CHAPTER 3
2-PHASE CONSENSUS WITH CUSTOMIZED FEEDBACK BASED GROUP DECISION MAKING INVOLVING HETEROGENEOUS DECISION MAKERS

In general, the DMs with expertise and knowledge in their respective area are invited to participate in decision-making. In cases of smart cities, social networks, or e-democracy, the acceptance and survival of the obtained decision are determined when the citizens' or users' preferences and concerns are also taken into account. For such cases, the potential solution needs to be drawn using the citizens' or users' (considered as non-expert's) perceptions along with that of experts in decision-making to ensure the concordance of the decision. Here we propose a novel GDM model that involves non-experts and experts to use the understanding of non-experts as well. For this purpose, two phases of the CRP are defined: the inter-consensus reaching phase, where consensus between experts and non-experts will be determined, and the intra-consensus reaching phase, where the experts negotiate among themselves to attain the consensus. In the existing GDM models, CRP overcomes the conflicts in the opinions of the DMs by providing feedback to DMs for modifying their preferences to achieve the required consensus. However, multiple feedback rounds increase the cost of CRP. The proposed GDM gives customized feedback to the experts only once at each phase, reducing the feedback cost. A numerical example is discussed to explain the effectiveness of the proposed model. The proposed approach is tested on different consensus thresholds to verify its practicality.

3.1 Background

A decision problem related to socio-technical systems (like urban infrastructure) usually requires DMs of different subgroups of society. The potential citizens, as non-experts, should also participate in the decision-making process. Therefore, the group decision process no longer depends on the experts. Rather, it involves experts and non-experts as a potential DMs set.

CRP in GDM is used to get a certain degree of agreement among the DMs who state, negotiate, and modify their preferences. It consists of a consensus measure followed by some feedback to provide recommendations to DMs to achieve consensus. Different consensus models providing feedback to the DMs to increase their consensus level have been proposed in such situations. However, there are situations where some of the DMs, identified above as non-experts, are not aware of their participation in decision-making and hence not interested in receiving the feedback recommendations. In such a context, it could be adequate to use them as resource persons to help experts make wise decisions.

Another issue is generating feedback recommendations for inconsistent DMs to improve the consensus among DMs. The feedback mechanisms to advise decision-makers to adjust their preferences may take several feedback rounds to get closer to the collective preference. According to Zhang et al. [52], the number of iterations to reach consensus is one of the criteria for measuring consensus efficiency, meaning thereby that the consensus efficiency is considered to be improved if a constant number of feedback advice, minimizing the feedback cost, is provided. Apart from this understanding of consensus efficiency, there are situations like emergency events when the decision is warranted at the earliest, and hence the challenge is to design a feedback mechanism that aids CRP in reaching consensus with minimal rounds of interaction.

To fill the aforementioned research gap, the CRP discussed in this chapter is about the feedback mechanism that helps experts reach consensus at once. We proposed a personalized and constant feedback mechanism-based CRP by respecting the initial preference provided by the DMs, which aims at helping DMs in reaching consensus at once.

3.2 Proposed Models

In the reported literature on CRP, some research gaps are identified. The proposed model attempts to fill the following two research gaps: lack of heterogeneity in the context of DMs and the feedback cost identified in the literature. Novelties of the proposed work are discussed as follows.

This work aims to model the decision-making process involving non-experts and experts, provide experts with some additional ideas/opinions generated by non-experts, and propose a high-quality solution to the decision problem. We first allow the experts to interact with the non-experts to get their wisdom about the issues (or the options available); hence, their influence gets reserved in decision-making. Next, the experts start negotiating among themselves to reach the final decision. The non-experts who can only contribute positively towards decision-making should be involved. As of now, it is assumed that the misleading non-experts are not involved in the decision-making process. The purpose is to involve suitable non-experts capable of enriching the decision process. In a way, it is an arrangement where suitable non-experts augment experts.

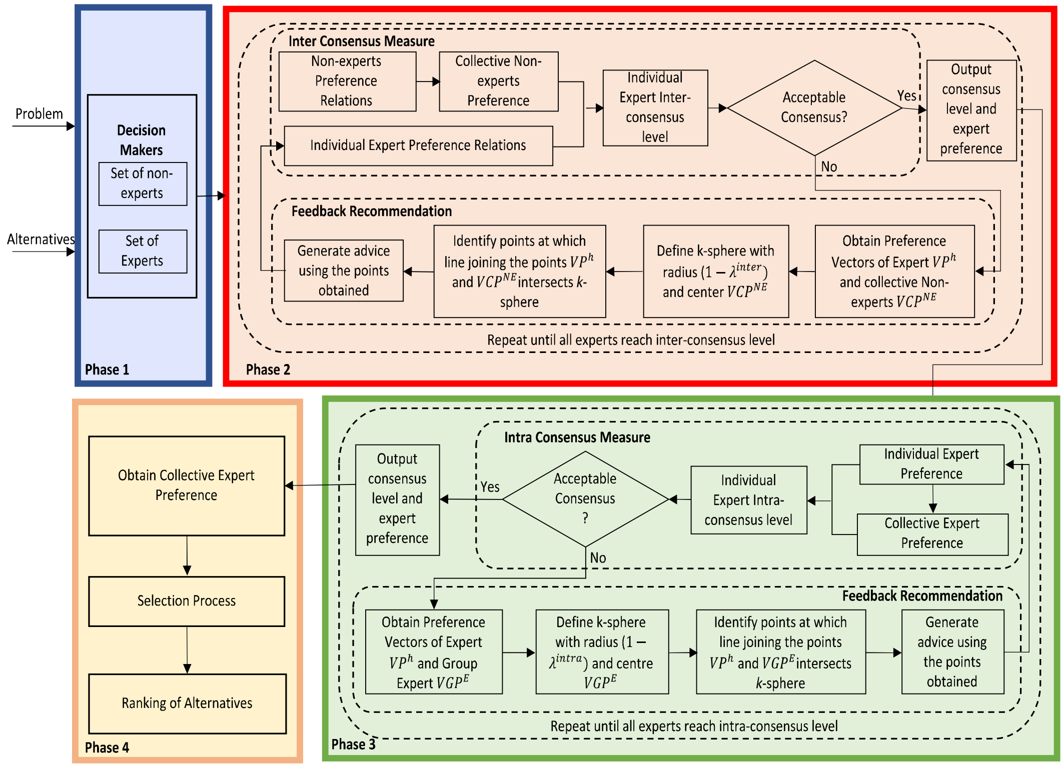


Fig. 3.1: Proposed Heterogenous Group Decision Making Framework

We propose a framework for GDM with two step consensus reaching process. The proposed work is divided into four phases, as shown in Fig.3.1. In the first phase, the experts and the non-experts provide their preferences. The second phase is the inter-consensus reaching phase, which deals with reaching the consensus between experts and non-experts. The third phase is the intra-consensus reaching phase that establishes consensus among the experts. In the fourth phase, the final decision is taken. For modelling the intra-consensus process, the various group decision-making approaches have been discussed in the literature, aiming to achieve high enough consensus for the experts [34][77]. However, the problem is to derive the consensus process for the inter-consensus phase with a unilateral feedback process where the aim is not to advise non-experts which reaching consensus; rather, use their opinions to guide experts to reach some level of agreement with them. Thereby, the preference provided by non-experts will not be modified. Therefore, the focus is on modifying experts' opinions concerning non-experts to achieve consensus. Each module in the proposed framework is discussed as follows.

3.2.1 Phase 1: Decision Makers Provide Preferences

Let $E = \{e_1, e_2, e_3, \ldots, e_m\}$ where $m \geq 2$ be the finite set of experts, and let $NE = \{ne_1, ne_2, ne_3, \ldots, ne_o\}$ be the set of non-experts, where $o \geq 1$. Both the experts and the non-experts express their opinions based on the knowledge, background and attitude, for the assessment of n alternatives given by, $X = \{x_1, x_2, x_3, \ldots, x_n\}$, $(n \geq 2)$ to achieve a satisfactory solution. Each expert e_h gives its opinion using fuzzy preference given by a preference matrix $P_E^h = (p_{ij}^h)_{n\times n}$ and each non-expert l gives its opinion using fuzzy preference given by a preference matrix $P_{NE}^l = (p_{ij}^l)_{n\times n}$. Since considered preference relation follows the property of reciprocity, i.e., $p_{ij}^h + p_{ji}^h = 1 \; \forall \; i, j \in \{1,2, \ldots, n\}$, out of $n \times n$ preference values, only $k = \frac{n\times(n-1)}{2}$ values can be used to initiate the decision-making process because the remaining $((n \times n) - k)$ values are the reciprocals which would be retained using these k values. This can be observed in the following preference matrix P_E^h given by the decision maker h for three alternatives in Eq. (3.1).

$$P_E^h = \begin{pmatrix} - & p_{12}^h & p_{13}^h \\ p_{21}^h & - & p_{23}^h \\ p_{31}^h & p_{32}^h & - \end{pmatrix} = \begin{pmatrix} - & p_{12}^h & p_{13}^h \\ 1 - p_{12}^h & - & p_{23}^h \\ 1 - p_{13}^h & 1 - p_{23}^h & - \end{pmatrix} \tag{3.1}$$

From the above matrix, it can be observed that p_{12}^h, p_{13}^h, and p_{23}^h are sufficient pairwise preference values to understand the complete preference relation of the expert e_h. That means only k-dimensional space would be sufficient to locate a decision-maker using its preference relation matrix of size $n \times n$.

In this work, to reach a consensus decision, we consider the set of decision-makers with their preference relation matrix as a set of points in a k-dimensional space. The points representing the experts in the form of preference relations can be repositioned in the k-dimensional space whereas the points representing the non-experts are fixed points, because experts may change their preference based on their degree of consensus whereas non-experts do not.

3.2.2 Phase 2: Inter-Consensus Reaching Phase

This section introduces an inter-consensus reaching process which accepts the opinion of non-experts to achieve the predefined consensus threshold λ^{inter}. The idea is to achieve consensus between experts and non-experts. Suitable non-experts are invited as resource persons where they do not continuously participate and modify their opinion. The non-experts' opinions are taken at once and used as advice to the experts to produce a consensual solution. This is because, in situations where the opinions of the non-experts are not requested directly but rather crawled from the available information on the internet, the non-experts are unaware of their opinions' involvement in the decision-making process. Hence, it would not be possible to provide feedback to the non-experts. In fact, their opinions would be used by the experts to modify the opinions rationally. This way, the influence of the non-experts can be reserved in the decision-making without making them participate continuously throughout the consensus reaching process.

After receiving the preferences from experts and non-experts, the inter-consensus reaching phase initiated to achieve consensus between non-experts and experts consists of a consensus measure followed by the feedback recommendation method. Both are discussed as follows.

3.2.2.1 Consensus Measure

The consensus measure in inter-consensus reaching process is used to find the degree of agreement between experts and non-experts. The non-expert's individual preference is integrated into a collective opinion, and experts will negotiate to achieve consensus with non-experts. The individual preferences of non-experts are fused using an aggregation operator. The aggregation operators such as weighted average and the ordered weighted are the most frequently used [19]. Depending upon the nature of problem and the knowledge of non-experts, the non-experts are prioritized, which is completely a subjective discussion. However, some research demonstrates the method for determining the weights of the decision-makers; the subjective methods and the objective methods [90]. In general, the moderator in group decision-making who serves a neutral role is held responsible for allocating weight to the decision-makers. In this study, we assume that all non-experts have the same weight. Without loss of generality, we employ the weighted average operator to obtain the collective preference of non-experts, i.e., CP^{NE} as it prioritizes the DMs. However, the findings of the paper do not get affected by the use of other aggregation operators.

Let $P_{NE}^{l} = (p_{ij}^{l})_{n\times n}$ be the preference matrix of non-expert ne_l and $w_{ne} = (w_{ne}^{1}, w_{ne}^{2}, \dots, w_{ne}^{o})$ be the non-experts' weight vector, where $w_{ne}^{l} \geq 0$ is the weight associated to the non-expert ne^l and $\sum_{l=1}^{o} w_{ne}^{l} = 1$.

$$P_{NE}^{l} = \begin{pmatrix} p_{11}^{l} & p_{12}^{l} & \cdots & p_{1n}^{l} \\ p_{21}^{l} & & \ddots & p_{2n}^{l} \\ \vdots & & & \vdots \\ p_{n1}^{l} & p_{n2}^{l} & \cdots & p_{nn}^{l} \end{pmatrix} \tag{3.2}$$

Collective preference of non-experts, i.e., $CP^{NE} = (cp_{ij}^{NE})_{n\times n}$, is calculated using weighted average operator as per the Eq. (2.1). We measure the inter-consensus level by calculating the distance between the preference matrix P_E^h of each expert $e_h \in E$ and the collective preference of non-experts CP^{NE} discussed in Definition 2. λ^{inter} is the inter-consensus threshold, i.e., minimum expected consensus between preference of an expert and collective preference of non-experts.

Definition 3.1: Let $P_E^h = (p_{ij}^h)_{n\times n}$ $\forall\, i,j \in \{1,2,\ldots,n\}$ & $i \neq j$ be the preference matrix of expert $e_h \in E$ and $CP^{NE} = (cp_{ij}^{NE})_{n\times n}$ $\forall\, i,j \in \{1,2,\ldots,n\}$ & $i \neq j$ be the collective preference matrix of non-experts. Degree of consensus between the expert and collective preference of non-experts can be defined in terms of degree of deviation between P_E^h and CP^{NE}. Degree of deviation, $D_{h,NE}^{inter}$, is calculated using the normalized Euclidean distance and given in Eq. (3.3).

$$D_{h,NE}^{inter} = \frac{1}{\sqrt{n(n-1)}}\Big(\sum_{i,j\,\in\{1,2,..,n\} and\ i\neq j}(p_{ij}^h - cp_{ij}^{NE})^2\Big)^{\frac{1}{2}} \tag{3.3}$$

Degree of consensus between the expert e_h and collective preference of non-experts CP^{NE}, i.e., $CD_{h,NE}^{inter}$, is calculated using the normalized Euclidean distance and given in Eq. (3.4).

$$CD_{h,NE}^{inter} = 1 - D_{h,NE}^{inter} \tag{3.4}$$

Since fuzzy preference relation in pairwise comparison model follows the property of reciprocity, we can simplify the Eq. (3.3) as given in Eq. (3.5).

$$D_{h,NE}^{inter} = \frac{1}{\sqrt{n(n-1)}}\Big(\sum_{i,j\,\in\{1,2,\ldots,n\} and\ i<j}(p_{ij}^h - cp_{ij}^{NE})^2 + \sum_{i,j\,\in\{1,2,\ldots,n\} and\ i<j}(p_{ji}^h - cp_{ji}^{NE})^2\Big)^{1/2} \tag{3.5}$$

Since $p_{ij}^h + p_{ji}^h = 1$ $\forall\, i,j \in \{1,2,\ldots,n\}$ and $cp_{ij}^{NE} + cp_{ji}^{NE} = 1$ $\forall\, i,j \in \{1,2,\ldots,n\}$, Eq. (3.5) can be rewritten as follows.

$$D_{h,NE}^{inter} = \frac{1}{\sqrt{n(n-1)}}\Big(\sum_{i,j\,\in\{1,2,\ldots,n\} and\ i<j}(p_{ij}^h - cp_{ij}^{NE})^2 + \sum_{i,j\,\in\{1,2,\ldots,n\} and\ i<j}((1-p_{ij}^h) - (1-cp_{ij}^{NE}))^2\Big)^{1/2} \tag{3.6}$$

$$D_{h,NE}^{inter} = \frac{1}{\sqrt{n(n-1)}} (\sum\nolimits_{i,j\in\{1,2,\ldots,n\} and\ i<j} (p_{ij}^{h} - cp_{ij}^{NE})^2 + \sum\nolimits_{i,j\in\{1,2,\ldots,n\} and\ i<j} (p_{ij}^{h} - cp_{ij}^{NE})^2)^{1/2} \tag{3.7}$$

$$D_{h,NE}^{inter} = \frac{1}{\sqrt{n(n-1)}} (2 \times \sum\nolimits_{i,j\in\{1,2,\ldots,n\} and\ i<j} (p_{ij}^{h} - cp_{ij}^{NE})^2)^{1/2} \tag{3.8}$$

$$D_{h,NE}^{inter} = \frac{\sqrt{2}}{\sqrt{n(n-1)}} (\sum\nolimits_{i,j\in\{1,2,\ldots,n\} and\ i<j} (p_{ij}^{h} - cp_{ij}^{NE})^2)^{1/2} \tag{3.9}$$

$$D_{h,NE}^{inter} = \frac{1}{\sqrt{\frac{n(n-1)}{2}}} (\sum\nolimits_{i,j\in\{1,2,\ldots,n\} and\ i<j} (p_{ij}^{h} - cp_{ij}^{NE})^2)^{1/2} \tag{3.10}$$

$$D_{h,NE}^{inter} = \frac{1}{\sqrt{k}} (\sum\nolimits_{i,j\in\{1,2,\ldots,n\} and\ i<j} (p_{ij}^{h} - cp_{ij}^{NE})^2)^{1/2} \tag{3.11}$$

From Eq. (3.11) and Eq. (3.1), one can observe that only k values in preference matrix P_E^h of expert e_h and collective preference of non-experts CP^{NE} are sufficient to calculate the degree of consensus in Eq. (3.4).

Obviously, $CD_{h,NE}^{inter} \in [0,1]$. The larger the value of $CD_{h,NE}^{inter}$ higher degree of consensus P_E^h with CP^{NE}. This obtained consensus degree is compared against the predefined inter-consensus threshold, $\lambda^{inter} \in [0,1]$ where 0 implies the complete dissimilarity and 1 implies the complete similarity. If $CD_{h,NE}^{inter} \geq \lambda^{inter}$ then expert e_h reached the expected consensus level; otherwise, undergoes a feedback process to improve the consensus level.

3.2.2.2 Feedback Recommendation

If the level of agreement between an expert and the collective preference of non-experts does not reach the required consensus level, we should adopt the feedback mechanism to reach the required consensus level. The proposed feedback mechanism generates recommendations to the experts based on their degree of deviation while preventing them from undergoing multiple feedback rounds to achieve the required consensus.

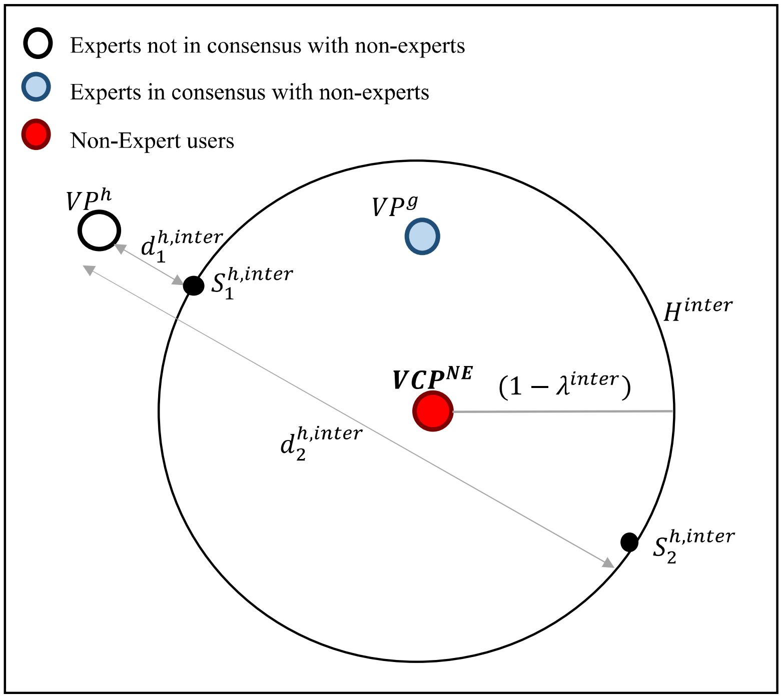


Fig. 3.2: Case of Inter-Consensus

It is known that, in general, threshold λ is the minimum required consensus among the experts where $\lambda \in [0,1]$. That means $\lambda = 0$ implies no agreement among decision-makers, whereas $\lambda = 1$ implies full agreement among experts, which is ideally not possible. Therefore, there will always be a little degree of disagreement among the experts, which is acceptable in every situation. Hence, we can say that in order to reach the consensus level, $(1-\lambda)$ is the maximum degree of disagreement that can be tolerated. If the disagreement degree of the expert is more than the specified degree of disagreement $(1-\lambda)$, we should adopt a feedback mechanism to improve the consensus level. The expert undergoes a feedback process; otherwise, the expert is in consensus. The advice generated is in the form of interval value defined by lower and upper bound for the respective k-preference values. It is worth noting that the recommendations generated for the experts are the most appropriate range of values to modify within. And if the received advice is properly considered, experts reach a consensus without undergoing any further rounds of negotiation. Hence, we propose to compute a customized amount of advice that varies in accordance with the experts' distance from the group value.

In view of the above, we define the feedback mechanism using some basic geometric solutions. To do so, we consider a k-dimensional Euclidean space where the experts and the collective non-expert constitute a set of $m+1$ points in Euclidean space. The preference value P^h of an expert e_h can be represented by a vector VP^h where $VP^h = (p_{ij}^h), \forall\, i,j \in \{1,2,\dots,n\}$ & $i<j$, representing the original preference of expert $e_h \in E$ and the collective non-experts preference CP^{NE} can be represented by a vector $VCP^{NE} = (cp_{ij}^{NE}), \forall\, i,j \in \{1,2,\dots,n\}$ & $i<j$ in a k-dimensional Euclidean space. Thereby the preference matrices of all experts and non-experts are present inside a unit k-cube.

For simplicity, we assume that there are two experts e_g and e_h with their preference matrices represented using vectors VP^g and VP^h, respectively. VP^g, VP^h, and the collective preference matrix of non-expert VCP^{NE} are shown in Fig. 3.2. It can be observed that an expert can have at most $(1-\lambda^{inter})$ as the maximum disagreement degree with VCP^{NE} to achieve the inter-consensus level. Thus, the curve enclosing the space of maximum degree of disagreement would be a $(k+1)$-sphere represented by H^{inter} with centre VCP^{NE} and radius $(1-\lambda^{inter})$. From the Fig. 3.2, it can be observed that the vector VP^h representing expert e_h falls outside the H^{inter}, which means e_h is not in agreement with non-experts whereas VP^g of e_g lies inside H^{inter}means e_g meets the required consensus. Thus, expert e_h needs to modify its preference to meet the required degree of consensus. For this, the proposed feedback mechanism that guides the expert in modifying their preference can be understood in steps given as follows:

step (i):In k-dimension space, we define $(k+1)-$sphere, H^{inter} (also called hypersphere) with centre VCP^{NE} and radius $\left(1-\lambda^{inter}\right)$.

step (ii): Each expert e_h who lies outside the hypersphere H^{inter} is not in consensus which means the consensus degree $CD_{h,NE}^{inter}$ is less than threshold λ^{inter}, i.e., $(1 - CD_{h,NE}^{inter}) > (1 - \lambda^{inter})$.

step (iii): To reach consensus threshold λ^{inter}, the expert e_h is guided in such a way that the distance $\left(1 - \lambda^{inter}\right) + d_1^{h,inter}$ gets reduced to at least $(1 - \lambda^{inter})$.

step (iv): To guide the expert e_h for each preference value, we find the point on the hypersphere H^{inter} where the line joining the point VP^h and VCP^{NE} intersects the hypersphere. Since the line passes through the centre, it will intersect hypersphere at two points say $S_1^{h,inter}$ and $S_2^{h,inter}$.

step (v): Using the points $S_1^{h,inter}$ and $S_2^{h,inter}$, generation of advice for each preference value will take place for the expert e_h.

Now we illustrate the method to find the points at which the line joining the experts and group decision intersects the $(k + 1) -$ sphere and how the advice is generated with lower and upper bound for each preference value as follows

(a). Calculation of intersection points

In order to find the intersection points, we find the equation of the line passing through the points VP^h and VCP^{NE}. Vector equation of the line through point VP^h is given in Eq. (3.12).

$$\begin{bmatrix} p_{12}^h \\ p_{13}^h \\ \vdots \\ p_{ij}^h \end{bmatrix} + t_h \times \begin{bmatrix} v_{12}^{h,NE} \\ v_{13}^{h,NE} \\ \vdots \\ v_{ij}^{h,NE} \end{bmatrix} \tag{3.12}$$

where t_h is a scalar, and $v_{ij}^{h,NE}$ is the direction vector of line through point VP^h given by $(v_{ij}^{h,NE}) = (p_{ij}^h - cp_{ij}^{NE}), \forall\, i,j \in \{1,2,\dots,n\}\ and\ i < j$. Since we have used normalized

Euclidean distance, for some general point $Z^{h,inter} = (z_{ij}^{h,inter}), \forall\, i,j \in \{1,2,\dots,n\}\ and\ i < j$, the equation of the $(k+1)$ −sphere can be written as:

$$(\frac{1}{\sqrt{k}}(\sum_{i,j\,\in\{1,2,\dots,n\}\ and\ i<j}(z_{ij}^{h,inter} - cp_{ij}^{NE}))^2 = (1-\lambda^{inter})^2 \qquad (3.13)$$

Eq. (3.13) can be simplified as:

$$\sum_{i,j\,\in\{1,2,\dots,n\}\ and\ i<j}(z_{ij}^{h,inter} - cp_{ij}^{NE})^2 = (1-\lambda^{inter})^2 \times k \qquad (3.14)$$

For searching the points that are present on the line and $(k+1)$ −sphere both, we combine Eqs. (3.12) and (3.14) to solve them. We substitute $z_{ij}^{h,inter} = p_{ij}^{h} + (t_h \times v_{ij}^{h,NE})$ in Eq. (3.14) and we get a quadratic equation in t_h. On solving the quadratic equation, we will get two values of t_h, say α_h^{inter} and β_h^{inter}. And for these two values, two different points on the hypersphere will be obtained. Let intersection points are $S_1^{h,inter} = (\gamma_{ij}^{h,inter}), \forall\, i,j \in \{1,2,\dots,n\}\ and\ i<j$ and $S_2^{h,inter} = (\delta_{ij}^{h,inter}), \forall\, i,j \in \{1,2,\dots,n\}\ and\ i<j$. The points $S_1^{h,inter}$ and $S_2^{h,inter}$ are the two distinct points lying on the hypersphere, using which the advice in form of interval value, i.e., the range of the preference value would be generated for the expert e_h.

(b). Advice Generation

We calculate the distance of point VP^h from point $S_1^{h,inter}$ and $S_2^{h,inter}$, which is $d_1^{h,inter}$ and $d_2^{h,inter}$, respectively. Depending on the distances $d_1^{h,inter}$ and $d_2^{h,inter}$, upper and the lower bound of the interval will be decided.

i. If $d_1^{h,inter} > d_2^{h,inter}$, then $S_2^{h,inter}$ will be used as a feedback matrix and e_h should modify the preference p_{ij}^{h} for pair of alternatives (x_i, x_j) as follows:

$$\begin{cases} \overline{p}_{ij}^{h} \in [min(cp_{ij}^{NE}, \delta_{ij}^{h,inter}), max(cp_{ij}^{NE}, \delta_{ij}^{h,inter})], & i \geq j \\ \overline{p}_{ji}^{h} = 1 - \overline{p}_{ij}^{h}, & i < j \end{cases} \qquad (3.15)$$

ii. If $d_1^{h,inter} < d_2^{h,inter}$, then $S_1^{h,inter}$ will be used as a feedback matrix and e_h should modify the preference p_{ij}^h for pair of alternatives (x_i, x_j) as follows:

$$\begin{cases} \overline{p}_{ij}^h \in [min(cp_{ij}^{NE}, \gamma_{ij}^{h,inter}), max(cp_{ij}^{NE}, \gamma_{ij}^{h,inter})], & i \geq j \\ \overline{p}_{ji}^h = 1 - \overline{p}_{ij}^h, & i < j \end{cases} \tag{3.16}$$

iii. $d_1^{h,inter} = d_2^{h,inter}$ is possible only when the identified line is the tangent to the surface, i.e., it touches the surface at only one point, which is not the case here because the line passes through the centre of the $(k+1)$-sphere.

The generated advice will help experts with lower consensus degree to reach consensus at once, if the expert modify the preference value in the advised range. The two extreme points $S_1^{h,inter}$ and $S_2^{h,inter}$ will behave as the maximum and the minimum value of preference.

3.2.3 Phase 3: Intra-Consensus Reaching Phase

After reaching the required degree of consensus with non-experts, each expert with new preference relation $\overline{P}_E^h = (\overline{p}_{ij}^h)_{n\times n}$ enter into the next phase where they negotiate to achieve a certain degree of agreement among them. Here again, CRP starts that involves consensus measure followed by the feedback recommendation method. In this phase, we are using the same procedure to obtain consensus, which has been proposed and used during the inter-consensus reaching phase. The only difference is the value of the degree of disagreement and the group preference value, while the remaining steps are the same.

3.2.3.1 Consensus Measure

Let $P_E^g = (p_{ij}^g)_{n\times n}$ be the preference matrix of the expert $e_g \in E$ in the intra-consensus reaching phase and let the weight vector is $w_e = (w_e^1, w_e^2, \ldots, w_e^m)$, where $w_e^g \geq 0$ is the weight associated with the expert e_g and $\sum_{g=1}^{m} w_e^g = 1$. The weighted average operator is used to obtain the group preference of the experts $GP^E = (gp_{ij}^E)_{n\times n}$ using Eq. (2.1).

We measure the intra-consensus level by calculating the distance between the preference matrix P_E^g of each expert $e_g \in E$ and the group preference of experts GP^E.

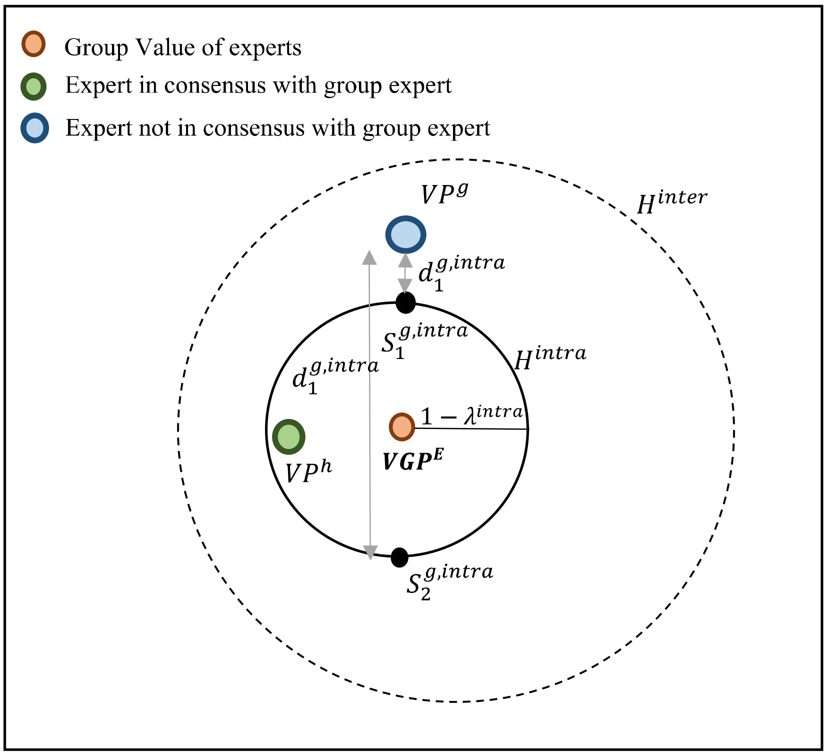


Fig. 3.3: Case of Intra-Consensus

λ^{intra} is the threshold, which is the minimum expected consensus among experts. Degree of consensus, i.e., $CD_{g,E}^{intra}$, between the expert e_g and group preference of experts GP^E is calculated using Eqs. (3.4) and (3.11) as given in Eqs. (3.17) and (3.18).

$$D_{g,E}^{intra} = \frac{1}{\sqrt{k}}\left(\sum_{i,j\,\in\{1,2,\ldots,n\} and\ i<j} (p_{ij}^g - gp_{ij}^E)^2\right)^{1/2} \tag{3.17}$$

$$CD_{g,E}^{intra} = 1 - D_{g,E}^{intra} \tag{3.18}$$

If the obtained $CD_{g,E}^{intra}$ is greater than or equal to the threshold, i.e., $CD_{g,E}^{intra} \geq \lambda^{intra}$, the expert e_g is in consensus. Whereas $CD_{g,E}^{intra} < \lambda^{intra}$ implies that the expert e_g needs to modify its preference. For the modification of the preference, the expert goes for the feedback recommendation phase to initiate a new consensus round.

3.2.3.2 Feedback Recommendation

Let $VGP^E = (gp_{ij}^E), \forall\, i,j\ \in \{1,2,\dots,n\}$ & $i < j$ is a vector that represents GP^E in k-dimensional Euclidean space. To reach the intra-consensus, experts can be at a maximum disagreement of $(1 - \lambda^{intra})$ from the VGP^E as shown in Fig. 3.3. From Fig. 3.3, we can see that the respective vector points of expert e_g is outside the hypersphere H^{intra} with center VGP^E and radius $\left(1 - \lambda^{intra}\right)$, this expert needs to modify its preference to achieve consensus. The feedback process to guide experts to modify their preference values is the same as that used in inter-consensus reaching phase, with the difference in the center and radius of the hypersphere. Steps of feedback recommendation process are given as follow:

step (i): In k-Euclidean space, we define k-sphere H^{intra} with radius $\left(1 - \lambda^{intra}\right)$ where VGP^E is the center.

step (ii): Each expert e_g lies outside the hypersphere H^{intra} at a distance greater than $(1 - \lambda^{intra})$ from the centre VGP^E (see Fig. 3) is not in consensus which means the consensus degree $CD_{g,E}^{intra}$ is less than threshold λ^{intra}, i.e., $(1 - CD_{g,E}^{intra}) > (1 - \lambda^{intra})$.

step (iii): To reach consensus threshold λ^{intra}, the expert e_g who is not in consensus must be guided so that the distance $(1 - \lambda^{intra}) + d_1^{g,intra}$ gets reduced to at least $(1 - \lambda^{intra})$.

step (iv): To guide the expert e_g, we find the points on the hypersphere H^{intra} where the line joining the point VP^g and VGP^E intersects the hypersphere. Since the line passes through the centre, it will intersect hypersphere at two points say $S_1^{g,intra}$ and $S_2^{g,intra}$.

step (v): Using the points $S_1^{g,intra}$ and $S_2^{g,intra}$, generation of advice will take place for the expert e_g. Advice will be generated in the same way as generated at inter-consensus phase.

(a). Calculation of intersection points

In order to find the intersection points, we find the equation of the line passing through the points VP^g and VGP^E as given in Equation of the line through point VP^g is given in Eq. (3.19).

$$\begin{bmatrix} p_{12}^g \\ p_{13}^g \\ \vdots \\ p_{ij}^g \end{bmatrix} + t_g \times \begin{bmatrix} v_{12}^{g,E} \\ v_{13}^{g,E} \\ \vdots \\ v_{ij}^{g,E} \end{bmatrix} \tag{3.19}$$

where t_g is a scalar, and $v_{ij}^{g,E}$ is the direction vector of line through point e_g given by $(v_{ij}^{g,E}) = (p_{ij}^g - gp_{ij}^E), \forall\, i,j \in \{1,2,\ldots,n\}\ and\ i < j.$

Since we have used the normalized Euclidean distance, for some general point $Z^{g,intra} = (z_{ij}^{g,intra}), \forall\, i,j \in \{1,2,\ldots,n\}\ and\ i < j$, the equation of the k-sphere can be written as follows:

$$(\frac{1}{\sqrt{k}}(\sum_{i,j\in\{1,2,\ldots,n\}\ and\ i<j}(z_{ij}^{g,intra} - gp_{ij}^E))^2 = (1-\lambda^{intra})^2 \tag{3.20}$$

Equation (3.20) can be simplified as:

$$\sum_{i,j\in\{1,2,\ldots,n\}\ and\ i<j}(z_{ij}^{g,intra} - gp_{ij}^E)^2 = (1-\lambda^{intra})^2 \times k \tag{3.21}$$

For searching the points that are present on the line and k-sphere both, we combine Eq. (3.20) and (3.21) to solve them. We substitute $z_{ij}^{g,intra} = p_{ij}^g + (t_g \times v_{ij}^{g,E})$ in Eq. (3.21) and we get a quadratic equation in t_g. On solving the quadratic equation, we will get two values of t_g, say α_g^{intra} and β_g^{intra}. And for these two values, two different points will be obtained on the hypersphere at which the line intersects the hypersphere. Let intersection points are $S_1^{g,intra} = (\gamma_{ij}^{g,intra}), \forall\, i,j \in \{1,2,\dots,n\}$ and $i < j$ and $S_2^{g,intra} = (\delta_{ij}^{g,intra}), \forall\, i,j \in \{1,2,\dots,n\}$ and $i < j$. The point $S_1^{g,intra}$ and $S_2^{g,intra}$ are the two distinct points lying on the hypersphere H^{intra}, using which the advice will be generated.

(b). Advice Generation

We calculate the distance $d_1^{g,intra}$ of point VP^g from point $S_1^{g,intra}$ and distance $d_2^{g,intra}$ of VP^g from $S_2^{g,intra}$. Depending on the distance of the points $S_1^{g,intra}$ and $S_2^{g,intra}$, upper and the lower bound of the interval will be decided.

i. If $d_1^{g,intra} > d_2^{g,intra}$, then $S_2^{g,intra}$ will be used as a feedback matrix and e_g should modify the preference p_{ij}^g for pair of alternatives (x_i, x_j) as follows:

$$\begin{cases} \overline{p}_{ij}^g \in [min(gp_{ij}^E, \delta_{ij}^{g,intra}), max(gp_{ij}^E, \delta_{ij}^{g,intra})], & i \geq j \\ \overline{p}_{ji}^g = 1 - \overline{p}_{ij}^g, & i < j \end{cases} \quad (3.22)$$

ii. If $d_1^{h,intra} < d_2^{h,intra}$, then then $S_1^{h,intra}$ will be used as a feedback matrix and e_h should modify the preference p_{ij}^h for pair of alternatives (x_i, x_j) as follows:

$$\begin{cases} \overline{p}_{ij}^g \in [min(gp_{ij}^E, \gamma_{ij}^{g,intra}), max(gp_{ij}^E, \gamma_{ij}^{g,intra})], & i \geq j \\ \overline{p}_{ji}^g = 1 - \overline{p}_{ij}^g, & i < j \end{cases} \quad (3.23)$$

iii. $d_1^{g,intra} = d_2^{g,intra}$ is possible only when the identified line is the tangent to the surface, i.e., it touches the surface of hypersphere at only one point, which is not the case here because the line passes through the centre of the $(k+1)$ −sphere.

In intra-consensus phase like inter-consensus phase, the generated advice helps experts with lower consensus degree to reach consensus at once, if the expert follows the advice. The two extreme points $S_1^{g,intra}$ and $S_2^{g,intra}$ are used to define the range of preference value. And once the experts modify their preference as per given advice, they certainly will be in agreement with each other.

3.2.4 Phase 4: Calculation of Final Decision

Once the expert's consensus level is acceptable at both the inter consensus level and intra-consensus level, let $\bar{P}_E^g = (\bar{p}_{ij}^g)_{n\times n}$ be the final preference matrix of the expert $e_g \in E$. Then we calculate the global collective matrix GP as given in Eq. (3.24).

$$GP = \sum_{g=1}^{m}(w_e^g \times \bar{P}_E^g) \quad (3.24)$$

Any selection operator can be applied to evaluate the ranking of the alternative using global collective experts' matrix $GP = \left(gp_{ij}\right)_{n\times n}$. For exemplary purposes, we apply the existing guided dominance degree (QGDD) operator [91]**.** For each alternative x_i, its associated score is calculated using Eq. (3.25).

$$S_i = \varphi(gp_{ij}),\ j = 1 \ldots . n \quad (3.25)$$

where φ is the mean operator and gp_{ij} is the value present at the position (i,j) of the GP matrix. The higher the evaluation score S_i, the higher the ranking of alternative will be in the final ordering of the set of alternatives.

3.3 Computational Complexity Analysis

The consensus reaching process (CRP) discussed in section 2.1.1 in Chapter 2 is the baseline design. In the baseline design, the consensus reaching process consists of a consensus measure followed by a feedback process. The feedback process is designed based on the Identification and Direction rule, which requires the individual and collective preference matrix. For the CRP, the best case would be when the experts with their initial preferences are already in consensus. Thus, the computational complexity for achieving consensus in the best case is $O(m[n(n-1)/2])$, where m is the number of experts, and n is the number of alternatives. However, the experts may hold different opinions and provide preferences accordingly. To minimize the difference in their opinions, the advice is generated for the experts with a lower consensus degree until consensus is reached. Therefore, for the baseline design, the worst-case computational complexity of the CRP is $O(m[n(n-1)/2]r)$, where r is the number of feedback rounds. Contrary to that, the proposed method with a novel feedback mechanism promises to achieve consensus by providing advice at once. The novel feedback process consists of calculating the intersection points and the advice generation, and computational complexity in the worst case is $O(\alpha[n(n-1)/2])$, where α is the number of non-experts, i.e., o in the inter-consensus reaching phase, and α is the number of experts, i.e., m, in the intra-consensus reaching phase. Furthermore, we can observe that computational complexity of the baseline model is equivalent to the proposed model in the best case only which is usually not guaranteed. Thus, computational complexity of the proposed model is better in compare to baseline design.

3.4 Numerical Example and Simulation Analysis

In this section, first we provide a numerical example to illustrate the working of proposed group decision making model. Then, we performed some simulated experiments to analyze our proposed model.

3.4.1 An Illustrative Example

The main purpose of the numerical example is to illustrate the use of the proposed two-phase group decision-making model. The proposed model is applied to the municipality budget allocation problem in the context of the citizens' participation as non-expert. To reduce the misuse of public funds and corruption, the government officials acting as leading experts would like to invite the citizens to influence the allocation decision. Here, the citizens will participate along with the government officials in part of an allocation of the public budget. We here consider the four alternative spaces where the budget needs to be allocated: roads and pavements planning (x_1), crime planning (x_2), healthcare decision-making (x_3) and children's play area (x_4). The concerned government officials and the non-elected citizens are entrusted to decide the allocation of the budget (or the municipality budget). The outcome of the decision process is the most preferred alternative by the experts and the citizens both. For this, we assume six equally important experts $\{e_1, e_2, e_3, e_4, e_5, e_6\}$, and eight equally important non-experts $\{ne_1, ne_2, ne_3, ne_4, ne_5, ne_6, ne_7, ne_8\}$ who provide their preferences over the defined alternatives. We know that the consensus threshold can be assigned any value between [0,1]. We keep $\lambda^{inter} = 0.75$ and $\lambda^{intra} = 0.90$. This combination of threshold values simulates the situation when the experts' opinions are given more importance even while taking the non-experts' opinion. Thus, it means that the degree of agreement required among the experts is greater than the degree of agreement among experts and non-experts. However, there are situations when the experts should give more importance to the non-experts' opinion and for such a situation the degree of agreement among experts and non-experts (i.e., the inter-consensus threshold) should be high. Experiment 3 in section 3.5 depicts the consensus values of experts for different combination of inter-consensus and intra-consensus threshold. Although the example is shown for six experts and eight non-experts to illustrate the proposed GDM model, it works well for any combination of $m \geq 2$, $o \geq 1$ and threshold values. The values of different parameters used here are enough to illustrate the proposed GDM model. All six experts and eight non-experts use fuzzy preference relation to express their preference information, which is given as follows:

Phase 1: Decision makers provide their preferences

Experts and Non-experts provide their preferences using fuzzy preference relation in the first phase as specified below:

$$P^1 = \begin{pmatrix} 0.50 & 0.45 & 0.38 & 0.78 \\ 0.55 & 0.50 & 0.36 & 0.53 \\ 0.62 & 0.64 & 0.50 & 0.71 \\ 0.22 & 0.47 & 0.29 & 0.50 \end{pmatrix} \quad P^2 = \begin{pmatrix} 0.50 & 0.87 & 0.33 & 0.65 \\ 0.13 & 0.50 & 0.97 & 0.08 \\ 0.67 & 0.03 & 0.50 & 0.59 \\ 0.35 & 0.92 & 0.41 & 0.50 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0.50 & 0.42 & 0.31 & 0.65 \\ 0.58 & 0.50 & 0.76 & 0.99 \\ 0.69 & 0.24 & 0.50 & 0.19 \\ 0.35 & 0.01 & 0.81 & 0.50 \end{pmatrix} \quad P^4 = \begin{pmatrix} 0.50 & 0.78 & 0.19 & 0.99 \\ 0.22 & 0.50 & 0.80 & 0.43 \\ 0.81 & 0.20 & 0.50 & 0.73 \\ 0.01 & 0.57 & 0.27 & 0.50 \end{pmatrix}$$

$$P^5 = \begin{pmatrix} 0.50 & 0.49 & 0.81 & 0.36 \\ 0.51 & 0.50 & 0.07 & 0.59 \\ 0.19 & 0.93 & 0.50 & 0.91 \\ 0.64 & 0.41 & 0.09 & 0.50 \end{pmatrix} \quad P^6 = \begin{pmatrix} 0.50 & 0.19 & 0.43 & 0.75 \\ 0.81 & 0.50 & 0.04 & 0.95 \\ 0.57 & 0.96 & 0.50 & 0.76 \\ 0.25 & 0.05 & 0.24 & 0.50 \end{pmatrix}$$

$$P_{ne}^1 = \begin{pmatrix} 0.50 & 0..56 & 0.18 & 0.50 \\ 0.44 & 0.50 & 0.52 & 0.99 \\ 0.82 & 0.48 & 0.50 & 0.85 \\ 0.50 & 0.01 & 0.15 & 0.50 \end{pmatrix} \quad P_{ne}^2 = \begin{pmatrix} 0.50 & 0.96 & 0.68 & 0.40 \\ 0.04 & 0.50 & 0.93 & 0.48 \\ 0.32 & 0.07 & 0.50 & 0.23 \\ 0.60 & 0.52 & 0.77 & 0.50 \end{pmatrix}$$

$$P_{ne}^3 = \begin{pmatrix} 0.50 & 0.40 & 0.71 & 0.56 \\ 0.60 & 0.50 & 0.76 & 0.99 \\ 0.29 & 0.24 & 0.50 & 0.96 \\ 0.44 & 0.01 & 0.04 & 0.50 \end{pmatrix} \quad P_{ne}^4 = \begin{pmatrix} 0.50 & 0.54 & 0.96 & 0.12 \\ 0.46 & 0.50 & 0.05 & 0.30 \\ 0.04 & 0.95 & 0.50 & 0.58 \\ 0.88 & 0.70 & 0.42 & 0.50 \end{pmatrix}$$

$$P_{ne}^5 = \begin{pmatrix} 0.50 & 0.53 & 0.90 & 0.54 \\ 0.47 & 0.50 & 0.43 & 0.54 \\ 0.10 & 0.57 & 0.50 & 0.71 \\ 0.46 & 0.46 & 0.29 & 0.50 \end{pmatrix} \quad P_{ne}^6 = \begin{pmatrix} 0.50 & 0.02 & 0.80 & 0.14 \\ 0.98 & 0.50 & 0.48 & 0.26 \\ 0.20 & 0.52 & 0.50 & 0.37 \\ 0.86 & 0.74 & 0.63 & 0.50 \end{pmatrix}$$

$$P_{ne}^7 = \begin{pmatrix} 0.50 & 0.66 & 0.17 & 0.28 \\ 0.34 & 0.50 & 0.20 & 0.20 \\ 0.83 & 0.80 & 0.50 & 0.33 \\ 0.72 & 0.80 & 0.67 & 0.50 \end{pmatrix} \quad P_{ne}^8 = \begin{pmatrix} 0.50 & 0.88 & 0.47 & 0.40 \\ 0.12 & 0.50 & 0.18 & 0.96 \\ 0.53 & 0.82 & 0.50 & 0.40 \\ 0.60 & 0.04 & 0.60 & 0.50 \end{pmatrix}$$

Meanwhile the suitable parameters provided in this example are as follows: $\lambda^{inter} = 0.75$ and $\lambda^{intra} = 0.90$. After getting the preferences from the experts and non-experts, the consensus process starts as follows.

Phase 2: Inter-Consensus Reaching Phase

Step (i): First, aggregate the preferences of non-experts using weighted average operator given in Eq. (2.1). The group decision matrix of non-experts CP^{NE} is:

$$CP^{NE} = \begin{pmatrix} 0.50 & 0.57 & 0.61 & 0.37 \\ 0.43 & 0.50 & 0.44 & 0.59 \\ 0.39 & 0.56 & 0.50 & 0.56 \\ 0.63 & 0.41 & 0.44 & 0.50 \end{pmatrix}$$

Step (ii): Using equations (3.11) and (3.4), we calculate the degree of consensus between each expert $e_h \in E$ and the non-experts group decision matrix CP^{NE}, and we have:

$$CD_{1,NE}^{inter} = 0.7851, \qquad CD_{2,NE}^{inter} = 0.6348,$$

$$CD_{3,NE}^{inter} = 0.7049, \qquad CD_{4,NE}^{inter} = 0.6343,$$

$$CD_{5,NE}^{inter} = 0.7735 \qquad CD_{6,NE}^{inter} = 0.6711$$

Based on the results obtained, we find that experts $e_2, e_3, e_4,$ and e_6 are not in consensus with the non-experts group decision. Therefore, the feedback mechanism is applied for the experts with lower consensus degree to adjust the original decision matrix.

Step (iii): Represent the preference value of individual experts' preference and non-experts' preference as a vector point in 6-dimensional space, as listed in Table 3.1.

Table 3.1. Vector Representation of Preferences in Phase 2

	p_{12}	p_{13}	p_{14}	p_{23}	p_{24}	p_{34}
VP^1	0.45	0.38	0.78	0.36	0.53	0.71
VP^2	0.87	0.36	0.65	0.97	0.08	0.59
VP^3	0.41	0.31	0.26	0.76	0.99	0.19
VP^4	0.78	0.19	0.99	0.80	0.42	0.73
VP^5	0.49	0.81	0.36	0.07	0.59	0.91
VP^6	0.19	0.43	0.75	0.04	0.95	0.76
VCP^{NE}	0.57	0.61	0.37	0.44	0.59	0.56

Step (iv): Use Eq. (3.14) to obtain the equation of a (k+1)-sphere with center VCP^{NE} and radius

$$(1-\lambda^{inter}):\left(z_{12}^{h,inter}-0.57\right)^2+\left(z_{13}^{h,inter}-0.61\right)^2+\left(z_{14}^{h,inter}-0.37\right)^2+\left(z_{23}^{h,inter}-0.44\right)^2+\left(z_{24}^{h,inter}-0.59\right)^2+\left(z_{34}^{h,inter}-0.56\right)^2=(1-0.75)^2\times 6$$

Step (v): Use Eq. (3.12) to obtain the vector points on (k+1)-sphere corresponding to each inconsistent expert. The obtained vector point is the point of intersection of the line joining the points VP^h and VCP^{NE} and the (k+1)-sphere. Using Eq. (3.12), for expert e_2, the equation of the line point joining the points VP^2 and VCP^{NE} is calculated as follows:

$$\begin{bmatrix}0.87\\0.36\\0.65\\0.97\\0.08\\0.59\end{bmatrix}+t_2\times\begin{bmatrix}-0.30\\0.25\\-0.38\\0.53\\0.51\\-0.03\end{bmatrix}$$

The above equation can be expanded as follows:

$$z_{12}^{2,inter}=0.87-t_2\times 0.30,\qquad z_{13}^{2,inter}=0.36+t_2\times 0.25,$$

$$z_{14}^{2,inter}=0.65-t_2\times 0.38,\qquad z_{23}^{2,inter}=0.97+t_2\times 0.53,$$

$$z_{24}^{2,inter}=0.08+t_2\times 0.51,\qquad z_{34}^{2,inter}=0.59-t_2\times 0.03,$$

Substitute the above values in the equation of sphere obtained in *step (iv)*, we find the points on sphere for expert e_2 given as: $S_1^{2,inter}$ = (0.7757, 0.4172, 0.5610, 0.8.72, 0.2388, 0.5771) and $S_2^{2,inter}$ = (0.3599, 0.8014, 0.1744, 0.0800, 0.9454, 0.5342). Similarly, for all the inconsistent experts the vector points can be obtained, listed in Table 3.2.

Table 3.2. Feedback Vector of Inconsistent Experts in Phase 2

	p_{12}	p_{13}	p_{14}	p_{23}	p_{24}	p_{34}
$S_1^{2,inter}$	0.7757	0.4172	0.5610	0.8072	0.2388	0.5771
$S_2^{2,inter}$	0.3599	0.8014	0.1744	0.0800	0.9454	0.5342
$S_1^{3,inter}$	0.4374	0.3550	0.2797	0.7106	0.9336	0.2430
$S_2^{3,inter}$	0.6982	0.8636	0.4556	0.1766	0.2507	0.8683
$S_1^{4,inter}$	0.7145	0.3251	0.7971	0.6902	0.4767	0.6747
$S_2^{4,inter}$	0.4211	0.8936	0.0617	0.1971	0.7076	0.4366
$S_1^{6,inter}$	0.2835	0.4748	0.6577	0.1362	0.8614	0.7138
$S_2^{6,inter}$	0.8521	0.7438	0.0777	0.7510	0.3229	0.3975

Step (vi): According to the closeness of the points, the lower and upper bound of the preference value for experts e_1 and e_2 could be determined. For the expert e_2, given range is as follows:

Range of values for p_{12}^2 : $[0.3599, 0.7757]$,

Range of values for p_{13}^2 : $[0.4172, 0.8014]$,

Range of values for p_{14}^2 : $[0.1744, 0.5610]$,

Range of values for p_{23}^2 : $[0.0800, 0.8072]$,

Range of values for p_{24}^2 : $[0.2388, 0.9454]$,

Range of values for p_{34}^2 : $[0.5342, 0.5771]$

Similarly, the range of values of all the inconsistent experts can be derived from Table 3.2. Suppose that all experts accept these suggestions and make the corresponding modifications. Four new preference matrices are given as:

$$\bar{P}^{2,inter} = \begin{pmatrix} 0.50 & 0.67 & 0.57 & 0.48 \\ 0.33 & 0.50 & 0.56 & 0.35 \\ 0.43 & 0.44 & 0.50 & 0.56 \\ 0.52 & 0.65 & 0.44 & 0.50 \end{pmatrix} \qquad \bar{P}^{3,inter} = \begin{pmatrix} 0.50 & 0.49 & 0.45 & 0.35 \\ 0.51 & 0.50 & 0.64 & 0.74 \\ 0.55 & 0.36 & 0.50 & 0.30 \\ 0.65 & 0.26 & 0.70 & 0.50 \end{pmatrix}$$

$$\bar{P}^{4,inter} = \begin{pmatrix} 0.50 & 0.59 & 0.33 & 0.49 \\ 0.41 & 0.50 & 0.52 & 0.55 \\ 0.67 & 0.48 & 0.50 & 0.67 \\ 0.51 & 0.45 & 0.33 & 0.50 \end{pmatrix} \qquad \bar{P}^{6,inter} = \begin{pmatrix} 0.50 & 0.49 & 0.81 & 0.36 \\ 0.51 & 0.50 & 0.07 & 0.59 \\ 0.19 & 0.93 & 0.50 & 0.91 \\ 0.64 & 0.41 & 0.09 & 0.50 \end{pmatrix}$$

Then according to (3.4), the degree of consensus are $CD_{e_2,CP^{NE}}^{inter} = 0.8691$, $CD_{e_3,CP^{NE}}^{inter} = 0.8552$, $CD_{e_4,CP^{NE}}^{inter} = 0.8625$ and $CD_{e_6,CP^{NE}}^{inter} = 0.8910$. Because the $CD_{e_h,CP^{NE}}^{inter} > \lambda^{inter}$ based on the feedback mechanism, the inconsistent experts e_2, e_3, e_4 and e_6 achieve consensus with the non-experts. Thus, all of the experts have reached consensus with the non-experts, so the Intra-consensus reaching process starts to obtain the consensus among experts.

Phase 3: Intra-Consensus Reaching Phase

In this phase, we evaluate the experts' decision after having accepted the non-expert's opinion, in order to help them to achieve consensus among them. The specific evaluation processes are shown as follows.

Step (i): Calculate the group decision matrix of experts GP^E using Eq. (2.1). Here we provide equal weights to each expert.

$$\overline{GP^E} = \begin{pmatrix} 0.50 & 0.54 & 0.52 & 0.48 \\ 0.46 & 0.50 & 0.43 & 0.59 \\ 0.48 & 0.57 & 0.50 & 0.65 \\ 0.52 & 0.41 & 0.35 & 0.50 \end{pmatrix}$$

Step (ii): Calculate the degree of consensus between each expert decision matrix and group decision matrix GP^E, using Eq. (3.17) and (3.18); we have

$$CD^{intra}_{e_1,GP^E} = 0.8520, \quad CD^{intra}_{e_2,GP^E} = 0.8637$$

$$CD^{intra}_{e_3,GP^E} = 0.8296, \quad CD^{intra}_{e_4,GP^E} = 0.9116,$$

$$CD^{intra}_{e_5,GP^E} = 0.7786\,, \quad CD^{intra}_{6,GP^E} = 0.9111.$$

Based on the results in step (ii), we find that, e_1, e_2, e_3 and e_5 do not achieve consensus. Therefore, the feedback mechanism is applied to update their decisions in step (iii).

Table 3.3. Vector Representation of Preferences in Phase 3

	p_{12}	p_{13}	p_{14}	p_{23}	p_{24}	p_{34}
VP^1	0.67	0.57	0.48	0.56	0.34	0.56
$\overline{VP}^2$	0.67	0.57	0.48	0.56	0.34	0.56
$\overline{VP}^3$	0.49	0.45	0.33	0.64	0.74	0.37
$\overline{VP}^4$	0.59	0.33	0.49	0.52	0.55	0.67
VP^5	0.49	0.81	0.36	0.07	0.59	0.91
$\overline{VP}^6$	0.49	0.81	0.36	0.07	0.59	0.91
$\overline{GP}^E$	0.54	0.52	0.48	0.42	0.59	0.65

Step (iii): Represent the preference value of individual experts' preference and group decision as a vector point in 6-dimensional space, as listed in Table 3.3.

Step (iv): Use Eq. (3.21) to generate advice to the experts. For that, we define the equation of sphere H^{intra} where experts' group decision GP^E is center and $(1-\lambda^{intra})$ is radius, as follows using Eq. (3.21).

$$\left(z_{12}^{h,intra}-0.54\right)^2+\left(z_{13}^{h,intra}-0.52\right)^2+\left(z_{14}^{h,intra}-0.48\right)^2+\left(z_{23}^{h,intra}-0.42\right)^2+\left(z_{24}^{h,intra}-0.59\right)^2+\left(z_{34}^{h,intra}-0.65\right)^2=(1-0.90)^2\times 6$$

Step (v): The equation of the line joining the points representing e_1 and GP^E is as follows:

$$z_{12}^{1,intra}=0.67-t_1\times 0.07,\quad z_{13}^{1,intra}=0.57-t_1\times 0.05$$
$$z_{14}^{1,intra}=0.48+t_1\times 0.00,\quad z_{23}^{1,intra}=0.56-t_1\times 0.14$$
$$z_{24}^{2,intra}=0.34+t_1\times 0.25,\quad z_{34}^{1,intra}=0.56-t_1\times 0.09$$

Substitute the above values in the equation of sphere obtained in step (iv), we find the points on sphere for expert e_1 given as: $S_1^{1,intra}$ = (0.4810, 0.4252, 0.6897, 0.3833, 0.5518, 0.6920) and $S_2^{1,intra}$ = (0.6044, 0.6027, 0.2729, 0.4626, 0.6328, 0.6102). Similarly, for all the inconsistent experts the feedback vector can be obtained, listed in Table 3.4.

Table 3.4. Feedback Vector of Inconsistent Experts in Phase 3

	p_{12}	p_{13}	p_{14}	p_{23}	p_{24}	p_{34}
$S_1^{1,intra}$	0.4810	0.4252	0.6897	0.3833	0.5518	0.6920
$S_2^{1,inter}$	0.6044	0.6027	0.2729	0.4626	0.6328	0.6102
$S_1^{2,inter}$	0.6421	0.5603	0.4827	0.5276	0.4105	0.5882
$S_2^{2,inter}$	0.4433	0.4676	0.4799	0.3183	0.7741	0.7141
$S_1^{3,inter}$	0.5132	0.4743	0.3918	0.5514	0.6772	0.4904
$S_2^{3,inter}$	0.5721	0.5535	0.5708	0.2945	0.5073	0.8119
$S_1^{5,inter}$	0.5226	0.6472	0.4249	0.2650	0.5917	0.7682
$S_2^{5,inter}$	0.5627	0.3807	0.5377	0.5810	0.5929	0.5341

The obtained points of intersection will be used to generate advice for the experts e_1, e_2, e_3 and e_5. Based on the distance of the intersecting points, the lower and upper bound of the preference value can be determined using Eq. (3.22) and (3.23). For the expert e_1, given range is as follows:

Range of values for $p_{12}^{1,intra}$: $[0.4810, 0.6044]$

Range of values for $p_{13}^{1,intra}$: $[0.4252, 0.6027]$

Range of values for $p_{14}^{1,intra}$: $[0.2729, 0.6897]$

Range of values for $p_{23}^{1,intra}$: $[0.3833, 0.4626]$

Range of values for $p_{24}^{1,intra}$: $[0.5518, 0.6328]$

Range of values for $p_{34}^{1,intra}$: $[0.6920, 0.6102]$

Similarly, the range of values of all the inconsistent experts can be derived from Table 3.4. Suppose that all experts accept these suggestions and make the corresponding modifications. Four new preference matrices are given as:

$$\bar{P}^{1,intra} = \begin{pmatrix} 0.50 & 0.53 & 0.43 & 0.62 \\ 0.47 & 0.50 & 0.40 & 0.56 \\ 0.57 & 0.60 & 0.50 & 0.65 \\ 0.38 & 0.44 & 0.34 & 0.50 \end{pmatrix} \quad \bar{P}^{2,intra} = \begin{pmatrix} 0.50 & 0.62 & 0.52 & 0.48 \\ 0.37 & 0.50 & 0.49 & 0.57 \\ 0.48 & 0.51 & 0.50 & 0.62 \\ 0.52 & 0.43 & 0.38 & 0.50 \end{pmatrix}$$

$$\bar{P}^{3,intra} = \begin{pmatrix} 0.50 & 0.53 & 0.48 & 0.48 \\ 0.47 & 0.50 & 0.49 & 0.65 \\ 0.52 & 0.51 & 0.50 & 0.50 \\ 0.52 & 0.35 & 0.50 & 0.50 \end{pmatrix} \quad \bar{P}^{5,intra} = \begin{pmatrix} 0.50 & 0.54 & 0.61 & 0.43 \\ 0.46 & 0.50 & 0.35 & 0.59 \\ 0.39 & 0.65 & 0.50 & 0.72 \\ 0.57 & 0.41 & 0.28 & 0.50 \end{pmatrix}$$

With the modified preference matrix of the experts e_1, e_2 and e_3, they reach the consensus.

$$CD_{1,GP}^{intra} = 0.9333,\ CD_{2,E}^{intra} = 0.9570\ CD_{3,E}^{intra} = 0.9356,\ CD_{4,E}^{intra} = 0.9212,\ CD_{5,E}^{intra} = 0.9237 \text{ and } CD_{6,E}^{intra} = 0.9163.$$

Phase 4: Calculating of Final Decision

The obtained preference matrix of all the three experts, we calculate the global collective preference using Eq. (3.24).

$$GP = \begin{pmatrix} 0.50 & 0.56 & 0.49 & 0.49 \\ 0.44 & 0.50 & 0.44 & 0.62 \\ 0.51 & 0.56 & 0.50 & 0.64 \\ 0.51 & 0.38 & 0.36 & 0.50 \end{pmatrix}$$

On applying the existing guided dominance degree (QGDD) operator, we obtain score for each alternative x_i using Eq. (3.25) and the obtained rank is: $x_2 \succ x_1 \succ x_4 \succ x_3$.

3.5 Simulated Experiments

In this section, we design some simulation experiments to verify the validity of the proposed group decision-making model. This section presents the simulation experiment to compare the consensus efficiency of the general CRP discussed in section 2.1.1 in Chapter 2 with the proposed CRP. The initial preferences of experts and non-experts are randomly generated to prevent the inclusion of any bias because no real data set is available for the defined problem. Moreover, the inter-consensus threshold $\lambda^{inter}$and the intra-consensus threshold λ^{intra} are randomly generated in the interval [0.70, 1]. Because there is no unified approach for determining the consensus threshold, a threshold value is decided depending on the When decision-making is critical, the threshold value can be chosen to have a high value like 0.9 or above. If the decision time is limited, i.e., the decision needs to be taken urgently, and the alternative has to be selected quickly, the threshold value can be a lower value such as 0.8 or smaller. Also, a very low consensus value may lead to conflicting opinions and non-cooperative behavior. A higher consensus value may cause a waste of resources because the choice of the final solution will not change after a certain consensus threshold. Hence considering these points regarding the selection of the consensus thresholds (i.e., the inter-consensus and intra-consensus threshold), we performed the simulation experiment for the range of values lying between 0.70 to 1. The idea is to simulate all possible situations, be it a situation where decision-making is critical, or the decision-time is limited. The findings of the simulation method will not get influenced by setting different combinations of the parameter m and n. Additionally, we also emphasize that with larger values of m and n, similar findings can be obtained. For simplicity, we provide equal weight to each expert and equal weight to each non-expert. For convenience, we assume the following three combinations of the parameters m and n: (i). $m = 6, n = 4, o = 8$ and $w_{ne} = (1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8)$ and $w_e = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6)$, (ii). $m = 4, n = 6, o = 8$ and $w_{ne} = (1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8)$ and $w_e = (1/4, 1/4, 1/4, 1/4)$, and (iii). $m = 6, n = 6, o = 8$ and $w_{ne} = (1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8)$ and $w_e = (1/6, 1/6, 1/6, 1/6, 1/6, 1/6)$. To achieve the desire consensus degree λ^{inter} and λ^{intra} for different input parameters, the simulation method is run 1000 times for each of the above combinations of parameters to generate the values of the number of rounds to reach consensus. We may obtain average of different numbers of rounds taken in the runs of 1000 times – this average number of rounds is being plotted against λ^{inter} and λ^{intra} separately to provide clarity.

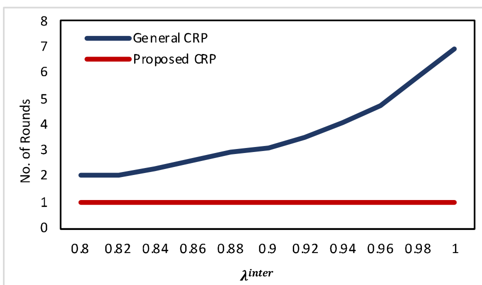
Experiment 1: Effect of change in inter-consensus threshold

To compare the general CRP with the proposed CRP regarding the number of rounds required to achieve the consensus between the experts and the non-experts, we perform the simulation experiment, and the following observations can be drawn. Fig. 3.4 represents the average number of rounds required to reach the consensus threshold λ^{inter} under different combinations of parameters shown in Fig 3.4(a), Fig 3.4(b) and Fig 3.4(c).

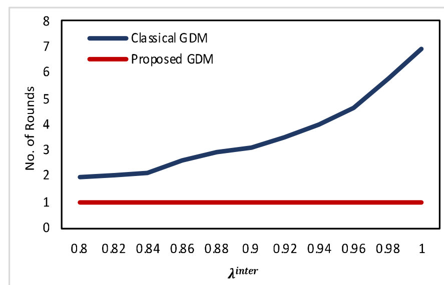
(i) We can observe that in using the general CRP discussed in Chapter 2, the number of rounds to reach consensus increases with the increase in the consensus threshold, whereas it remains constant in the case of the proposed CRP. That means the efficiency of the proposed CRP is better than the efficiency of the general CRP for the inter-consensus reaching process.

Table 3.5. Inter-Consensus level of Experts at different λ^{inter}

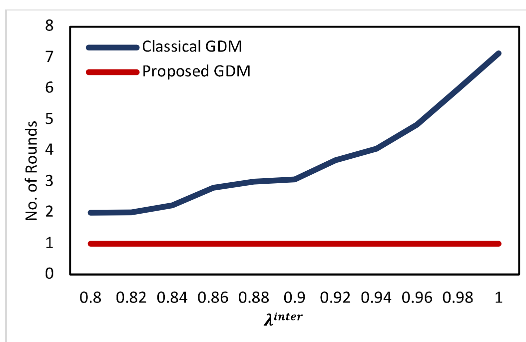
λ^{inter}	e_1	e_2	e_3	e_4	e_5	e_6
INITIAL	0.644	0.594	0.678	0.602	0.792	0.622
0.7	0.843	0.798	0.829	0.788	0.792	0.814
0.75	0.845	0.824	0.918	0.875	0.792	0.871
0.8	0.893	0.964	0.928	0.891	0.91	0.894
0.85	0.887	0.944	0.948	0.901	0.962	0.952
0.9	0.915	0.947	0.931	0.947	0.955	0.932
0.95	0.968	0.962	0.976	0.978	0.966	0.987



(a). $m = 6, n = 4$



(b). $m = 4, n = 6$



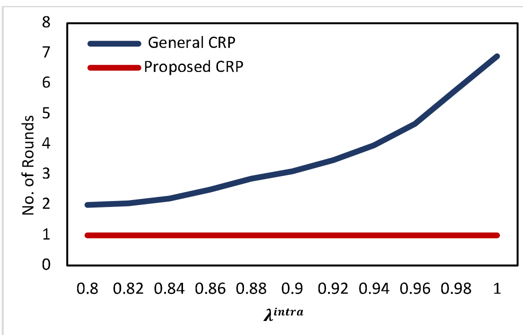
(c). $m = 6, n = 6$

Fig. 3.4: Simulation Analysis: Average No. of Rounds versus λ^{inter}

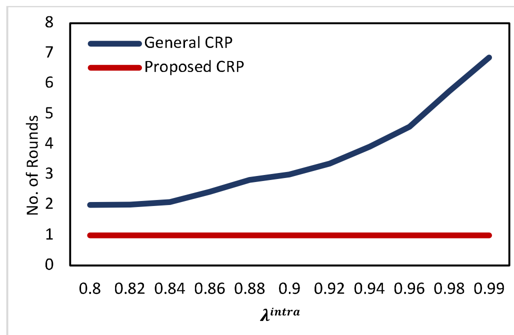
(ii) Table 3.5 presents the consensus value of experts at different values of the inter-consensus threshold λ^{inter}. We found that the experts with lower initial consensus value reach the consensus threshold faster. For instance, expert e_2 with initial consensus value 0.594 reflects 0.798 consensus value at $\lambda^{inter} = 0.7$. Similarly, e_3 with initial consensus value 0.678 reflects 0.829 consensus value at $\lambda^{inter} = 0.7$. Thus, it is indicated that the feedback mechanism in the proposed GDM model converges to consensus more rapidly as it is expected.

Experiment 2: Effect of change in intra-consensus threshold

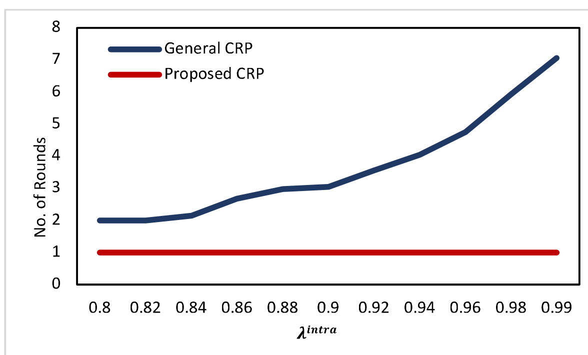
To compare the number of rounds required to achieve the consensus among the experts using the general CRP with the proposed CRP, we perform the simulation experiment, and the following observations can be drawn.



(a). m=6, n=4



(b). m=4, n=6



(c). m=6, n=6

Fig. 3.5: Simulation Analysis: Average No. of Rounds versus λ^{intra}

(i) Fig. 3.5 represents the average number of rounds required to reach the consensus threshold λ^{intra} under different combinations of parameters shown in Fig. 3.5(a), Fig. 3.5(b) and Fig. 3.5(c). We can observe that in the case of the general CRP, the number of rounds to reach consensus increases as the value of λ^{intra} increases, whereas using the proposed CRP, it remains constant. This means that using proposed CRP the consensus can be reached at once, as it is expected. Thus, the efficiency of the proposed CRP is better than the efficiency of the general CRP.

(ii) Table 3.6 presents the consensus value of experts at different values of the inter-consensus threshold λ^{intra}. We found that the experts with lower initial consensus values reach the consensus threshold faster. For instance, expert e_2 with initial consensus value 0.594 reflects 0.798 consensus value at $\lambda^{intra} = 0.7$. Similarly, e_3 with initial consensus value 0.678 reflects 0.829 consensus value at $\lambda^{intra} = 0.7$. Thus, it is indicated the feedback mechanism in the proposed GDM model converges to consensus more rapidly and that too in a single round of feedback than the general CRP.

Table 3.6. Inter-Consensus level of Experts at different λ^{intra}

λ^{intra}	e_1	e_2	e_3	e_4	e_5	e_6
INITIAL	0.7737	0.6578	0.8180	0.7245	0.6587	0.8204
0.7	0.7765	0.7736	0.7794	0.8047	0.7616	0.7551
0.75	0.8402	0.8381	0.8312	0.8376	0.8202	0.8443
0.8	0.8778	0.8634	0.8880	0.8818	0.8670	0.8769
0.85	0.8810	0.8776	0.8995	0.8919	0.8873	0.8747
0.9	0.9193	0.9171	0.9166	0.9162	0.9184	0.9219
0.95	0.9589	0.9581	0.9648	0.9796	0.9520	0.9821

Experiment 3: Effect of change in inter-consensus threshold and intra-consensus threshold

In this experiment, we observed the impact of the inter-consensus threshold and the intra-consensus threshold together on the number of rounds to reach consensus, as shown in Fig. 3.6, and obtained consensus values are shown in Table 3.7. From Fig. 3.6, we can observe that number of rounds to reach consensus increases with an increase in threshold values for the general CRP, whereas it takes a maximum of two rounds to reach consensus for the proposed CRP. This means that in the proposed CRP the expert may take one feedback for inter-consensus reaching process and another one for intra-consensus reaching phase. Thus, we can say that for any value of λ^{inter}, experts always reach λ^{intra} with one round of modification only.

Table 3.7 shows the consensus value of experts for different combinations of λ^{inter} and λ^{intra} thresholds represented in the form of a heat map. The density of colors defines the consensus value of experts. Here, the lighter colour represents the lower consensus value, and the darker colour represents the higher consensus value. Table 3.7 shows that the color of the experts' consensus value is always darker than the given threshold values, which means that for any combination of λ^{inter} and λ^{intra} thresholds, the experts always have achieved consensus irrespective of their initial consensus values. It can also be observed that even when the experts achieve consensus among them, the consensus between experts and non-experts does not get violated for any of the threshold values. A quick visual summary is that the color of the experts' threshold value is always darker than the λ^{inter} and λ^{intra} thresholds. Hence, the proposed GDM model, which promises to achieve consensus with the predefined λ^{inter} and λ^{intra} thresholds, can be verified after looking at the consensus results in Table 3.7.

Table 3.7. Heat Map for Consensus Value of Experts at different λ^{inter} and λ^{intra}

λ^{inter}	λ^{intra}	e_1	e_2	e_3	e_4	e_5	e_6
		0.613	0.631	0.788	0.712	0.773	0.648
0.7	0.85	0.947	0.923	0.911	0.95	0.959	0.963
	0.87	0.929	0.926	0.906	0.921	0.928	0.929
	0.89	0.964	0.925	0.919	0.927	0.938	0.96
	0.91	0.966	0.947	0.984	0.957	0.944	0.974
	0.93	0.961	0.97	0.972	0.944	0.959	0.966
	0.95	0.979	0.973	0.984	0.978	0.969	0.977
	0.97	0.991	0.987	0.989	0.98	0.986	0.987
	0.99	0.995	0.995	0.993	0.993	0.995	0.997
0.75	0.85	0.935	0.908	0.967	0.896	0.941	0.866
	0.87	0.94	0.933	0.917	0.898	0.936	0.92
	0.89	0.936	0.949	0.919	0.959	0.969	0.943
	0.91	0.969	0.96	0.938	0.942	0.945	0.955
	0.93	0.961	0.942	0.965	0.957	0.959	0.977
	0.95	0.968	0.992	0.994	0.972	0.985	0.977
	0.97	0.99	0.983	0.99	0.984	0.987	0.989
	0.99	0.997	0.997	0.996	0.996	0.997	0.993
0.8	0.85	0.894	0.957	0.922	0.916	0.901	0.919

0.700

0.99

	0.87	0.894	0.957	0.922	0.916	0.901	0.919
	0.89	0.894	0.957	0.922	0.916	0.901	0.919
	0.91	0.967	0.945	0.93	0.916	0.931	0.927
	0.93	0.954	0.962	0.955	0.956	0.967	0.974
	0.95	0.99	0.965	0.974	0.983	0.976	0.975
	0.97	0.985	0.98	0.982	0.991	0.986	0.984
	0.99	0.994	0.996	0.996	0.999	0.994	0.996
0.85	0.85	0.917	0.93	0.965	0.924	0.955	0.927
	0.87	0.917	0.93	0.965	0.924	0.955	0.927
	0.89	0.917	0.93	0.965	0.924	0.955	0.927
	0.91	0.917	0.93	0.965	0.924	0.955	0.927
	0.93	0.968	0.937	0.957	0.978	0.97	0.975
	0.95	0.971	0.968	0.973	0.987	0.961	0.963
	0.97	0.985	0.981	0.983	0.984	0.984	0.984
	0.99	0.993	0.994	0.994	0.995	0.994	0.995
0.9	0.85	0.929	0.94	0.925	0.953	0.956	0.941
	0.87	0.929	0.94	0.925	0.953	0.956	0.941
	0.89	0.929	0.94	0.925	0.953	0.956	0.941
	0.91	0.929	0.94	0.925	0.953	0.956	0.941
	0.93	0.948	0.941	0.97	0.953	0.958	0.947
	0.95	0.979	0.974	0.97	0.957	0.958	0.981
	0.97	0.987	0.984	0.986	0.986	0.994	0.992
	0.99	0.998	0.993	0.996	0.994	0.995	0.996
0.95	0.85	0.976	0.966	0.976	0.974	0.971	0.982
	0.87	0.976	0.966	0.976	0.974	0.971	0.982
	0.89	0.976	0.966	0.976	0.974	0.971	0.982
	0.91	0.976	0.966	0.976	0.974	0.971	0.982
	0.93	0.976	0.966	0.976	0.974	0.971	0.982
	0.95	0.976	0.966	0.976	0.974	0.971	0.982
	0.97	0.977	0.987	0.978	0.974	0.971	0.983
	0.99	0.994	0.997	0.996	0.994	0.994	0.996

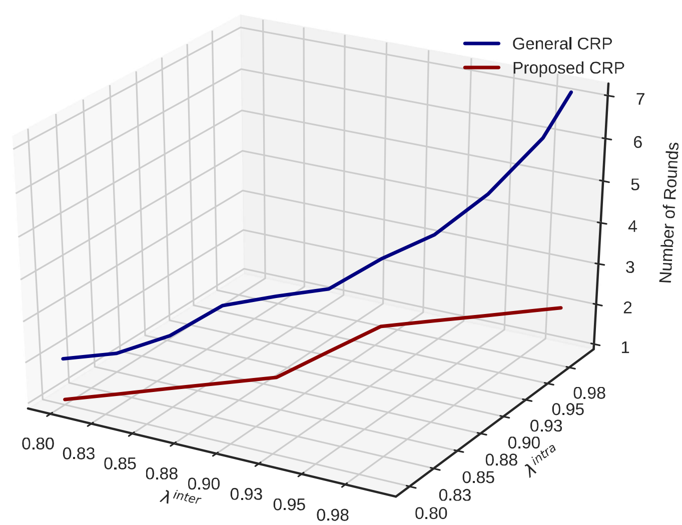


Fig. 3.6: Average no. of Rounds under λ^{inter} and λ^{intra}

Experiment 4: Effect of thresholds on the number of experts involved in feedback process

We observe the number of experts involved in the feedback process for different values λ^{inter} and λ^{intra}, and obtained results are shown in Table 3.8. From Table 3.8, we can observe that the closer the value of λ^{inter} to λ^{intra}, the smaller the number of experts going under the feedback process. For instance, when $\lambda^{inter} = 0.7$, the number of experts who need to modify their preferences increases with an increase in the λ^{intra}, and when we fix the λ^{intra} say at 0.9 then on increasing λ^{inter}, the number of experts undergoing modifications decreases. Thus, at a higher consensus threshold, a large number of experts may need to modify their preferences, which eventually distorts their original opinions. Thus, we need to set the consensus thresholds according to the requirement of the decision problem.

Table 3.8. Number of Experts to be given Feedback

λ^{intra} \ λ^{inter}	0.8	0.82	0.84	0.86	0.88	0.9	0.92	0.94	0.96	0.98	0.99
0.6	5	6	6	6	6	6	6	6	6	6	6
0.65	4	5	5	6	6	6	6	6	6	6	6
0.7	1	1	3	5	5	6	6	6	6	6	6
0.75	0	1	4	4	5	6	6	6	6	6	6
0.8	0	0	0	0	0	1	4	5	6	6	6
0.85	0	0	0	0	0	0	1	4	5	6	6
0.9	0	0	0	0	0	0	0	2	4	6	6
0.95	0	0	0	0	0	0	0	0	0	5	6

3.5.1 Comparison from Theoretical Perspectives

This study mainly contributes to addressing two challenges: insufficiency of only experts, which has been addressed through the involvement of non-experts, particularly citizens in case smart city, and iteration control in CRP, which has been addressed through proposed CRP. The proposed CRP helps improve the efficiency, shown using simulation experiments. In the subsection, we compare our proposed method with existing group decision-making models based on some distinctive criteria, shown in Table 3.9. We first briefly introduce the criteria for comparison analysis and then show the comparison results.

1. **Decision Context:** Though the proposed method is discussed in the context of GDM with only two classes of decision-makers, it can also be used in the LSGDM context with multiple clusters of decision-makers. Thus, the proposed CRP can be discussed in the context of both the GDM and LSGDM.
2. **Definition of Group:** This study considers the heterogeneous group of decisions. In traditional group decision-making, a group of experts participate in solving the decision-making problem. But with the evolution of time, this criterion has been used to identify the decision-making structure.
3. **GDM process:** There are two main methods to measure the consensus among decision-makers. The first is based on the distance from the collective opinions, and the other is based on the pairwise comparison among decision-makers. The proposed method in this work measures consensus based on the distance to the collective preferences.
4. **Feedback Provided:** In the existing literature, two kinds of feedback mechanisms exist that aim to provide preference-modifications suggestions: the identification and direction rule and the feedback mechanism with minimum adjustment or cost (FMMA/C). The FMMA/C aims to minimize the consensus cost by minimizing the adjustment between the original and adjusted preferences.
5. **Number of Rounds (Z):** The number of iterations required to reach the defined consensus degree is an important measure, especially for emergency decision-making. The proposed method undergoes a single round of feedback when only experts are considered, while it takes one more iteration when experts and non-experts both are considered. Thus, the proposed work meets the timeliness

requirements of the decision-making; hence, the proposed CRP is best suitable for the Emergency Decision Making (EDM) situation.

6. **Consensus Cost:** We define the consensus cost in terms of the following: the distance between the original and adjusted preferences denoted by $D(P, \bar{P})$ and the number of rounds to reach the consensus denoted by Z. These two-cost metrics will be used to differentiate the GDM models based on consensus cost.

From Table 3.9. we observe that all discussed GDM models with feedback provided are based either on the minimum cost model or the identification and direction rule. Also, the GDM models discussed in [92] [59] [93] consider experts as the only group of decision-makers, whereas GDM in [46] considers the internal experts and external experts as the group of decision-makers. Also, Tang et al. [46] and [78] implemented a delegation mechanism for LSGDM, allowing external experts to allocate trust weights to the internal expert. This way, the external experts are refrained from participating in the feedback mechanism. Similarly, here in this study, the non-experts' opinion is made use of, but explicitly they are not participating in the decision process. However, in the proposed model, the non-experts' influence in decision-making is controlled by the λ^{inter}. Furthermore, in the case of non-experts being extended as experts, the delegation approach used in [78] and [46] of giving weight may also be considered in the proposed work.

With respect to the consensus cost criterion, we find that the research discussed in the literature mainly focuses minimizing the cost incurred due to the adjustment between the original and final preference, whereas the proposed model minimizes the cost by reducing the number of rounds to reach consensus. It is to note that each model has its own importance. For example, Tang et al. [46] and Labella et al. [92] focussed on minimizing cost based on distance, while our model mainly addressed improving consensus efficiency by minimizing the number of rounds to reach consensus and for heterogeneous decision-makers. Also, the CRP in the proposed model can be very effectively applied in emergency decision situations with the constraint of limited time. Thus, it is too ideal to propose a model that can perform well regarding all perspectives of CRP efficiency.

Table 3.9. Comparison of Proposed GDM Approach with other GDM Approaches based on Distinctive Criteria

	Decision Context	Definition of Group	GDM Process	Feedback Provided	Number of Rounds	Consensus Cost
Labella et al. [92]	GDM and LSGDM	Experts only	Consensus among experts	Minimum cost model	Multiple	Defined by $D(P,\bar{P})$
Wu et al. [59]	LSGDM	Experts Only	Consensus among experts	Minimum cost model	Multiple	Defined by $D(P,\bar{P})$
Tang et al. [46]	LSGDM	Internal Experts and External Experts	Consensus among experts	Double reference consensus (DRC) with control parameter θ	Multiple	Defined by $D(P,\bar{P})$
Xu et al. [78]	LGEDM	Experts only	Consensus among clusters based on exit-delegation mechanism	Identification and Direction rule	Multiple	-
Zha et al. [93]	GDM	Experts Only	Consensus among experts guided by experts	Identification and Direction rule	Multiple	Defined by $D(P,\bar{P})$
Proposed Model	GDM	Experts and Non-Experts	Consensus among experts guided by cumulative feedback by experts and non-experts	Identification and Direction rule	Single	Defined by Z

LGEDM: Large Group Emergency decision-making, LSGDM: Large Scale Group Decision Making

3.6 Summary

Systems such as e-democracy and smart cities consider citizens' support and assistance in decision-making. To achieve this, decision-makers are categorized into two groups. One group is called the experts, and another is called the non-experts consisting of citizens or end-users. The proposed GDM model considers the two phases of the consensus-reaching process. The first phase is the Inter-consensus reaching phase, which establishes the consensus between the experts and the non-experts. The consensus among the experts is achieved in the second phase, called the Intra-consensus reaching phase. In both these phases, the feedback is provided to the experts only, considering that non-experts act as a catalyst promoting the achievement of the consensus without undergoing any change. One threshold is given in each phase that helps control the role of experts and non-experts in the final decision. Our model provides a range of values for each decision-maker to shift within. As they change their preference in the provided range, they are promised to reach a consensus at once. The advantages and limitations of the proposed GDM model are as follows.

Advantages:

a. The proposed GDM model is equally effective in reaching consensus for the classical GDM (as discussed in Chapter 2) when only experts are considered and for emergency decision making, which is to be done in a limited time.

b. With this proposed model, we overcome the challenge of fixing the maximum number of consensus rounds to reach the global consensus by providing the way to generate a customized range of advice that will be accurate enough to reach consensus at once for the experts with lower consensus degree.

c. The proposed model proposes a feedback mechanism that generates customized advice to the experts on each preference value, based on the initial value provided.

Further, we point out the following drawbacks that future works will address.

a. The proposed model considers homogeneous preference structure but not heterogeneous preference structure.

b. In general, in GDM, there may be experts who may factually lack cooperation from each other for convergence because of reluctance to accept the advice provided. Therefore, this remains an issue for this proposed model particularly and any GDM in general.

c. This model does not deal with the misleading behavior of non-experts. In future work, we propose to incorporate a mechanism identifying them as outliers and managing those outliers so that their input does not affect the final decision.

d. In the proposed model, at most, one feedback round is required in the inter-consensus and intra-consensus phases. Therefore, it lacks providing enough flexibility for the experts. The experts do not have enough degree of freedom to adjust their opinions. They are bound to adjust their opinion over the given range of advice if they accept it.