithm 3 is tested on various test problems (listed in Tables 2.3 and 2.4) from the literature. It is shown in Theorem 2.4 that the primal infeasibilities, dual infeasibility and complementarity reduce at every iteration. Therefore, combined reduction of the primal infeasibilities, dual infeasibility and complementarity is calculated by the merit function $\nu(\Omega)$, and hence implementation uses the stopping criterion $\nu(\Omega) < \epsilon$ for some prespecified precision value $\epsilon > 0$. If the problem (2.4) is such that the Hessian matrix $H(\mathrm{x}, y, z)$, is positive definite then Theorem 2.4 ensures that a reduction in primal infeasibilities $\|\varrho_{\hat{\beta}}\|_1$ and $\|\rho\|_1$, dual infeasibility $\|\sigma_{\hat{\beta}}\|_1$ and complementarity γ can be made at each iteration point. However, in case of nonconvex problem, the algorithm perturbs the Hessian (see (2.38)) and based on this perturbed Hessian a reduction in primal infeasibilities $\|\varrho_{\hat{\beta}}\|_1$ and $\|\rho\|_1$ and complementarity γ can be seen in Theorem 2.5. But Theorem 2.5 does not guarantee a reduction in dual infeasibility. To overcome this deficiency, we give a method to compute the value of λ (see Algorithm 2). In Note 2.4, we show that the the value of λ, computed by Algorithm 2 produces the reduction in dual infeasibility. Empirical outcomes in Section 2.8 are obtained either by taking $\lambda = 0$ or the method given in



(a) (b)

Figure 2.4: Obtained nondominated points (left) of the comet problem by Algorithm 3 and the feasible region in the objective space (right)

Subsection 2.7.1.

For each test problem, we use $\xi_1 = 1$ and $\xi_2 = 1$ to initiate the values of slack variables v and w (see (2.23)). Also, we use $\epsilon = 10^{-6}$ as the precision value. The results in this section are obtained by taking $r = 0.1$, $\kappa = 0.5$ and $\delta = 0.95$.

Generated nondominated solutions of the constrained test problems by Algorithm 3 are shown in Figures 2.5 and 2.6. For unconstrained test problems, the generated nondominated set along with reduction in infeasibiliy for a fixed $\hat{\beta}$, are shown in Figures 2.7–2.14.

We solve some widely used multiobjective constrained test problems (see Table 2.3) and unconstrained test problems (see Table 2.4) to test the performance of Algorithm 3. The test problem Kita in Table 2.3 is a maximization problem and remaining are the minimization problems. The nondominated sets of the constrained test problems in Table 2.3 are priorly unknown, i.e., the closed form of nondominated sets are unknown and the nondominated sets of unconstrained test problems are priorly known to us. The generated nondominated points of these problems together with feasible objective space are shown in Figures 2.5 and 2.6.

Table 2.3: Constrained test problems used in this study

Problem	n	$\mathcal{P}$	Nondominated set type	Number of $\hat{\beta}$'s	Average iteration	Nondominated set
BNH [94]	2	2	convex	200	8.6	Not known a priori
TNK [95]	2	2	nonconvex and disconnected	100	9.8	Not known a priori
SRN [96]	2	2	convex	75	8.1	Not known a priori
Kita [97]	2	3	nonconvex	150	10.2	Not known a priori

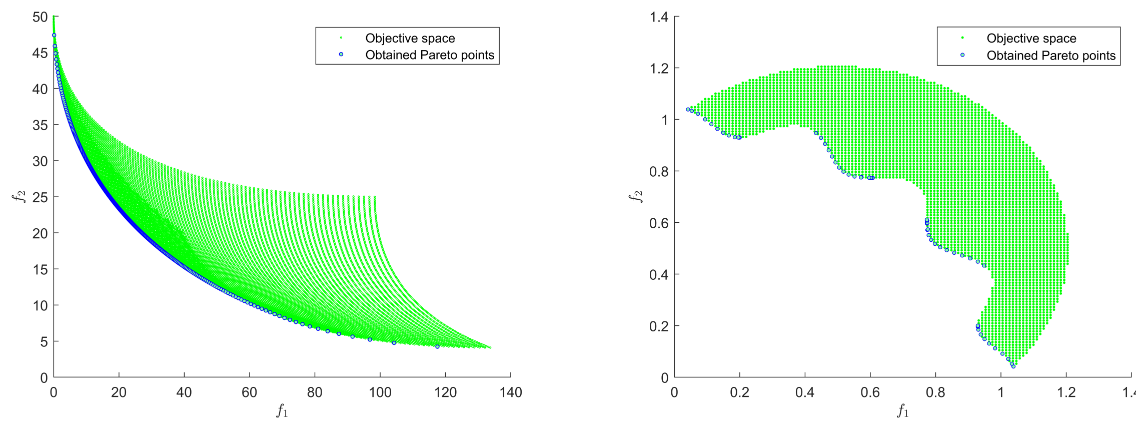


Figure 2.5: Obtained nondominated points of BNH and TNK problems by Algorithm 3

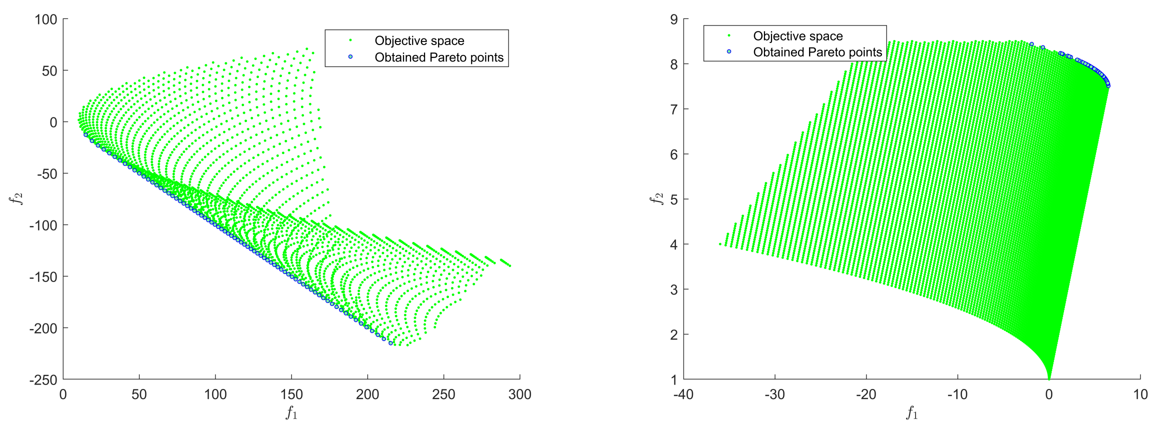


Figure 2.6: Obtained nondominated points of SRN and Kita problems by Algorithm 3

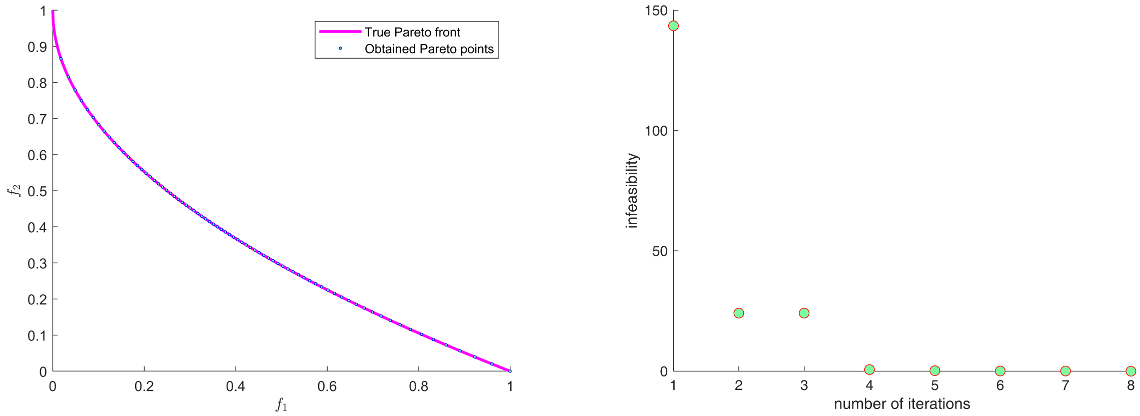


Figure 2.7: Obtained nondominated points of ZDT1 by Algorithm 3, and the reduction of infeasiblity for $\hat{\beta} = (0.978, 0.207)^\top$

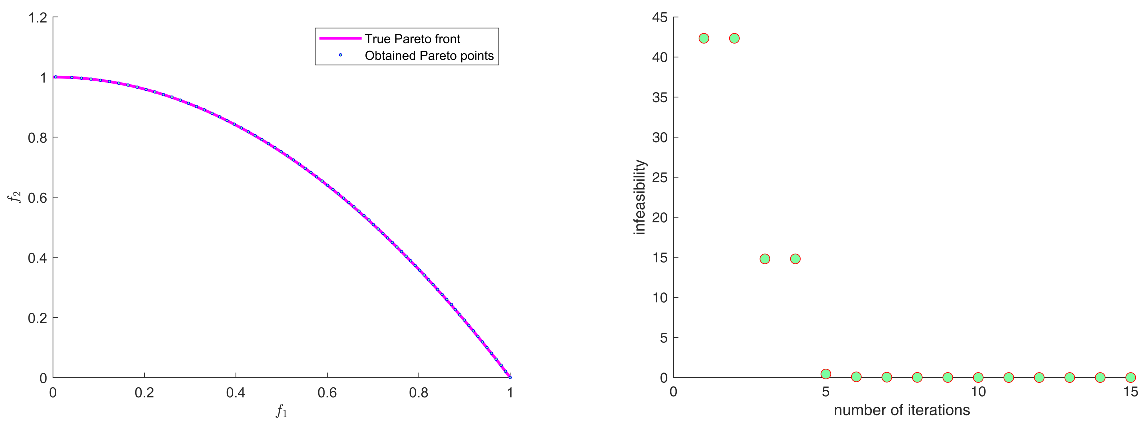


Figure 2.8: Obtained nondominated points of ZDT2 by Algorithm 3, and the reduction of infeasiblity for $\hat{\beta} = (0.923, 0.382)^\top$

2.8.1 Performance metrics and relative efficiency

To check the performance of Algorithm 3, two performance metrics, namely Inverted Generational Distance (IGD) and Hyper-Volume (HV) [98] are used to assess the quality of the generated solution set in terms of optimality and diversity. For the IGD value, the lesser, the better. For the HV value, the bigger, the better.

Table 2.4 reports the considered test problems and average number of iterations (per one $\hat{\beta}$) taken by Algorithm 3 to generate a nondominated point.

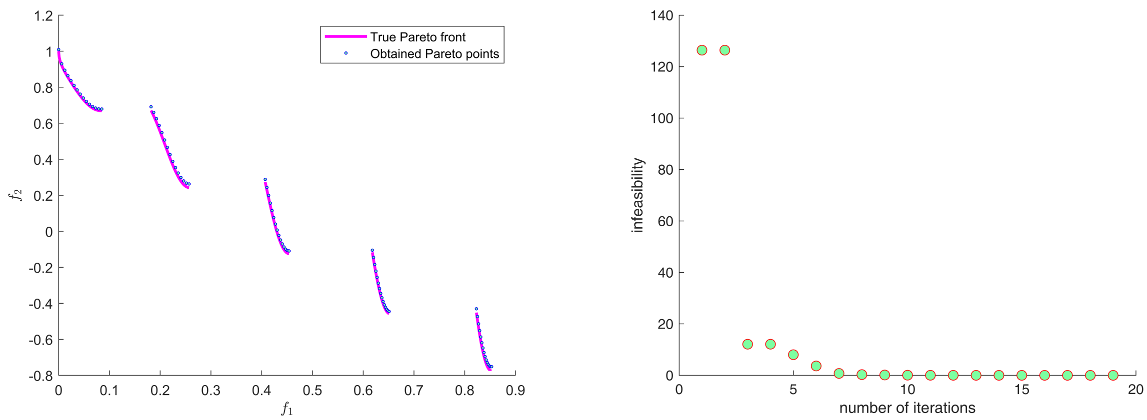


Figure 2.9: Obtained nondominated points of ZDT3 by Algorithm 3, and the reduction of infeasiblity for $\hat{\beta} = (0.707, 0.707)^\top$

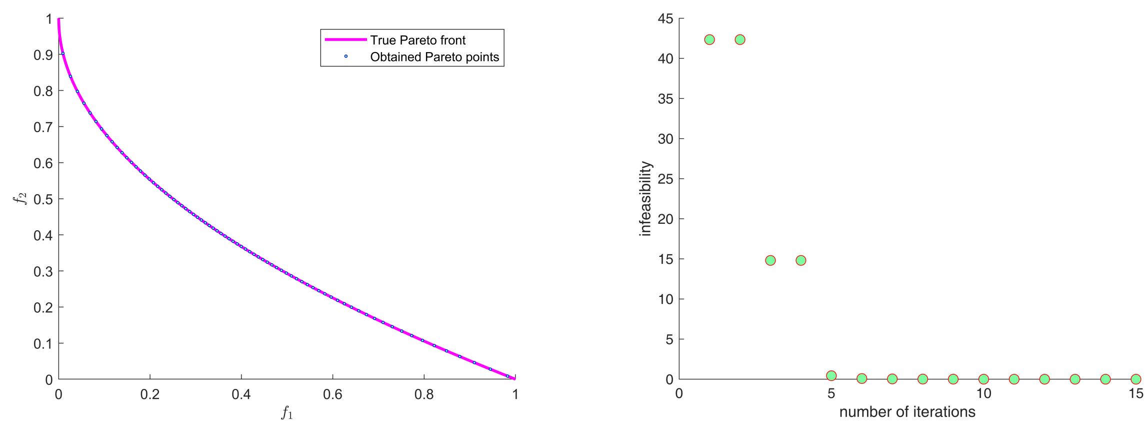


Figure 2.10: Obtained nondominated points of ZDT4 by Algorithm 3, and the reduction of infeasiblity for $\hat{\beta} = (0.341, 0.940)^\top$

In Table 2.4, the parameter n is the number of decision variables and s is the number of objectives. In order to calculate the performance metrices for Algorithm 3, we use the same initial point for different $\hat{\beta}$'s to solve all the parametrized problems.

For ZDT and DTLZ test suits (listed in Table 2.4), the median of the IGD values and HV values of the generated solution sets by Algorithm 3 and other existing efficient solvers are reported in Table 2.5 and Table 2.6, respectively. From Table 2.5, we see

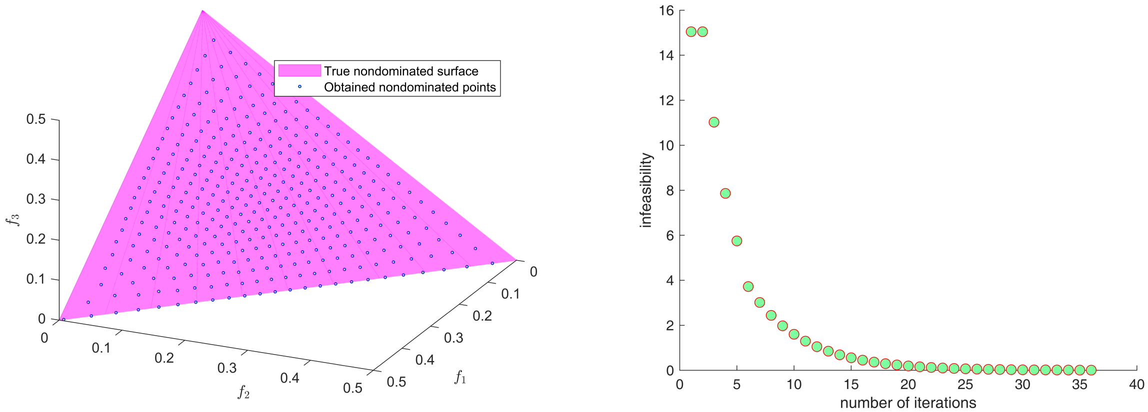


Figure 2.11: Obtained nondominated set of DTLZ1 by Algorithm 3, and the reduction of infeasiblity for $\hat{\beta} = (0.877, 0.421, 0.229)^\top$

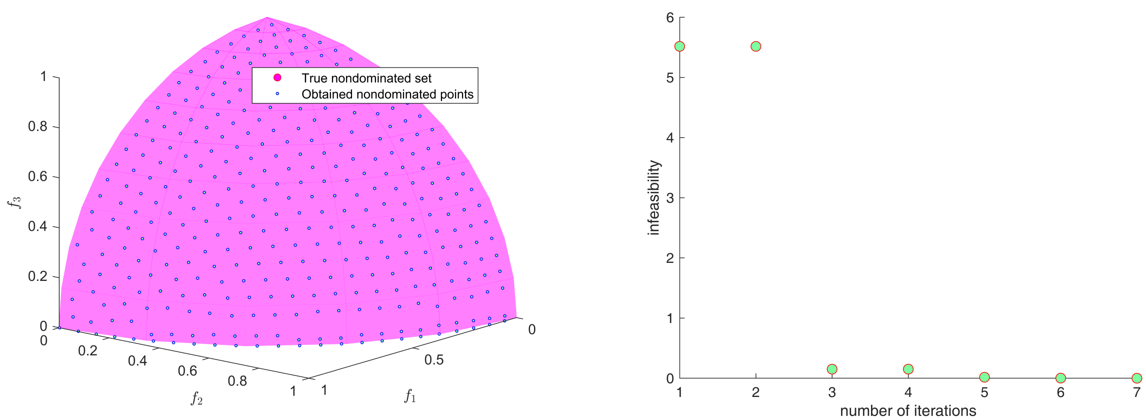


Figure 2.12: Obtained nondominated set of DTLZ2 by Algorithm 3, and the reduction of infeasiblity for $\hat{\beta} = (1, 0)^\top$

that the IIPM (CM) has least values for all the test problems. From Table 2.6, we observe that the IIPM (CM) has greatest values for all the test problems. Hence, the proposed method outperforms the existing efficient methods.

Based on the result reported in Table 2.5, next, we evaluate the relative efficiency of the proposed method. We compare MOEA/D(WS), MOEA/D(TE), MOEA/D(PBI), NSGA-II and $pa\lambda$-MOEA/D methods with the IIPM (CM) method as follows: for each i^{th} test problem, we compute the median of IGD values by the j^{th} solver and denote it

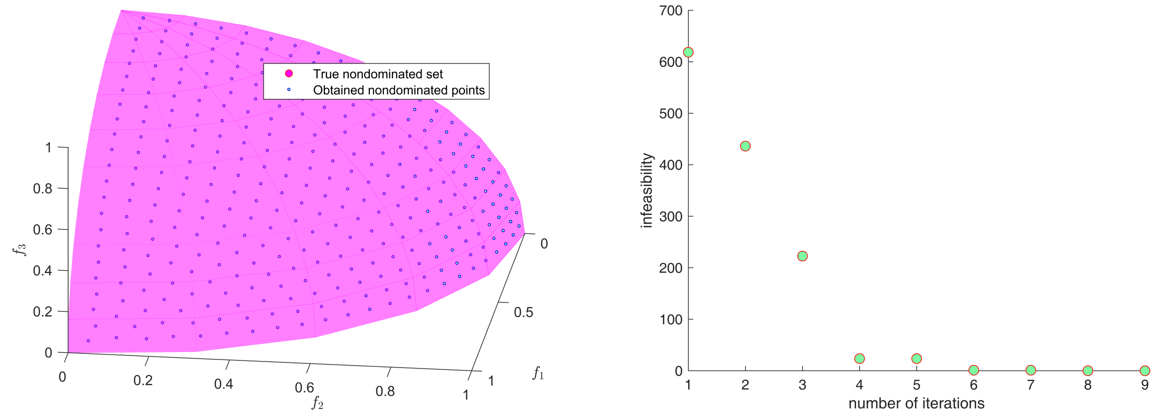


Figure 2.13: Obtained nondominated set of DTLZ3 by Algorithm 3, and the reduction of infeasiblity for $\hat{\beta} = (1, 0, 0)^\top$

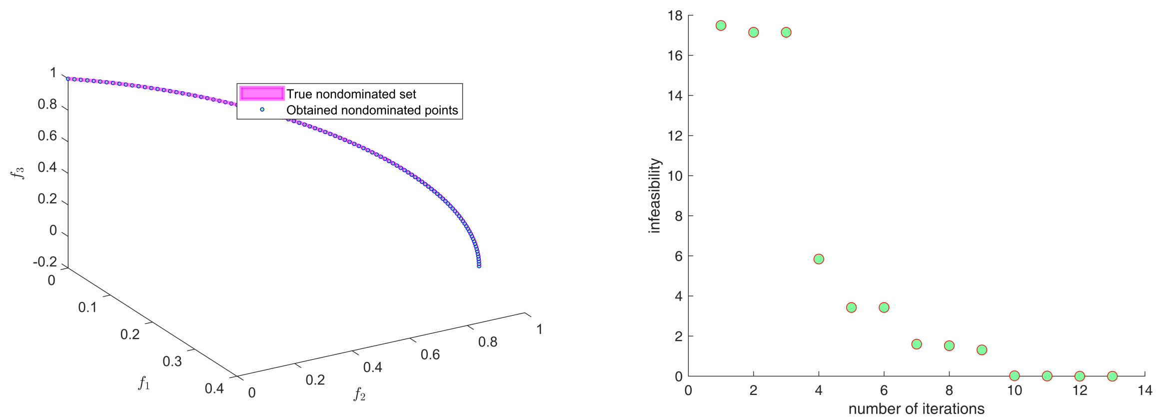


Figure 2.14: Obtained nondominated set of DTLZ5 by Algorithm 3, and the reduction of infeasiblity for $\hat{\beta} = (0.809, 0.558, 0.181)^\top$.

by MIGD(i, j). Then, we calculate the ratio

$$r(i, j) = \frac{\text{MIGD}(i, j)}{\text{MIGD}(i, \text{IIPM(CM)})}.$$

Geometric mean of these ratios for the j^{th} solver over all the test problems is defined

Table 2.4: Data for the test problems that are used to check performance of Algorithm 3

Problem	n	$\mathcal{P}$	Average iteration	No. of $\hat{\beta}$	Nondominated set type	Nondominated set	Source
ZDT1	10	2	8.1	75	Convex	A priori Known	[10]
ZDT2	10	2	13.4	75	Nonconvex	A priori Known	[10]
ZDT3	10	2	16.4	75	Nonconvex and disconnected	A priori Known	[10]
ZDT4	10	2	14.4	75	Convex	A priori Known	[10]
DTLZ1	5	3	33.2	359	Convex	A priori Known	[99]
DTLZ2	5	3	6.9	285	Nonconvex	A priori Known	[99]
DTLZ3	5	3	9.9	285	Nonconvex	A priori Known	[99]
DTLZ5	5	3	12.7	100	Nonconvex	A priori Known	[99]

Table 2.5: Median of IGD values for the ZDT and DTLZ benchmark suite obtained by different algorithms

Problem	MOEA/D (WS) [100]	MOEA/D (TE) [100]	MOEA/D (PBI) [100]	NSGA-II [100]	$pa\lambda$-MOEA/D [100]	IIPM (CM)
ZDT1	$5.42E-4$	$6.84E-4$	$1.14E-3$	$7.94E-4$	$5.80E-4$	$1.14E-4$
ZDT2	$1.30E-2$	$5.84E-4$	$7.03E-4$	$8.16E-4$	$6.02E-4$	$2.95E-6$
ZDT3	$4.93E-3$	$2.01E-3$	$2.06E-3$	$1.20E-3$	$1.97E-3$	$2.04E-4$
ZDT4	$7.02E-3$	$6.54E-4$	$7.86E-4$	$8.14E-4$	$5.94E-4$	$1.98E-7$
DTLZ1	$4.03E-3$	$6.93E-4$	$4.21E-4$	$7.91E-4$	$4.52E-4$	$4.74E-6$
DTLZ2	$5.35E-3$	$7.42E-4$	$6.15E-4$	$7.58E-4$	$5.78E-4$	$1.43E-4$
DTLZ3	$1.42E-2$	$1.24E-3$	$1.90E-3$	$3.59E-3$	$1.18E-3$	$1.27E-4$
DTLZ5	$1.38E-3$	$5.22E-5$	$1.11E-4$	$1.89E-5$	$2.79E-5$	$3.78E-5$

Table 2.6: Median of HV values for the ZDT and DTLZ benchmark suite obtained by different algorithms

Problem	MOEA/D (WS) [100]	MOEA/D (TE) [100]	MOEA/D (PBI) [100]	NSGA-II [100]	$pa\lambda$-MOEA/D [100]	IIPM (CM)
ZDT1	0.6521	0.6392	0.6057	0.6381	0.6412	0.6585
ZDT2	0.0000	0.3097	0.2957	0.3060	0.3107	0.3218
ZDT3	0.4863	0.4807	0.4642	0.5066	0.4873	0.5101
ZDT4	0.3534	0.6360	0.6244	0.6359	0.6391	0.6593
DTLZ1	0.2259	0.7434	0.7835	0.7262	0.7814	0.9119
DTLZ2	0.0000	0.3777	0.3812	0.3766	0.4074	0.4562
DTLZ3	0.0000	0.3605	0.2633	0.1901	0.3817	0.4002
DTLZ5	0.0000	0.0894	0.0779	0.0930	0.0916	0.2010

by

$$r(j) = \left(\prod_{i \in P} r(i,j) \right)^{\frac{1}{|P|}}, \tag{2.51}$$

which is referred to as *relative efficiency*, where P denotes the set of the test problems and $|P|$ is the cardinality of P.

The values of $r(\text{MOEA/D(WS)})$, $r(\text{MOEA/D(TE)})$, $r(\text{MOEA/D(PBI)})$, $r(\text{NSGA-II})$

and $r(pa\lambda$-MOEA/D) are mentioned in Table 2.7.

MOEA/D(WS)	MOEA/D(TE)	MOEA/D(PBI)	NSGA-II	$pa\lambda$-MOEA/D	IIPM (CM)
148.228	28.156	33.542	29.555	23.041	1

Table 2.7: Relative efficiency of MOEA/D(WS), MOEA/D(TE), MOEA/D(PBI), NSGA-II, $pa\lambda$-MOEA/D and IIPM (CM) methods with respect to median of IGD values

Similarly, based on the median HV values reported in Table 2.6, we calculate the relative efficiency with respect to median of HV values, which is mentioned in Table 2.8.

MOEA/D(WS),	MOEA/D(TE)	MOEA/D(PBI)	NSGA-II	$pa\lambda$-MOEA/D	IIPM (CM)
0	0.831	0.776	0.772	0.855	1

Table 2.8: Relative efficiency of MOEA/D(WS), MOEA/D(TE), MOEA/D(PBI), NSGA-II, $pa\lambda$-MOEA/D and IIPM (CM) methods with respect to median of HV values

From Tables 2.7 and 2.8 for relative efficiencies, we notice that IIPM (CM) has the least relative efficiency with respect to IGD values and has the greatest relative efficiency with respect to HV values. Hence, the proposed IIPM (CM) outperforms the existing efficient techniques.

2.9 Conclusion

This chapter has introduced an infeasible interior-point approach, with the help of the cone method, to find a discrete subset of nondominated points of an MOPs. Towards the derivation, a log-barrier problem corresponding to each parametrized problem of the cone method has been formulated and solved by the Newton method. To find a solution to the Newton system, two merit functions are used in the proposed algorithms: one is to find an appropriate step length ($\psi_{\eta,\mu}$) and another for the stopping criterion ($\nu(\Omega)$).

Theorem 2.2 has shown that the search directions (2.15) computed by Theorem 2.1 are descent for the merit function $\psi_{\eta,\mu}$. Reductions in primal infeasibilities ($\varrho_{\hat{\beta}},\ \rho$), dual infeasibility ($\sigma_{\hat{\beta}}$) and complementarity ($\gamma$) after one iteration are shown in Theorem 2.3. Theorem 2.4 extends the conclusion of Theorem 2.3 for the k-th iteration and provides the total number of iterations needed to obtain an ϵ-precise solution.

It is important to notice that the proposed infeasible interior-point method is capable of handling convex (Algorithm 1) and nonconvex multiobjective optimization (Algorithm 3). Whenever the problem is not convex, a diagonal perturbation to the Hessian matrix has been applied to make the search directions descent. Consequently, the reductions in primal infeasibilities ($\varrho_{\hat{\beta}},\ \rho$), dual infeasibility ($\sigma_{\hat{\beta}}$) and complementarity ($\gamma$) have been shown in Theorem 2.5.

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