Chapter 3

Effects of heterogeneous impulses on synchronization of complex-valued neural networks with mixed time-varying delays

3.1 Introduction

In this chapter, the Complex-valued neural networks (CVNNs) have been analysed. Due to complicated dynamics, one neuron CVNN solves the XOR and the detection symmetry problem which can not be solved by one neuron real-valued neural network (RVNN). Therefore, the dynamics of CVNNs have been investigated innumerable times in research publications [60, 61, 62, 63, 64, 65, 66]. There are two types of complex-valued activation functions (CVAFs), which are usually considered in literature [60, 61, 62, 63, 64, 65, 66, 67, 68]. Type I contains a class of CVAFs whose real and imaginary parts are dependent on two variables. Type II contains those CVAFs whose real and imaginary parts are dependent on single variable. In [64], Type II CVAFs have been considered for investigating global stability of complex-valued recurrent neural networks using a matrix measure approach. In [67], the stability analysis of CVNNs with Type I CVAFs has been done by employing a nonlinear measure mathematical technique. To find more general results, both types of CVAFs have been considered with CVNNs to investigate synchronization problem in the research articles [60, 62, 63, 68, 66, 61].

Impulsive control is an efficient and economic control strategy compared to the other control strategies as it controls the state of system only at certain discrete moments of impulsive sequence. Impulses are usually categorized into two categories: (i) synchronizing/stabilizing impulses ($|\mu| < 1$) which can enhance synchronization, (ii) desynchronizing/destabilizing impulses ($|\mu| > 1$) which can destroy synchronization. The impulsive sequence which can allow synchronizing and desynchronizing impulses in the same sequence is called hybrid impulsive sequence. There are many results related to the implications of hybrid impulses on synchronization of neural networks, see [23, 69, 60, 70, 71]. All existing results are confined to the impulsive effect which varies only in time domain. In practical applications, each node of complex neural networks or multi-agent systems has specific dynamics. Thus, impulsive effects in each node of neural networks are not only different in time domain but also non-identical in space domain. These types of impulses are called heterogeneous impulses. Motivated by the above discussions, the focus is on the exponential synchronization of CVNNs with mixed time-varying delays and heterogeneous impulses. The concepts of AII and AIG are implemented to cope with the supremum/infimum of impulsive intervals and the simultaneous existence of synchronizing and desynchronizing heterogeneous impulses, respectively. A heterogeneous impulsive controller is deesigned to achieve the exponential synchronization of the CVNNs by using the matrix measure method together with Halanay inequality.

3.2 Model Description and Preliminaries

In this article, we consider the following Complex-valued neural network (CVNN) with discrete and distributed time-varying delays as the master system

$$\dot{w}(t) = -Cw(t) + Af(w(t)) + Bg(w(t-\sigma_1(t))) + D\int_{t-\sigma_2(t)}^{t} h(w(s))ds + I(t), \tag{3.1}$$

where $w(t) = [w_1(t), w_2(t), ..., w_n(t)]^T \in \mathbb{C}^n$ refers to the state vector of n neurons; $C = \text{diag}\{c_1, c_2, ..., c_n\} \in \mathbb{R}^{n\times n}$ denotes the self-feedback connection weights matrix with $c_i > 0$; $A = [a_{ij}]_{n\times n} \in \mathbb{C}^{n\times n}$, $B = [b_{ij}]_{n\times n} \in \mathbb{C}^{n\times n}$ and $D = [d_{ij}]_{n\times n} \in \mathbb{C}^{n\times n}$ denote connection weights matrices for $i,\ j = 1, 2, ..., n$; $f(.) = [f_1(.), f_2(.), ..., f_n(.)]^T : \mathbb{C}^n \to \mathbb{C}^n$, $g(.) = [g_1(.), g_2(.), ..., g_n(.)]^T : \mathbb{C}^n \to \mathbb{C}^n$, and $h(.) = [h_1(.), h_2(.), ..., h_n(.)]^T : \mathbb{C}^n \to \mathbb{C}^n$ are the complex-valued nonlinear activation functions; $\sigma_1(t)$ and $\sigma_2(t)$ are the time-varying delays satisfying $0 \le \sigma_1(t) \le \sigma_1$ and $0 \le \sigma_2(t) \le \sigma_2$, respectively, where $\sigma = \max\{\sigma_1, \sigma_2\}$; $I(t) = [I_1(t), I_2(t), ..., I_n(t)]^T$ denotes an external input vector of the network.

Corresponding to the master system (3.1), the slave system is given as

$$\dot{\hat{w}}(t) = -C\hat{w}(t) + Af(\hat{w}(t)) + Bg(\hat{w}(t-\sigma_1(t))) + D\int_{t-\sigma_2(t)}^{t} h(\hat{w}(s))ds + I(t) + U(t), \tag{3.2}$$

where $U(t) = [U_1(t), U_2(t), ..., U_n(t)]^T$ is a control function to be defined later.

The complex vectors w and $\hat{w}$ can be written as $w = w^R + iw^I$ and $\hat{w} = \hat{w}^R + i\hat{w}^I$, where $w^R, w^I, \hat{w}^R, \hat{w}^I \in \mathbb{R}^n$ are real and imaginary parts.

Complex-valued functions can be of two types. Type I contains the functions whose real and imaginary parts are of two variables. Type II contains the functions whose real and imaginary parts are of one variable. In this chapter, both types of complex-valued activation functions have been considered. Type I CVAFs $f(.)$, $g(.)$, and $h(.)$ can be written as

$$f(w) = f^R(w^R, w^I) + if^I(w^R, w^I), \tag{3.3}$$

$$g(w) = g^R(w^R, w^I) + ig^I(w^R, w^I), \tag{3.4}$$

$$h(w) = h^R(w^R, w^I) + ih^I(w^R, w^I), \tag{3.5}$$

where $f^R(.\ ,.) = [f_1^R, f_2^R, ..., f_n^R]^T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $f^I(.\ ,.) = [f_1^I, f_2^I, ..., f_n^I]^T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $g^R(.\ ,.) = [g_1^R, g_2^R, ..., g_n^R]^T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $g^I(.\ ,.) = [g_1^I, g_2^I, ..., g_n^I]^T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $h^R(.\ ,.) = [h_1^R, h_2^R, ..., h_n^R]^T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, and $h^I(.\ ,.) = [h_1^I, h_2^I, ..., h_n^I]^T : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are real valued functions which satisfy the following assumptions.

Assumption 3. For all $w_j^R,\ w_j^I,\ \hat{w}_j^R,\ \hat{w}_j^I \in \mathbb{R}, j = 1, 2, ..., n$, the real and imaginary parts of CVAFs satisfy Lipschitz condition as

$$|f_j^R(\hat{w}_j^R, \hat{w}_j^I) - f_j^R(w_j^R, w_j^I)| \leq L_{qj}^{RR}|\hat{w}_j^R - w_j^R| + L_{qj}^{RI}|\hat{w}_j^I - w_j^I|,$$
$$|f_j^I(\hat{w}_j^R, \hat{w}_j^I) - f_j^I(w_j^R, w_j^I)| \leq L_{qj}^{IR}|\hat{w}_j^R - w_j^R| + L_{qj}^{II}|\hat{w}_j^I - w_j^I|,$$
$$|g_j^R(\hat{w}_j^R, \hat{w}_j^I) - g_j^R(w_j^R, w_j^I)| \leq M_{qj}^{RR}|\hat{w}_j^R - w_j^R| + M_{qj}^{RI}|\hat{w}_j^I - w_j^I|,$$
$$|g_j^I(\hat{w}_j^R, \hat{w}_j^I) - g_j^I(w_j^R, w_j^I)| \leq M_{qj}^{IR}|\hat{w}_j^R - w_j^R| + M_{qj}^{II}|\hat{w}_j^I - w_j^I|,$$
$$|h_j^R(\hat{w}_j^R, \hat{w}_j^I) - h_j^R(w_j^R, w_j^I)| \leq H_{qj}^{RR}|\hat{w}_j^R - w_j^R| + H_{qj}^{RI}|\hat{w}_j^I - w_j^I|,$$
$$|h^I(\hat{w}_j^R, \hat{w}_j^I) - h^I(w_j^R, w_j^I)| \leq H_{qj}^{IR}|\hat{w}_j^R - w_j^R| + H_{qj}^{II}|\hat{w}_j^I - w_j^I|,$$

where $L_q^{RR} = \max_{1\leq j\leq n}\{L_{qj}^{RR}\},\ L_q^{RI} = \max_{1\leq j\leq n}\{L_{qj}^{RI}\},\ L_q^{IR} = \max_{1\leq j\leq n}\{L_{qj}^{IR}\},\ L_q^{II} = \max_{1\leq j\leq n}\{L_{qj}^{II}\},\ M_q^{RR} = \max_{1\leq j\leq n}\{M_{qj}^{RR}\},\ M_q^{RI} = \max_{1\leq j\leq n}\{M_{qj}^{RI}\},$
$M_q^{IR} = \max_{1\leq j\leq n}\{M_{qj}^{IR}\},\ M_q^{II} = \max_{1\leq j\leq n}\{M_{qj}^{II}\},\ H_q^{RR} = \max_{1\leq j\leq n}\{H_{qj}^{RR}\},\ H_q^{RI} = \max_{1\leq j\leq n}\{H_{qj}^{RI}\},\ H_q^{IR} = \max_{1\leq j\leq n}\{H_{qj}^{IR}\}$, and $H_q^{II} = \max_{1\leq j\leq n}\{H_{qj}^{II}\}$.

Assumption 4. Type II of the CVAFs can be written as

$$f_j(w) = f_j^R(w_j^R) + if_j^I(w_j^I), \quad g_j(w) = g_j^R(w_j^R) + ig_j^I(w_j^I), \quad h_j(w) = h_j^R(w_j^R) + ih_j^I(w_j^I),$$

where $f_j^R(.),\ f_j^I(.) : \mathbb{R} \to \mathbb{R},\ g_j^R(.),\ g_j^I(.) : \mathbb{R} \to \mathbb{R},$ and $h_j^R(.),\ h_j^I(.) : \mathbb{R} \to \mathbb{R}$ which satisfy the following inequalities for all $w_1^R,\ w_2^R,\ w_1^I,\ w_2^R \in \mathbb{R}$ as

$$\begin{aligned}
&0 \leq \frac{f_j^R(w_1^R) - f_j^R(w_2^R)}{w_1^R - w_2^R} \leq L_{qj}^{RR}, \qquad 0 \leq \frac{f_j^I(w_1^I) - f_j^I(w_2^I)}{w_1^I - w_2^I} \leq L_{qj}^{II},\\
&0 \leq \frac{g_j^R(w_1^R) - g_j^R(w_2^R)}{w_1^R - w_2^R} \leq M_{qj}^{RR}, 0 \leq \frac{g_j^I(w_1^I) - g_j^I(w_2^I)}{w_1^I - w_2^I} \leq M_{qj}^{II},\\
&0 \leq \frac{h_j^R(w_1^R) - h_j^R(w_2^R)}{w_1^R - w_2^R} \leq H_{qj}^{RR}, \qquad 0 \leq \frac{h_j^I(w_1^I) - h_j^I(w_2^I)}{w_1^I - w_2^I} \leq H_{qj}^{II},
\end{aligned}$$

where $L_{qj}^{RR},\ L_{qj}^{II},\ M_{qj}^{RR},\ M_{qj}^{II},\ H_{qj}^{RR}$, and H_{qj}^{II} are positive constants for all $j = 1, 2, ..., n$.

Separating into the real and imaginary parts, the systems (3.1) and (3.2) can be written as

$$\begin{cases} \dot{w}^R(t) = & -Cw^R(t) + A^R f^R(w^R(t), w^I(t)) - A^I f^I(w^R(t), w^I(t)) \\ & +B^R g^R(w^R(t-\sigma_1(t)), w^I(t-\sigma_1(t))) - B^I g^I(w^R(t-\sigma_1(t)), w^I(t-\sigma_1(t))) \\ & +D^R \int_{t-\sigma_2(t)}^{t} h^R(w^R(s), w^I(s))ds - D^I \int_{t-\sigma_2(t)}^{t} h^I(w^R(s), w^I(s))ds + I^R(t), \\ \dot{w}^I(t) = & -Cw^I(t) + A^R f^I(w^R(t), w^I(t)) + A^I f^R(w^R(t), w^I(t)) \\ & +B^R g^I(w^R(t-\sigma_1(t)), w^I(t-\sigma_1(t))) + B^I g^R(w^R(t-\sigma_1(t)), w^I(t-\sigma_1(t))) \\ & +D^R \int_{t-\sigma_2(t)}^{t} h^I(w^R(s), w^I(s))ds + D^I \int_{t-\sigma_2(t)}^{t} h^R(w^R(s), w^I(s))ds + I^I(t), \\ \dot{\hat{w}}^R(t) = & -C\hat{w}^R + A^R f^R(\hat{w}^R(t), \hat{w}^I(t)) - A^I f^I(\hat{w}^R(t), \hat{w}^I(t)) \\ & +B^R g^R(\hat{w}^R(t-\sigma_1(t)), \hat{w}^I(t-\sigma_1(t))) - B^I g^I(\hat{w}^R(t-\sigma_1(t)), \hat{w}^I(t-\sigma_1(t))) \\ & +D^R \int_{t-\sigma_2(t)}^{t} h^R(\hat{w}^R(s), \hat{w}^I(s))ds - D^I \int_{t-\sigma_2(t)}^{t} h^I(\hat{w}^R(s), \hat{w}^I(s))ds + I^R(t) + U^R(t), \\ \dot{\hat{w}}^I(t) = & -C\hat{w}^I + A^R f^I(\hat{w}^R(t), \hat{w}^I(t)) + A^I f^R(\hat{w}^R(t), \hat{w}^I(t)) \\ & +B^R g^I(\hat{w}^R(t-\sigma_1(t)), \hat{w}^I(t-\sigma_1(t))) + B^I g^R(\hat{w}^R(t-\sigma_1(t)), \hat{w}^I(t-\sigma_1(t))) \\ & +D^R \int_{t-\sigma_2(t)}^{t} h^I(\hat{w}^R(s), \hat{w}^I(s))ds - D^I \int_{t-\sigma_2(t)}^{t} h^R(\hat{w}^R(s), \hat{w}^I(s))ds + I^I(t) + U^I(t), \end{cases} \tag{3.6}$$

where $w^R(t) = Re\big(w(t)\big) = [w_1^R(t), w_2^R(t), ..., w_n^R(t)]^T$, $w^I(t) = Im\big(w(t)\big) = [w_1^I(t), w_2^I(t), ..., w_n^I(t)]^T$, $\hat{w}^R(t) = Re\big(\hat{w}(t)\big) = [\hat{w}_1^R(t), \hat{w}_2^R(t), ..., \hat{w}_n^R(t)]^T$, $\hat{w}^I(t) = Im\big(\hat{w}(t)\big) = [\hat{w}_1^I(t), \hat{w}_2^I(t), ..., \hat{w}_n^I(t)]^T$, $f^R(.,\ .) = Re\big(f(.)\big)$, $f^I(.,\ .) = Im\big(f(.)\big)$, $g^R(.,\ .) = Re\big(g(.)\big)$, $g^I(.,\ .) = Im\big(g(.)\big)$, $h^R(.,\ .) = Re\big(h(.)\big)$, $g^I(.,\ .) = Im\big(g(.)\big)$, $A^R = Re\big(A\big) = [a_{ij}^R]_{n\times n}$, $A^I = Im\big(A\big) = [a_{ij}^I]_{n\times n}$, $B^R = Re\big(B\big) = [b_{ij}^R]_{n\times n}$, $B^I = Im\big(B\big) = [b_{ij}^I]_{n\times n}$, $D^R = Re\big(D\big) = [d_{ij}^R]_{n\times n}$, $D^I = Im\big(D\big) = [d_{ij}^I]_{n\times n}$, $I^R(t) = Re\big(I(t)\big)$, $I^I = Im\big(I(t)\big)$, $U^R(t) = Re\big(U(t)\big)$, and $U^I(t) = Im\big(U(t)\big)$.

In order to synchronize the slave system (3.2) with the master system (3.1), an impulsive controller $U(t)$ is designed as

$$U_i(t) = \sum_{k=1}^{\infty} \gamma_{ik}\big(\hat{w}_i(t_k^-) - w_i(t_k^-)\big)\delta(t - t_k), \quad i = 1, 2, ...n, \tag{3.7}$$

where $\gamma_{ik} \in \mathbb{R}$ be the impulsive strength at the impulse time instant t_k, $k \in \mathbb{N}$ of the impulsive sequence $\mathcal{S} = \{t_k : k \in \mathbb{N}\}$ satisfying $0 = t_0 < t_1 < t_2 < ... < t_k < t_{k+1} < ...$, and $t_k \to \infty$ as $k \to \infty$. The suffix i in γ_{ik} corresponds to the i^{th} neuron of the neural network, i.e., the impulsive strength not only varies with impulsive time t_k but also with the states of the systems. $\delta(t)$ represents the Dirac delta function. The master-slave systems (3.1) and (3.2) are transformed into a complex-valued error system $\chi(t) = \hat{w}(t) - w(t)$, where $\chi(t) = [\chi_1(t), \chi_2(t), ..., \chi_n(t)]^T = \chi^R(t) + i\chi^I(t)$. The real and imaginary parts of the complex-valued error system $\chi(t)$ are defined as $\chi^R(t) = \hat{w}^R(t) - w^R(t)$ and $\chi^I(t) = \hat{w}^I(t) - w^I(t)$, which can be written in the following form using equations (3.6) and (3.7).

$$\begin{cases} \dot{\chi}^R(t) = & -C\chi^R(t) + A^R\hat{f}^R\big(\chi^R(t), \chi^I(t)\big) - A^I\hat{f}^I\big(\chi^R(t), \chi^I(t)\big) \\ & +B^R\hat{g}^R\big(\chi^R(t-\sigma_1(t)), \chi^I(t-\sigma_1(t))\big) - B^I\hat{g}^I\big(\chi^R(t-\sigma_1(t)), \chi^I(t-\sigma_1(t))\big) \\ & +D^R\int_{t-\sigma_2(t)}^{t}\hat{h}^R\big(\chi^R(s), \chi^I(s)\big)ds - D^I\int_{t-\sigma_2(t)}^{t}\hat{h}^I\big(\chi^R(s), \chi^I(s)\big)ds, \quad t \neq t_k, \\ \chi^R(t_k) = & \mathbf{\Gamma}_k\chi^R(t_k^-), \quad t = t_k, \\ \dot{\chi}^I(t) = & -C\chi^I(t) + A^R\hat{f}^I\big(\chi^R(t), \chi^I(t)\big) + A^I\hat{f}^R\big(\chi^R(t), \chi^I(t)\big) \\ & +B^R\hat{g}^I\big(\chi^R(t-\sigma_1(t)), \chi^I(t-\sigma_1(t))\big) + B^I\hat{g}^R\big(\chi^R(t-\sigma_1(t)), \chi^I(t-\sigma_1(t))\big) \\ & +D^R\int_{t-\sigma_2(t)}^{t}\hat{h}^I\big(\chi^R(s), \chi^I(s)\big)ds + D^I\int_{t-\sigma_2(t)}^{t}\hat{h}^R\big(\chi^R(s), \chi^I(s)\big)ds, \quad t \neq t_k, \\ \chi^I(t_k) = & \mathbf{\Gamma}_k\chi^I(t_k^-), \quad t = t_k, \end{cases} \tag{3.8}$$

where $\hat{f}^R\big(\chi^R(t),\chi^I(t)\big) = f^R\big(\hat{w}^R(t),\hat{w}^I(t)\big) - f^R\big(w^R(t),w^I(t)\big)$, $\hat{f}^I\big(\chi^R(t),\chi^I(t)\big) = f^I\big(\hat{w}^R(t),\hat{w}^I(t)\big) - f^I\big(w^R(t),w^I(t)\big)$, $\hat{g}^R\big(\chi^R(t-\sigma_1(t)),\chi^I(t-\sigma_1(t))\big) = g^R\big(\hat{w}^R(t-\sigma_1(t)),\hat{w}^I(t-\sigma_1(t))\big) - g^R\big(w^R(t-\sigma_1(t)),w^I(t-\sigma_1(t))\big)$, $\hat{g}^I\big(\chi^R(t-\sigma_1(t)),\chi^I(t-\sigma_1(t))\big) = g^I\big(\hat{w}^R(t-\sigma_1(t)),\hat{w}^I(t-\sigma_1(t))\big) - g^I\big(w^R(t-\sigma_1(t)),w^I(t-\sigma_1(t))\big)$, $\hat{h}^R\big(\chi^R(s),\chi^I(s)\big) = h^R\big(\hat{w}^R(s),\hat{w}^I(s)\big) - h^R\big(w^R(s),w^I(s)\big)$, and $\hat{h}^I\big(\chi^R(s),\chi^I(s)\big) = h^I\big(\hat{w}^R(s),\hat{w}^I(s)\big) - h^I\big(w^R(s),w^I(s)\big)$. It is assumed that the state variables of the impulsive dynamical system (3.8) satisfies $\chi^R(t_k^+) = \chi^R(t_k) \neq \chi^R(t_k^-)$, and $\chi^I(t_k^+) = \chi^I(t_k) \neq \chi^I(t_k^-)$ for all $t_k \in \mathcal{S}$, i.e., their trajectories exhibit a jump kind of discontinuity from the left side and continuity from the right at t_k. $\Gamma_k = \text{diag}\{\mu_{1k},\mu_{2k},...,\mu_{nk}\}$, where $\mu_{ik} = 1+\gamma_{ik}$, $i = 1,2,...,n, k \in \mathbb{N}$ and the initial conditions of real and imaginary parts of the error system (3.8) are $\chi^R(s) = \phi^R(s) \in \text{PC}\big([t_0-\sigma,t_0],\mathbb{R}^n\big)$ and $\chi^I(s) = \varphi^I(s) \in \text{PC}\big([t_0-\sigma,t_0],\mathbb{R}^n\big)$, respectively.

Remark 3.2.1. The impulses γ_{ik} considered in the impulsive controller (3.7) are not only varying in time domain but also in space domain, i.e., impulsive effects γ_{ik} will be different for distinct impulse moments $t_k \in \mathcal{S}$ and state variables χ_i^R or χ_i^I, $i = 1,2,...,n$. Such types of impulses are called heterogeneous impulse which were first proposed in [72]. The heterogeneous impulses encompass two different kinds of impulsive effects. If $\gamma_{ik} = \gamma_k$, then the heterogeneous impulses are characterized as time-varying impulses which have been extensively investigated in [60, 69, 73, 70]. If $\gamma_{ik} = \gamma_i$, then the heterogeneous impulses are called nonidentical impulses, i.e., the impulses occuring on the distinct nodes of the network are distinct. Therefore, our results on the exponential synchronization of CVNN systems demonstrate more general kind of impulsive effects than the existing ones [60, 69, 73, 70, 62] .

Remark 3.2.2. The heterogeneous impulses γ_{ik} can be characterized into three categories according to their behaviour in the states of the systems. If $|\gamma_{ik}| > 1$, then the impulses enlarge the absolute value of the state at impulsive points and do not favour synchronization process, hence, these types of impulses are called desynchronizing impulses. If $|\gamma_{ik}| < 1$, then the impulses shorten the absolute value of the state and favour synchronization process, hence, the impulses are called synchronizing impulses. If $|\gamma_{ik}| = 1$, then the impulsive effects on the states of system are neutral, i.e., neither favour nor disfavour synchronization process, hence, the impulses are called inactive impulses. Our main concern in this chapter is to study the simultaneous effects of synchronizing and desynchronizing heterogeneous impulses on the synchronization of CVNN systems having mixed time-varying delays.

Lemma 3.1. *[74] The useful properties of the matrix measure $\mu_q(.)$ $(q = 1, 2, \infty)$ with respect to the induced matrix norm $\|.\|_q$ on $\mathbb{R}^{n\times n}$ can be described in the following points as*

(i) $-\|\mathcal{W}\|_q \leq \mu_q(\mathcal{W}) \leq \|\mathcal{W}\|_q$;

(ii) In general $\mu_q(-\mathcal{W}) \neq \mu_q(\mathcal{W})$, *but* $\mu_q(\gamma\mathcal{W}) = \gamma\mu_q(\mathcal{W})$, $\forall\gamma \geq 0$ *and* $\forall\mathcal{W} \in \mathbb{R}^{n\times n}$;

(iii) The matrix measure $\mu_q(.)$ *is a convex function, i.e.,* $\mu_q(\gamma\mathcal{A} + (1-\gamma)\mathcal{B}) \leq \gamma\mu_q(\mathcal{A}) + (1-\gamma)\mu_q(\mathcal{B})$, $\forall\gamma \in [0, 1]$, $\forall\mathcal{A}, \mathcal{B} \in \mathbb{R}^{n\times n}$;

(iv) $max\{\mu_q(\mathcal{A}) - \mu_q(-\mathcal{B}),\ \mu_q(\mathcal{B}) - \mu_q(-\mathcal{A})\} \leq \mu_q(\mathcal{A}+\mathcal{B}) \leq \mu_q(\mathcal{A}) + \mu_q(\mathcal{B})$, $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{n\times n}$.

Lemma 3.2. *[75] Let* $\xi_1 \in \mathbb{R}$, $\xi_2 \geq 0$, $\delta > 1$, $b_k \geq \frac{1}{\delta}$ *are real constants and* $0 \leq W(t) \in PC(\mathbb{R},\ \mathbb{R}_+)$ *such that*

$$\begin{cases} D^+(W(t)) \leq \xi_1 W(t) + \xi_2 \sup_{t-\sigma\leq s\leq t} W(s), \quad t \neq t_k, k \in \mathbb{N}, \\ W(t_k^+) \leq b_k W(t_k^-), \quad t = t_k, \\ W(s) = H(s), s \in [-\sigma,\ 0]. \end{cases} \tag{3.9}$$

Further, suppose $\xi_1 + \xi_2\delta - \frac{\ln\delta}{\varsigma} < 0$*, where* $\varsigma = \sup_{k\in\mathbb{N}}\{t_{k+1} - t_k\} < \infty$ *and* $0 < \lambda < \frac{\ln\delta}{\varsigma} - \xi_1 - \xi_2\delta e^{\lambda\sigma}$*. If there exist* $\eta \geq 0$ *and* $\mathfrak{M} \geq \delta$ *such that* $\delta^{k+1}\prod_{t_0<t_k<t} b_k \leq \mathfrak{M}e^{\eta(t_k-t_0)}$*,* $\forall\ k \in \mathbb{N}$*, then* $W(t) \in PC(\mathbb{R},\ \mathbb{R}_+)$ *is estimated as*

$$W(t) \leq \mathfrak{M} \sup_{t_0-\sigma\leq s\leq t_0} W(s)e^{-(\lambda-\eta)(t-t_0)},\ \forall t \geq t_0. \tag{3.10}$$

Proof. The detailed proof of Lemma 3.2 can be found in [75]. □

Remark 3.2.3. The results in Lemma 3.2 are extension of the lemmas used in the previous results [60, 61, 62]. In Lemma 3.2, inequality (3.10) is true for all $\xi_1 \in \mathbb{R}$ and $\xi_2 \geq 0$ whereas the previous results given in [60, 61, 62] are only true for $-\xi_1 < -\xi_2 \leq 0$. Therefore, our results are less conservative.

Remark 3.2.4. The inequality $\xi_1 + \xi_2\delta - \frac{\ln\delta}{\varsigma} < 0$ implies that there exists $\lambda \in \mathbb{R}$ such that $0 < \lambda < \frac{\ln\delta}{\varsigma} - \xi_1 - \xi_2\delta e^{\lambda\sigma}$ holds. For this, assume $\xi_1 + \xi_2\delta - \frac{\ln\delta}{\varsigma} < 0$ and define $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\phi(x) = \frac{\ln\delta}{\varsigma} - \xi_1 - \xi_2\delta e^{x\sigma} - x,\quad \phi(0) = \frac{\ln\delta}{\varsigma} - \xi_1 - \xi_2\delta > 0,$$

then there exists $\epsilon > 0$ such that $\phi(x) > 0$ for all $x \in (-\epsilon,\ \epsilon)$. One needs to just pick $\lambda \in (0,\ \epsilon)$ to verify $0 < \lambda < \frac{\ln\delta}{\varsigma} - \xi_1 - \xi_2\delta e^{\lambda\sigma}$.

Lemma 3.3. *[61] (C_q inequality) For all* $u, v \in \mathbb{R},\ q > 0$*, the inequality* $(|u|+|v|)^q \leq C_q(|u|^q + |v|^q)$ *holds, where* $C_q = \begin{cases} 1, & 0 < q \leq 1, \\ 2^{q-1}, & q > 1. \end{cases}$

The impulsive neural network (3.8) can be written in a compact form as

$$\begin{aligned}\dot{\mathcal{E}}(t) =& -\bar{C}\mathcal{E}(t) + \bar{A}_1\bar{f}_1(\mathcal{E}(t)) + \bar{A}_2\bar{f}_2(\mathcal{E}(t)) + \bar{B}_1\bar{g}_1(\mathcal{E}(t-\sigma_1(t))) + \bar{B}_2\bar{g}_2(\mathcal{E}(t-\sigma_1(t))) \\ &+ \bar{D}_1\int_{t-\sigma_2(t)}^{t}\bar{h}_1(\mathcal{E}(s))ds + \bar{D}_2\int_{t-\sigma_2(t)}^{t}\bar{h}_2(\mathcal{E}(s))ds, \quad t \neq t_k, \\ \mathcal{E}(t_k) =& \bar{\mathbf{\Gamma}}_k\mathcal{E}(t_k^-), \quad t = t_k, \end{aligned} \tag{3.11}$$

where $\mathcal{E}(t) = \begin{pmatrix}\chi^R(t) \\ \chi^I(t)\end{pmatrix}$, $\bar{C} = \begin{pmatrix}C & 0 \\ 0 & C\end{pmatrix}$, $\bar{A}_1 = \begin{pmatrix}A^R & 0 \\ 0 & A^I\end{pmatrix}$, $\bar{A}_2 = \begin{pmatrix}-A^I & 0 \\ 0 & A^R,\end{pmatrix}$, $\bar{B}_1 = \begin{pmatrix}B^R & 0 \\ 0 & B^I\end{pmatrix}$, $\bar{B}_2 = \begin{pmatrix}-B^I & 0 \\ 0 & B^R\end{pmatrix}$, $\bar{D}_1 = \begin{pmatrix}D^R & 0 \\ 0 & D^I\end{pmatrix}$, $\bar{D}_2 = \begin{pmatrix}-D^I & 0 \\ 0 & D^R\end{pmatrix}$, $\bar{f}_1(\mathcal{E}(t)) = \begin{pmatrix}\hat{f}^R(\mathcal{E}(t)) \\ \hat{f}^R(\mathcal{E}(t))\end{pmatrix}$, $\bar{f}_2(\mathcal{E}(t)) = \begin{pmatrix}\hat{f}^I(\mathcal{E}(t)) \\ \hat{f}^I(\mathcal{E}(t))\end{pmatrix}$, $\bar{g}_1(\mathcal{E}(t-\sigma_1(t))) = \begin{pmatrix}\hat{g}^R(\mathcal{E}(t-\sigma_1(t))) \\ \hat{g}^R(\mathcal{E}(t-\sigma_1(t)))\end{pmatrix}$, $\bar{g}_2(\mathcal{E}(t-\sigma_1(t))) = \begin{pmatrix}\hat{g}^I(\mathcal{E}(t-\sigma_1(t))) \\ \hat{g}^I(\mathcal{E}(t-\sigma_1(t)))\end{pmatrix}$, $\bar{h}_1(\mathcal{E}(t)) = \begin{pmatrix}\hat{h}^R(\mathcal{E}(t)) \\ \hat{h}^R(\mathcal{E}(t))\end{pmatrix}$, $\bar{h}_2(\mathcal{E}(t)) = \begin{pmatrix}\hat{h}^I(\mathcal{E}(t)) \\ \hat{h}^I(\mathcal{E}(t))\end{pmatrix}$,
and $\bar{\mathbf{\Gamma}}_k = \begin{pmatrix}\mathbf{\Gamma}_k & 0 \\ 0 & \mathbf{\Gamma}_k\end{pmatrix}$.

3.3 Exponential synchronization criteria of CVNN system

In this section, sufficient criteria are given which ensure the exponential synchronization between the master and slave systems (3.1) and (3.2) under the effect of heterogeneous impulses.

Theorem 3.3.1. Suppose that Assumption 3 holds and T_a is AII of impulsive sequence $\mathcal{S} = \{t_k : k \in \mathbb{N}\}$. The trajectories of the impulsive error system (3.11) converge globally exponentially to a zero equilibrium point at the convergence rate $\tilde{r}_1 > 0$ if there exist a matrix measure $\mu_q(.)$ for $q = 1, 2, \infty$, and a positive constant $\theta > 1$ such that

$$\Pi_1 + \Pi_2\theta - \frac{\ln\theta}{\kappa} < 0, \tag{3.12}$$

where $\Pi_1 = \mu_q(-\bar{C}) + 2\Psi_1\|\bar{A}_1\|_q + 2\Psi_2\|\bar{A}_2\|_q$, $\Pi_2 = 2\|\bar{B}_1\|_q\Phi_1 + 2\|\bar{B}_2\|_q\Phi_2 + 2\Omega_1\sigma_2\|\bar{D}_1\|_q + 2\Omega_2\sigma_2\|\bar{D}_2\|_q$, $\Psi_1 = \max\{L_q^{RR}, L_q^{RI}\}$, $\Psi_2 = \max\{L_q^{IR}, L_q^{II}\}$, $\Phi_1 = \max\{M_q^{RR}, M_q^{RI}\}$, $\Phi_2 = \max\{M_q^{IR}, M_q^{II}\}$, $\Omega_1 = \max\{H_q^{RR}, H_q^{RI}\}$, $\Omega_2 = \max\{H_q^{IR}, H_q^{II}\}$, $\kappa = \sup_{k\in\mathbb{N}}\{t_{k+1} - t_k\} < \infty$, and the convergence rate $\tilde{r}_1 > 0$ will depend on AIG of heterogeneous impulses, which is described through the following cases:

Case 1: If $\frac{1}{\theta} \leq \varpi < 1$, then the convergence rate will be $\tilde{r}_1 = \lambda_1 - \tilde{\beta}_1 \geq 0$ such that

$$\|\mathcal{E}(t)\|_q \leq M_1 \sup_{-\sigma\leq s\leq 0} \|\mathcal{E}(s)\|_q e^{-(\lambda_1 - \tilde{\beta}_1)t}, \forall t \geq 0, \tag{3.13}$$

where $\varpi = \frac{|\varpi_1| + |\varpi_2| + \ldots + |\varpi_{N_\mathcal{S}(s,t)}|}{N_\mathcal{S}(s,t)}$, $\varpi_k = \|\bar{\Gamma}_k\|_q$, $M_1 = \theta^{N_0+1}\varpi^{-N_0}$, $\tilde{\beta}_1 = \frac{\ln(\theta\varpi)}{T_a} \geq 0$, and $\lambda_1 \in \big(0, \ \frac{\ln\theta}{\kappa} - \Pi_1 - \Pi_2\theta e^{\lambda_1\sigma}\big)$.

Case 2: If $\varpi \geq 1$, then the convergence rate will be $\tilde{r}_1 = \lambda_1 - \tilde{\beta}_2 \geq 0$ such that

$$\|\mathcal{E}(t)\|_q \leq M_2 \sup_{-\sigma\leq s\leq 0} \|\mathcal{E}(s)\|_q e^{-(\lambda_1 - \tilde{\beta}_2)t}, \forall t \geq 0, \tag{3.14}$$

where $M_2 = \theta^{N_0+1}\varpi^{N_0}$ and $\tilde{\beta}_2 = \frac{\ln(\theta\varpi)}{T_a}$.

Proof. For $t \neq t_k$, the right hand Dini derivative of $\|\mathcal{E}(t)\|_q$ with respect to t along the solution of impulsive system (3.11) is

$$\begin{aligned} D^+\big(\|\mathcal{E}(t)\|_q\big) &= \lim_{h\to 0^+} \frac{\|\mathcal{E}(t+h)\|_q - \|\mathcal{E}(t)\|_q}{h} \\ &= \lim_{h\to 0^+} \frac{\|\mathcal{E}(t) + h\dot{\mathcal{E}}(t) + o(h)\|_q - \|\mathcal{E}(t)\|_q}{h}. \end{aligned} \tag{3.15}$$

From impulsive system (3.11), we obtain

$$\begin{aligned} \|\mathcal{E}(t) + h\dot{\mathcal{E}}(t) + o(h)\|_q =& \Bigg\| \mathcal{E}(t) + h\Big(- \bar{C}\mathcal{E}(t) + \bar{A}_1\bar{f}_1(\mathcal{E}(t)) + \bar{A}_2\bar{f}_2(\mathcal{E}(t)) \\ & + \bar{B}_1\bar{g}_1(\mathcal{E}(t-\sigma_1(t))) + \bar{B}_2\bar{g}_2(\mathcal{E}(t-\sigma_1(t))) \\ & + \bar{D}_1\int_{t-\sigma_2(t)}^{t} \bar{h}_1(\mathcal{E}(s))ds + \bar{D}_2\int_{t-\sigma_2(t)}^{t} \bar{h}_2(\mathcal{E}(s))ds\Big) + o(h)\Bigg\|_q \end{aligned} \tag{3.16}$$

$$\begin{aligned} \leq& \|I + h(-\bar{C})\|_q\|\mathcal{E}(t)\|_q + h\Big(\|\bar{A}_1\|_q\|\bar{f}_1(\mathcal{E}(t))\|_q \\ & + \|\bar{A}_2\|_q\|\bar{f}_2(\mathcal{E}(t))\|_q + \|\bar{B}_1\|_q\|\bar{g}_1(\mathcal{E}(t-\sigma_1(t)))\|_q \\ & + \|\bar{B}_2\|_q\|\bar{g}_2(\mathcal{E}(t-\sigma_1(t)))\|_q + \|\bar{D}_1\|_q\int_{t-\sigma_2(t)}^{t} \|\bar{h}_1(\mathcal{E}(s))\|_q ds \\ & + \|\bar{D}_2\|_q\int_{t-\sigma_2(t)}^{t} \|\bar{h}_2(\mathcal{E}(s))\|_q ds\Big) + \|o(h)\|_q. \end{aligned} \tag{3.17}$$

Using Assumption 3 and Lemma 3.3, we obtain

$$\begin{aligned} \|\bar{f}_1(\mathcal{E}(t))\|_q^q \leq 2\|\hat{f}^R(\mathcal{E}(t))\|_q^q \leq& 2(L_q^{RR}\|\chi^R(t)\|_q + L_q^{RI}\|\chi^I(t)\|_q)^q \\ \leq& 2\psi_1^q(\|\chi^R(t)\|_q + \|\chi^I(t)\|_q)^q \\ \leq& 2^q\psi_1^q(\|\chi^R(t)\|_q^q + \|\chi^I(t)\|_q^q) \\ \|\bar{f}_1(\mathcal{E}(t))\|_q \leq& 2\psi_1\|\mathcal{E}(t)\|_q. \end{aligned} \tag{3.18}$$

Similar to inequality (3.18), we can obtain

$$\begin{aligned}
&\|\bar{f}_2(\mathcal{E}(t))\|_q \leq 2\Psi_2\|\mathcal{E}(t)\|_q,\ \|\bar{g}_1(\mathcal{E}(t-\sigma_1(t)))\|_q \leq 2\Phi_1\|\mathcal{E}(t-\sigma_1(t))\|_q,\\
&\|\bar{g}_2(\mathcal{E}(t-\sigma_1(t)))\|_q \leq 2\Phi_2\|\mathcal{E}(t-\sigma_1(t))\|_q,\\
&\|\bar{h}_1(\mathcal{E}(t))\|_q \leq 2\Omega_1\|\mathcal{E}(t)\|_q,\ \|\bar{h}_2(\mathcal{E}(t))\|_q \leq 2\Omega_2\|\mathcal{E}(t)\|_q. \qquad (3.19)
\end{aligned}$$

Substituting inequalities (3.18) and (3.19) in inequality (3.17), we get

$$\begin{aligned}
\|\mathcal{E}(t)+h\dot{\mathcal{E}}(t)+o(h)\|_q \leq &\|I+h(-\bar{C})\|_q\|\mathcal{E}(t)\|_q + h\Big(2\Psi_1\|\bar{A}_1\|_q\|\mathcal{E}(t)\|_q + 2\Psi_2\|\bar{A}_2\|_q\|\mathcal{E}(t)\|_q\\
&+2\Phi_1\|\bar{B}_1\|_q\|\mathcal{E}(t-\sigma_1(t))\|_q + 2\Phi_2\|\bar{B}_2\|_q\|\mathcal{E}(t-\sigma_1(t))\|_q\\
&+2\Omega_1\|\bar{D}_1\|_q\int_{t-\sigma_2(t)}^{t}\|\mathcal{E}(s)\|_q ds + 2\Omega_2\|\bar{D}_2\|_q\int_{t-\sigma_2(t)}^{t}\|\mathcal{E}(s)\|_q ds\Big)\\
&+\|o(h)\|_q. \qquad (3.20)
\end{aligned}$$

Using inequality (3.20), we get

$$\begin{aligned}
D^+\big(\|\mathcal{E}(t)\|_q\big) \leq &\lim_{h\to 0^+}\left(\frac{\|I+h(-\bar{C})\|_q-1}{h}\right)\|\mathcal{E}(t)\|_q + 2\Psi_1\|\bar{A}_1\|_q\|\mathcal{E}(t)\|_q\\
&+2\Psi_2\|\bar{A}_2\|_q\|\mathcal{E}(t)\|_q + 2\Phi_1\|\bar{B}_1\|_q\|\mathcal{E}(t-\sigma_1(t))\|_q + 2\Phi_2\|\bar{B}_2\|_q\|\mathcal{E}(t-\sigma_1(t))\|_q\\
&+2\Omega_1\|\bar{D}_1\|_q\int_{t-\sigma_2(t)}^{t}\|\mathcal{E}(s)\|_q ds + 2\Omega_2\|\bar{D}_2\|_q\int_{t-\sigma_2(t)}^{t}\|\mathcal{E}(s)\|_q ds\\
\leq&\Big(\mu_q(-\bar{C}) + 2\Psi_1\|\bar{A}_1\|_q + 2\Psi_2\|\bar{A}_2\|_q\Big)\|\mathcal{E}(t)\|_q + \Big(2\Phi_1\|\bar{B}_1\|_q + 2\Phi_2\|\bar{B}_2\|_q\\
&+2\Omega_1\sigma_2\|\bar{D}_1\|_q + 2\sigma_2\Omega_2\|\bar{D}_2\|_q\Big)\sup_{t-\sigma\leq s\leq t}\|\mathcal{E}(s)\|_q\\
\leq&\Pi_1\|\mathcal{E}(t)\|_q + \Pi_2\sup_{t-\sigma\leq s\leq t}\|\mathcal{E}(s)\|_q, \qquad (3.21)
\end{aligned}$$

where $\Pi_1 = \mu_q(-\bar{C}) + 2\Psi_1\|\bar{A}_1\|_q + 2\Psi_2\|\bar{A}_2\|_q$ and $\Pi_2 = 2\|\bar{B}_1\|_q\Phi_1 + 2\|\bar{B}_2\|_q\Phi_2 + 2\Omega_1\sigma_2\|\bar{D}_1\|_q + 2\Omega_2\sigma_2\|\bar{D}_2\|_q$.

Now, at $t = t_k$,

$$\|\mathcal{E}(t_k^+)\|_q = \left\| \begin{pmatrix} \chi^R(t) \\ \chi^I(t) \end{pmatrix}_q \right\| = \left\| \begin{pmatrix} \Gamma_k \chi^R(t_k^-) \\ \Gamma_k \chi^I(t_k^-) \end{pmatrix} \right\|_q = \left\| \begin{pmatrix} \Gamma_k & 0 \\ 0 & \Gamma_k \end{pmatrix} \begin{pmatrix} \chi^R(t_k^-) \\ \chi^I(t_k^-) \end{pmatrix} \right\|_q$$
$$\leq \left\| \begin{pmatrix} \Gamma_k & 0 \\ 0 & \Gamma_k \end{pmatrix} \right\|_q \left\| \begin{pmatrix} \chi^R(t_k^-) \\ \chi^I(t_k^-) \end{pmatrix} \right\|_q,$$

i.e.,

$$\|\mathcal{E}(t_k)\|_q \leq \varpi_k \|\mathcal{E}(t_k^-)\|_q. \tag{3.22}$$

Assembling inequalities (3.21) and (3.22), we get the following system

$$\begin{aligned} &D^+(\|\mathcal{E}(t)\|_q) \leq \Pi_1 \|\mathcal{E}(t)\|_q + \Pi_2 \sup_{t-\sigma \leq s \leq t} \|\mathcal{E}(s)\|_q, \quad && t \neq t_k, \\ &\|\mathcal{E}(t_k)\|_q \leq \varpi_k \|\mathcal{E}(t_k^-)\|_q, && t = t_k, \\ &\|\mathcal{E}(s)\|_q = \left\| \begin{pmatrix} \phi^R(s) \\ \varphi^I(s) \end{pmatrix} \right\|_q, \quad \forall s \in [-\sigma, 0]. \end{aligned} \tag{3.23}$$

From inequality (3.12), we have $\Pi_1 + \Pi_2\theta - \frac{\ln \theta}{\kappa} < 0$. Now, using Lemma 3.2, we can estimate the solution of the system (3.23) through the following cases.

Case 1: If $\frac{1}{\theta} \leq \varpi < 1$, then from Definitions of AII and AIG defined in the chapter 1, we have

$$\begin{aligned}
\theta^{N_{\mathcal{S}}(s,t)+1} \prod_{s<t_k<t} \varpi_k &\leq \theta^{N_{\mathcal{S}}(s,t)+1} \left(\frac{|\varpi_1| + |\varpi_2| + \ldots + |\varpi_{N_{\mathcal{S}}(s,t)}|}{N_{\mathcal{S}}(s,t)} \right)^{N_{\mathcal{S}}(s,t)} \\
&= \theta^{N_{\mathcal{S}}(s,t)+1} \varpi^{N_{\mathcal{S}}(s,t)} \\
&\leq \theta^{N_{\mathcal{S}}(s,t)+1} \varpi^{\frac{t-s}{T_a} - N_0} \\
&= \theta^{N_{\mathcal{S}}(s,t)+1} \varpi^{-N_0} e^{\frac{\ln \varpi}{T_a}(t-s)} \\
&\leq \theta^{N_0+1} \varpi^{-N_0} e^{\frac{\ln(\theta\varpi)}{T_a}(t-s)}. \qquad (3.24)
\end{aligned}$$

Now, the inequality (3.24) can be written as

$$\theta^{N_{\mathcal{S}}(s,t)+1} \prod_{s<t_k<t} \varpi_k \leq M_1 e^{\tilde{\beta}_1 (t-s)}. \qquad (3.25)$$

Using Lemma 3.2, we get

$$\|\mathcal{E}(t)\|_q \leq M_1 \sup_{-\sigma \leq s \leq 0} \|\mathcal{E}(s)\|_q e^{-(\lambda_1 - \tilde{\beta}_1)t}, \forall t \geq 0. \qquad (3.26)$$

Case 2: If $\varpi \geq 1$, then from Definitions of AII and AIG defined in the chapter 1, we have

$$\begin{aligned}
\theta^{N_{\mathcal{S}}(s,t)+1} \prod_{s<t_k<t} \varpi_k &\leq \theta^{N_{\mathcal{S}}(s,t)+1} \varpi^{N_{\mathcal{S}}(s,t)} \\
&\leq \theta^{N_{\mathcal{S}}(s,t)+1} \varpi^{\frac{t-s}{T_a} + N_0} \\
&= \theta^{N_{\mathcal{S}}(s,t)+1} \varpi^{N_0} e^{\frac{\ln \varpi}{T_a}(t-s)} \\
&\leq \theta^{N_0+1} \varpi^{N_0} e^{\frac{\ln(\theta\varpi)}{T_a}(t-s)}.
\end{aligned}$$

i.e.,

$$\theta^{N_{\mathcal{S}}(s,t)+1} \prod_{s<t_k<t} \varpi_k \leq M_2 e^{\tilde{\beta}_2(t-s)}. \tag{3.27}$$

Using Lemma 3.2, we get

$$\|\mathcal{E}(t)\|_q \leq M_2 \sup_{-\sigma \leq s \leq 0} \|\mathcal{E}(s)\|_q e^{-(\lambda_1-\tilde{\beta}_2)t} \ \forall t \geq 0. \tag{3.28}$$

From inequalities (3.26) and (3.28), we can conclude that the trajectory of the impulsive error system (3.11) is converging globally exponentially at the convergent rates $\tilde{r}_1 = \lambda_1 - \tilde{\beta}_1$ and $\tilde{r}_1 = \lambda_1 - \tilde{\beta}_2$ to its unique zero equilibrium point for two different ranges of average impulsive gain: $\frac{1}{\theta} \leq \varpi < 1$, and $\varpi > 1$, respectively. Hence, the proof of Theorem 3.3.1 is completed. □

Remark 3.4. The inequalities (3.25) and (3.27) are being derived by using the concept of AIG defined in the chapter 1. One can observe from the calculations that synchronizing ($\varpi_k < 1$) and desynchronizing ($\varpi_k > 1$) heterogeneous impulses can exist simultaneously in $\varpi = \frac{|\varpi_1|+|\varpi_2|+...+|\varpi_{N_{\mathcal{S}}(s,t)}|}{N_{\mathcal{S}}(s,t)}$, i.e., the impulsive sequence is heterogeneous. Unfortunately, the authors in [69, 73, 23] have discussed the effect of synchronizing and desynchronizing impulses on synchronization of neural networks, separately. On the other hand, using the concept of AIG, one can consider the synchronizing and desynchronizing impulses in the same impulsive sequence.

Remark 3.5. It is worth to mention that the result in Theorem 3.3.1 does not contain Lipschitz constants of each $f_j^R(.)$, $f_j^I(.)$ for all $j = 1, 2, ..., n$. Therefore, one can proceed to find more precise exponential synchronization criteria which contain information about each L_{qj}^{RR}, L_{qj}^{II} for all $j = 1, 2, ..., n$. For this, the following lemma is introduced which will be useful in the proof of the next theorem.

Lemma 3.6. *[76] Assume that $\mu_q(.)$ be a matrix measure corresponding to an induced matrix norm $\|.\|_q$ on $\mathbb{R}^{n\times n}$, then $\mu_q(AF(e(t))) \leq \mu_q(A^*D)$ if*

$$0 \leq \frac{f_j(e_j(t))}{e_j(t)} \leq d_j, \quad j = 1, 2, ...n,$$

where $F(e(t)) = diag\Big\{\frac{f_1(e_1(t))}{e_1(t)},\ \frac{f_2(e_2(t))}{e_2(t)}, ..., \frac{f_n(e_n(t))}{e_n(t)}\Big\}$, $D = diag\{d_1, d_2, ..., d_n\}$, $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, $A^* = (a^*_{ij})_{n\times n} \in \mathbb{R}^{n\times n}, q = 1, \infty,$ and

$$a^*_{ij} = \begin{cases} \max(0,\ a_{ij}), & i = j, \\ a_{ij}, & i \neq j. \end{cases}$$

Theorem 3.7. *Suppose that Assumption 4 holds and the average impulsive interval of impulsive sequence $\mathcal{S} = \{t_k : k \in \mathbb{N}\}$ is T_a. The impulsive error system (3.11) converges globally exponentially to a zero equilibrium point at the convergence rate $\bar{r}_2 > 0$ if there exist a matrix measure $\mu_q(.)$ for $q = 1, \infty$ and a positive constant $\theta > 1$ such that*

$$\widehat{\Pi_1} + \widehat{\Pi_2}\theta - \frac{\ln\theta}{\kappa} < 0, \tag{3.29}$$

where $\widehat{\Pi_1} = \mu_q(-\bar{C}) + \mu_q(\hat{A}^*\Gamma) + (L^{RR}_q + L^{II}_q)\|\bar{A}_1\|_q + (L^{RR}_q + L^{II}_q)\|\bar{A}_2\|_q$, $\widehat{\Pi_2} = 2\Phi_1\|\bar{B}_1\|_q + 2\Phi_2\|\bar{B}_2\|_q + 2\sigma_2\Omega_1\|\bar{D}_1\|_q + 2\sigma_2\Omega_2\|\bar{D}_2\|_q$ in which $\Gamma = diag\{L^{RR}_{q1}, L^{RR}_{q2}, ..., L^{RR}_{qn}, L^{II}_{q1}, L^{II}_{q2}, ..., L^{II}_{qn}\}$, $\hat{A} = [\hat{a}_{ij}]_{2n\times 2n} = \bar{A}_1 + \bar{A}_2$, and $\hat{A}^* = [\hat{a}^*_{ij}]_{2n\times 2n}$ with $\hat{a}^*_{ij} = \max\{0, \hat{a}_{ij}\}$ for $i = j$ otherwise $\hat{a}^*_{ij} = \hat{a}_{ij}$. The convergence rate $\bar{r}_2 > 0$ will depend on AIG of heterogeneous impulses, which is described through the following cases.

Case 1: If $\frac{1}{\theta} \leq \varpi < 1$, then $\bar{r}_2 = \lambda_2 - \tilde{\beta}_1 \geq 0$ such that

$$\|\mathcal{E}(t)\|_q \leq M_1 \sup_{-\sigma\leq s\leq 0} \|\mathcal{E}(s)\|_q e^{-(\lambda_2-\tilde{\beta}_1)t}, \forall t \geq 0, \tag{3.30}$$

where $\lambda_2 \in \big(0,\ \frac{\ln\theta}{\kappa} - \widehat{\Pi_1} - \widehat{\Pi_2}\theta e^{\lambda_2\sigma}\big)$.

Case 2: If $\varpi \geq 1$*, then* $\tilde{r}_2 = \lambda_2 - \tilde{\beta}_2 \geq 0$ *such that*

$$\|\mathcal{E}(t)\|_q \leq M_2 \sup_{-\sigma \leq s \leq 0} \|\mathcal{E}(s)\|_q e^{-(\lambda_2 - \tilde{\beta}_2)t}, \forall t \geq 0. \tag{3.31}$$

All the remaining symbols are same as defined in Theorem 3.3.1.

Proof. For $t \neq t_k$, we have the following from the equality (3.16).

$$\begin{aligned}\|\mathcal{E}(t) + h\dot{\mathcal{E}(t)} + o(h)\|_q = \bigg\| & \mathcal{E}(t) + h\bigg(- \bar{C}\mathcal{E}(t) + \bar{A}_1\bar{f}_1(\mathcal{E}(t)) + \bar{A}_2\bar{f}_2(\mathcal{E}(t)) \\ & + \bar{B}_1\bar{g}_1(\mathcal{E}(t - \sigma_1(t))) + \bar{B}_2\bar{g}_2(\mathcal{E}(t - \sigma_1(t))) \\ & + \bar{D}_1 \int_{t-\sigma_2(t)}^{t} \bar{h}_1(\mathcal{E}(s))ds + \bar{D}_2 \int_{t-\sigma_2(t)}^{t} \bar{h}_2(\mathcal{E}(s))ds\bigg) + o(h)\bigg\|_q.\end{aligned} \tag{3.32}$$

We know from system (3.11) that $\bar{f}_1(\mathcal{E}(t)) = \begin{pmatrix} \hat{f}^R(\mathcal{E}(t)) \\ \hat{f}^R(\mathcal{E}(t)) \end{pmatrix}$, which can be rewritten as

$$\bar{f}_1(\mathcal{E}(t)) = \begin{pmatrix} \hat{f}^R(\mathcal{E}(t)) \\ \hat{f}^R(\mathcal{E}(t)) \end{pmatrix} = \Xi_1(t) - \Xi_2(t) + \Xi_3(t), \tag{3.33}$$

where $\Xi_1(t) = \begin{pmatrix} \hat{f}^R(\mathcal{E}(t)) \\ \hat{f}^I(\mathcal{E}(t)) \end{pmatrix} = \bar{F}(\mathcal{E}(t))\mathcal{E}(t)$, $\Xi_2(t) = \begin{pmatrix} 0_{n\times 1} \\ \hat{f}^I(\mathcal{E}(t)) \end{pmatrix}$, $\Xi_3(t) = \begin{pmatrix} 0_{n\times 1} \\ \hat{f}^R(\mathcal{E}(t)) \end{pmatrix}$,
and
$\bar{F}(\mathcal{E}(t)) = \text{diag}\Big\{ \frac{f_1^R(\mathcal{E}_1(t))}{\mathcal{E}_1(t)}, \frac{f_2^R(\mathcal{E}_2(t))}{\mathcal{E}_2(t)}, ..., \frac{f_n^R(\mathcal{E}_n(t))}{\mathcal{E}_n(t)}, \frac{f_1^I(\mathcal{E}_1(t))}{\mathcal{E}_1(t)}, \frac{f_2^I(\mathcal{E}_2(t))}{\mathcal{E}_2(t)}, ..., \frac{f_n^I(\mathcal{E}_n(t))}{\mathcal{E}_n(t)} \Big\}$.

Similarly, we can write

$$\bar{f}_2(\mathcal{E}(t)) = \begin{pmatrix} \hat{f}^I(\mathcal{E}(t)) \\ \hat{f}^I(\mathcal{E}(t)) \end{pmatrix} = \bar{F}(\mathcal{E}(t))\mathcal{E}(t) - \Xi_4(t) + \Xi_5(t), \tag{3.34}$$

where $\Xi_4(t) = \begin{pmatrix} \hat{f}^R(\mathcal{E}(t)) \\ 0_{n\times 1} \end{pmatrix}$ and $\Xi_5(t) = \begin{pmatrix} \hat{f}^I(\mathcal{E}(t)) \\ 0_{n\times 1} \end{pmatrix}$. Substituting the right-hand sides of identities (3.33) and (3.34) in (3.32), we get

$$\begin{aligned}
\|\mathcal{E}(t) + h\dot{\mathcal{E}(t)} + o(h)\|_q = \Bigg\| & \mathcal{E}(t) + h\Big(- \bar{C}\mathcal{E}(t) + \bar{A}_1\big(\bar{F}(\mathcal{E}(t))\mathcal{E}(t) - \Xi_2(t) + \Xi_3(t)\big) \\
& + \bar{A}_2\big(\bar{F}(\mathcal{E}(t))\mathcal{E}(t) - \Xi_3(t) + \Xi_4(t)\big) + \bar{B}_1\bar{g}_1(\mathcal{E}(t-\sigma_1(t))) \\
& + \bar{B}_2\bar{g}_2(\mathcal{E}(t-\sigma_1(t))) + \bar{D}_1\int_{t-\sigma_2(t)}^{t}\bar{h}_1(\mathcal{E}(s))ds \\
& + \bar{D}_2\int_{t-\sigma_2(t)}^{t}\bar{h}_2(\mathcal{E}(s))ds\Big) + o(h)\Bigg\|_q
\end{aligned} \tag{3.35}$$

$$\begin{aligned}
\le\ & \|I + h\big(- \bar{C} + \bar{A}_1\bar{F}(\mathcal{E}(t)) + \bar{A}_2\bar{F}(\mathcal{E}(t))\big)\|_q\|\mathcal{E}(t)\|_q \\
& + h\Big(\|\bar{A}_1\|_q\|\Xi_2(t)\|_q + \|\bar{A}_1\|_q\|\Xi_3(t)\|_q + \|\bar{A}_2\|_q\|\Xi_4(t)\|_q \\
& + \|\bar{A}_2\|_q\|\Xi_5(t)\|_q + \|\bar{B}_1\|_q\|\bar{g}_1(\mathcal{E}(t-\sigma_1(t)))\|_q \\
& + \|\bar{B}_2\|_q\|\bar{g}_2(\mathcal{E}(t-\sigma_1(t)))\|_q + \|\bar{D}_1\|_q\int_{t-\sigma_2(t)}^{t}\|\bar{h}_1(\mathcal{E}(s))\|_q ds \\
& + \|\bar{D}_2\|_q\int_{t-\sigma_2(t)}^{t}\|\bar{h}_2(\mathcal{E}(s))\|_q ds\Big) + \|o(h)\|_q.
\end{aligned} \tag{3.36}$$

Using inequality (3.36) in (3.15), we get

$$D^+\big(\|\mathcal{E}(t)\|_q\big) \leq \lim_{h\to 0^+}\left(\frac{\|I+h\big(-\bar{C}+\bar{A}_1\bar{F}(\mathcal{E}(t))+\bar{A}_2\bar{F}(\mathcal{E}(t))\big)\|_q-1}{h}\right)\|\mathcal{E}(t)\|_q$$
$$+(L_q^{RR}+L_q^{II})\|\bar{A}_1\|_q\|\mathcal{E}(t)\|_q \ +(L_q^{RR}+L_q^{II})\|\bar{A}_2\|_q\|\mathcal{E}(t)\|_q$$
$$+2\Phi_1\|\bar{B}_1\|_q\|\mathcal{E}(t-\sigma_1(t))\|_q+2\Phi_2\|\bar{B}_2\|_q\|\mathcal{E}(t-\sigma_1(t))\|_q$$
$$+2\Omega_1\|\bar{D}_1\|_q\int_{t-\sigma_2(t)}^{t}\|\mathcal{E}(s)\|_q ds+2\Omega_2\|\bar{D}_2\|_q\int_{t-\sigma_2(t)}^{t}\|\mathcal{E}(s)\|_q ds \tag{3.37}$$

$$\leq \big(\mu_q(-\bar{C}+\bar{A}_1\bar{F}(\mathcal{E}(t))+\bar{A}_2\bar{F}(\mathcal{E}(t)))+(L_q^{RR}+L_q^{II})\|\bar{A}_1\|_q$$
$$+(L_q^{RR}+L_q^{II})\|\bar{A}_2\|_q\big)\|\mathcal{E}(t)\|_q+\big(2\Phi_1\|\bar{B}_1\|_q+2\Phi_2\|\bar{B}_2\|_q+2\sigma_2\Omega_1\|\bar{D}_1\|_q$$
$$+2\sigma_2\Omega_2\|\bar{D}_2\|_q\big)\sup_{t-\sigma\leq s\leq t}\|\mathcal{E}(s)\|_q. \tag{3.38}$$

Let $\hat{A}=\bar{A}_1+\bar{A}_2$, then from Lemmas 3.1 & 3.6 and Assumption 4, we have

$$\mu_q(-\bar{C}+\hat{A}\bar{F}(\mathcal{E}(t)))\leq \mu_q(-\bar{C})+\mu_q(\hat{A}\bar{F}(\mathcal{E}(t)))\leq \mu_q(-\bar{C})+\mu_q(\hat{A}^*\Gamma). \tag{3.39}$$

Using inequality (3.39) in equation (3.38), we get

$$D^+\big(\|\mathcal{E}(t)\|_q\big)\leq \big(\mu_q(-\bar{C})+\mu_q(\hat{A}^*\Gamma)+(L_q^{RR}+L_q^{II})\|\bar{A}_1\|_q+(L_q^{RR}+L_q^{II})\|\bar{A}_2\|_q\big)\|\mathcal{E}(t)\|_q$$
$$+\big(2\Phi_1\|\bar{B}_1\|_q+2\Phi_2\|\bar{B}_2\|_q+2\sigma_2\Omega_1\|\bar{D}_1\|_q+2\sigma_2\Omega_2\|\bar{D}_2\|_q\big)\sup_{t-\sigma\leq s\leq t}\|\mathcal{E}(s)\|_q$$
$$\leq \widehat{\Pi_1}\|\mathcal{E}(t)\|_q+\widehat{\Pi_2}\sup_{t-\sigma\leq s\leq t}\|\mathcal{E}(s)\|_q, \tag{3.40}$$

where $\widehat{\Pi_1}=\mu_q(-\bar{C})+\mu_q(\hat{A}^*\Gamma)+(L_q^{RR}+L_q^{II})\|\bar{A}_1\|_q+(L_q^{RR}+L_q^{II})\|\bar{A}_2\|_q$ and $\widehat{\Pi_2}=2\Phi_1\|\bar{B}_1\|_q+2\Phi_2\|\bar{B}_2\|_q+2\sigma_2\Omega_1\|\bar{D}_1\|_q+2\sigma_2\Omega_2\|\bar{D}_2\|_q$.

From inequality (3.22), we have $\|\mathcal{E}(t_k)\|_q\leq \varpi_k\|\mathcal{E}(t_k^-)\|_q$ for $t=t_k$. Similar to the proof of theorem 3.3.1, we get the following results.

Case 1: If $\frac{1}{\theta} \leq \varpi < 1$, then we have

$$\|\mathcal{E}(t)\|_q \leq M_1 \sup_{-\sigma \leq s \leq 0} \|\mathcal{E}(s)\|_q e^{-(\lambda_2 - \tilde{\beta}_1)t}, \forall t \geq 0. \tag{3.41}$$

Case 2: If $\varpi > 1$, then we have

$$\|\mathcal{E}(t)\|_q \leq M_2 \sup_{-\sigma \leq s \leq 0} \|\mathcal{E}(s)\|_q e^{-(\lambda_2 - \tilde{\beta}_2)t} \forall t \geq 0, \tag{3.42}$$

where $\lambda_2 \in \left(0, \ \frac{\ln \theta}{\kappa} - \widehat{\Pi_1} - \widehat{\Pi_2} \theta e^{\lambda_2 \sigma}\right)$. It is clear from the inequalities (3.41) and (3.42) that the trajectory of the impulsive system (3.11) is converging globally exponentially at the convergence rates $\bar{r}_2 = \lambda_2 - \tilde{\beta}_1$ and $\bar{r}_2 = \lambda_2 - \tilde{\beta}_2$. Hence, the proof is completed. □

Remark 3.8. According to definition of matrix measure given in the chapter 1, the value of a matrix measure $\mu_q(\mathcal{W})$ $(q = 1, 2, \infty)$ can be negative, positive, or zero, whereas a matrix norm can only have a non-negative value. Thus, using matrix measure one can better utilize the elements of connection weight matrix. Moreover, most of the previous results on the stability of neural networks have been derived by constructing Lyapunov functional and using LMI approach, whereas in the matrix measure approach there is no need to construct Lyapunov functional.

3.4 Examples

In order to verify the results obtained in the previous section, two examples are considered in this section.

Example 3.1. *The following two-dimensional CVNN system with mixed time-varying delays is considered as the master system as*

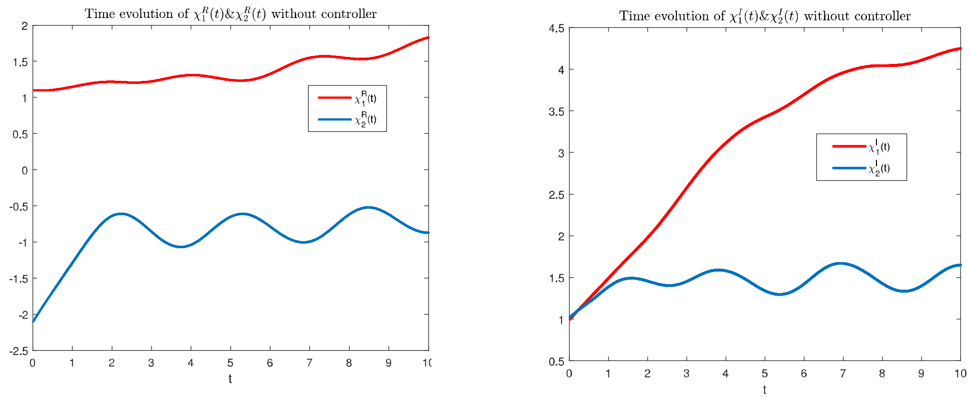
$$\dot{w}(t) = -Cw(t) + Af(w(t)) + Bg(w(t - \sigma_1(t))) + D\int_{t-\sigma_2(t)}^{t} h(w(s))ds + I(t), \tag{3.43}$$

where $w_j = w_j^R + iw_j^I$ *for* $j = 1, 2$, $C = \begin{pmatrix} 0.35 & 0 \\ 0 & 0.38 \end{pmatrix}$, $A = \begin{pmatrix} 0.2 - 0.3i & -0.2 - 0.3i \\ -0.2 + 0.05i & -0.4 + 0.1i \end{pmatrix}$, $B = \begin{pmatrix} 0.4 - 0.15i & 0.2 + 0.2i \\ 0.3 + 0.5i & -0.3 + 0.1i \end{pmatrix}$, $D = \begin{pmatrix} -0.5 - 0.2i & 0.1 + 0.3i \\ 0.8 - 1.2i & 0.6 + 0.5i \end{pmatrix}$, $\sigma_1(t) = 0.5 + 0.5\sin^2(t)$, $\sigma_2(t) = \cos^2(t)$, $\sigma = 1$, $f_j(w_j) = g_j(w_j) = h_j(w_j) = \frac{1}{1+\exp\left(-w_j^R + 2w_j^I\right)} + i\frac{1-\exp\left(-2w_j^R - w_j^I\right)}{1+\exp\left(-2w_j^R - w_j^I\right)}$, *and* $I(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. *By a simple calculation, we obtain* $L_{qj}^{RR} = M_{qj}^{RR} = H_{qj}^{RR} = 1$, $L_{qj}^{RI} = M_{qj}^{RI} = H_{qj}^{RI} = 2$, $L_{qj}^{IR} = M_{qj}^{IR} = H_{qj}^{IR} = 4$, *and* $L_{qj}^{II} = M_{qj}^{II} = H_{qj}^{II} = 2$; *using these values , we get* $\Psi_1 = 2$, $\Psi_2 = 4$, $\Phi_1 = 2$, $\Phi_2 = 4$, $\Omega_1 = 2$, *and* $\Omega_2 = 4$.

The slave system identical to the master system (3.43) *is considered as*

$$\dot{\hat{w}}(t) = -C\hat{w}(t) + Af(\hat{w}(t)) + Bg(\hat{w}(t - \sigma_1(t))) + D\int_{t-\sigma_2(t)}^{t} h(\hat{w}(s))ds + I(t) + U(t), \tag{3.44}$$

where $\hat{w}_j = \hat{w}_j^R + i\hat{w}_j^I$ *for all* $j = 1, 2$ *and* $I(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. *Suppose* $q = 2$. *From Theorem 3.3.1, we have* $\mu_2(-\bar{C}) = -0.35$, $\|\bar{A}_1\|_2 = 0.4606$, $\|\bar{A}_2\|_2 = 0.4606$, $\|\bar{B}_1\|_2 = 0.5237$, $\|\bar{B}_2\|_2 = 0.5237$, $\|\bar{D}_1\|_2 = 1.3349$, $\|\bar{D}_2\|_2 = 1.3349$, $\Pi_1 = 5.1772$, *and* $\Pi_2 = 22.3032$. *Set* $\theta = 2$, *then the supremum of the impulsive interval of the hybrid impulsive sequence* $\mathcal{S} = \{t_k : k \in \mathbb{N}\}$ *is* $\kappa = 0.0139$ *for which the inequality* $\Pi_1 + \Pi_2\theta -$

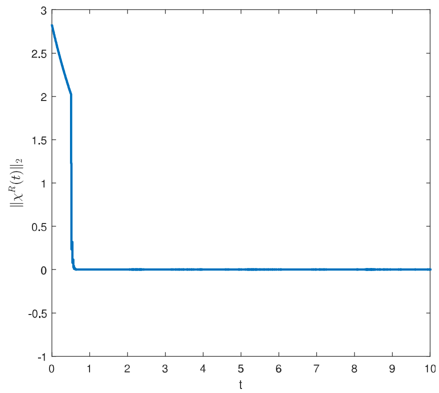


(a) (b)

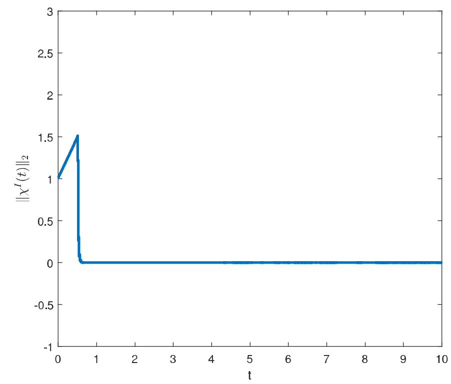
FIGURE 3.1: The time evolution of real and imaginary parts of the error system (3.8) without impulsive control (a) Real part (b) Imaginary part.

$\frac{\ln\theta}{\kappa} = -0.0831 < 0$ *holds, i.e., for given* $\theta = 2$*, the hybrid impulsive sequence whose supremum length of impulsive interval is* $\kappa > 0.0139$*, the inequality* (3.12) *will not be satisfied which further implies that the synchronization is not possible. Using Remark 3.2.4, we can choose* $\lambda_1 = 0.2561 \in (0, 0.3337)$*. Suppose that the hybrid impulsive sequence* $\mathcal{S} = \{\epsilon_0,\ 2\epsilon_0, ..., (N_0-1)\epsilon_0,\ N_0T_a,\ N_0T_a+\epsilon_0,\ N_0T_a+2\epsilon_0, ..., N_0T_a+(N_0-1)\epsilon_0, 2N_0T_a, ...\}$*, which is chosen from [23] for the distribution of heterogeneous impulses, where* $\kappa = \sup_{k\in\mathbb{N}}\{t_{k+1}-t_k\} = N_0(T_a-\epsilon_0)+\epsilon_0$ *and* $\inf_{k\in\mathbb{N}}\{t_{k+1}-t_k\} = \epsilon_0$*. For chosen random initial condition* $\chi(s) = \begin{pmatrix} 1.1+0.99i \\ -2.11+1.02i \end{pmatrix} \in \mathbb{C}^2$ *for all* $s \in [-1, 0]$*, the time evolutions of the real and imaginary parts of the error system* (3.8) *without impulsive control are depicted in Figure 3.1.*

Case 1: Let the impulsive strength matrices at distinct points of the impulsive sequence $\mathcal{S}$ *are* $\bar{\Gamma}_1 =$*diag*$\{1.2,\ 0.5,\ 1.1,\ 0.6\}$ *and* $\bar{\Gamma}_2 =$*diag*$\{0.5,\ 0.2,\ 0.5,\ 0.3\}$*, i.e.,* $\varpi_1 = \|\bar{\Gamma}_1\|_2 = 1.2$ *and* $\varpi_2 = \|\bar{\Gamma}_2\|_2 = 0.5$*, then the corresponding average impulsive gain is* $\varpi = \frac{1.2+0.5}{2} = 0.85 < 1$*. Suppose that* $N_0 = 2,\ T_a = 0.011,$ *, then* $\tilde{\beta}_1 = \frac{\ln(\theta\varpi)}{T_a} = 0.1112$*. One can calculate* $\|\chi(s)\|_2 = 2.7718$ *and* $\tilde{r}_1 = 0.1449$*. From*



(a)



(b)

FIGURE 3.2: The time evolutions of (a) Real and (b) Imaginary parts of the error system (3.8) with the impulsive control for $T_a = 0.011$ and $\varpi = 0.85$.

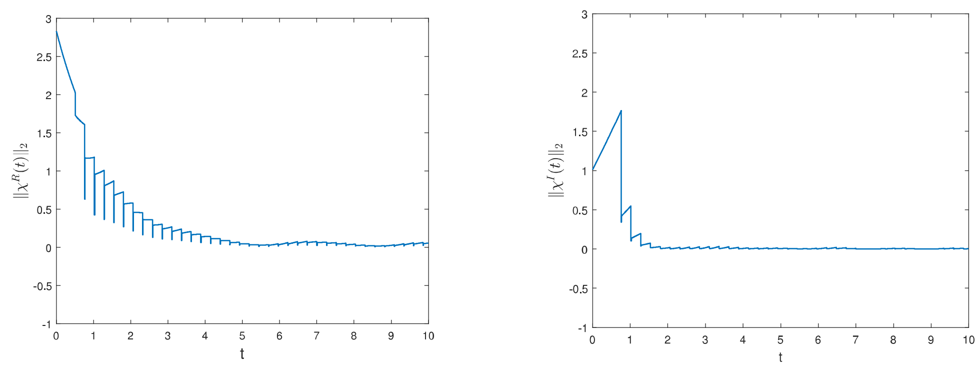
the inequality (3.13), *we have*

$$\|\mathcal{E}(t)\|_2 \leq 11.0721e^{-0.1449t}, \quad \forall t \geq 0. \tag{3.45}$$

One can conclude from inequality (3.45) *that the slave system* (3.2) *is synchronizing globally exponentially at the convergence rate* $\tilde{r}_1 = 0.1449$ *with the master system* (3.1) *under the hybrid impulsive controller* (3.7) *containing heterogeneous type impulses. Further, the time evolutions of the real and imaginary parts of the error system* (3.8) *with impulsive control* (3.7) *activated at the discrete points of the hybrid impulsive sequence* $\mathcal{S}$ *are depicted in Figure* 3.2.

Case 2: Let $\bar{\Gamma}_1 = diag\{1.5,\ 0.5,\ 0.6,\ 1.3\}$ *and* $\bar{\Gamma}_2 = diag\{0.3,\ 0.4,\ 0.506,\ 0.4\}$, *i.e.,* $\varpi_1 = \|\bar{\Gamma}_1\|_2 = 1.5$ *and* $\varpi_2 = \|\bar{\Gamma}_2\|_2 = 0.506$, *then the corresponding average impulsive gain is* $\varpi = \frac{1.5+0.506}{2} = 1.003 > 1$. *Suppose that the average impulsive interval* $T_a = 0.013$, $N_0 = 2$, *and* $\epsilon_0 = 0.01$, *then we have* $\tilde{\beta}_2 = 0.2304$ *and* $\bar{r}_1 = 0.0257$. *From the inequality* (3.14), *we can write the following.*

$$\|\mathcal{E}(t)\|_2 \leq 2.7718M_2e^{-0.0257t}, \quad \forall t \geq 0. \tag{3.46}$$



(a) (b)

FIGURE 3.3: The time evolutions of real and imaginary parts of the error system (3.8) with the impulsive control for $T_a = 0.013$ and $\varpi = 1.003$.

A similar conclusion can be deduced from the inequality (3.46) *as it has been deduced in Case 1 from the inequality* (3.45) *except the rate of convergence which is* $\bar{r}_1 = 0.0257$*. That is, the synchronization of CVNN systems with time-varying discrete and distributed delays can be achieved by impulsive control with heterogeneous synchronizing, and desynchronizing impulses provided the occurrence of the impulses at the points of impulsive sequence should obey Definition of AII given in the chapter 1. The exponential convergence of the real and imaginary parts of the error system* (3.8) *are illustrated in Figure 3.3. Hence, Theorem 3.3.1 is verified.*

Remark 3.9. It is worth noting that the Halanay inequality applied in the previous results [60, 61, 62, 63, 64] to solve the stability or the synchronization problems of CVNN systems with time-delay is only valid for $\Pi_1 > \Pi_2 \geq 0$. On the contrary, in Example 3.1, we have $\Pi_1 = 5.1772 < \Pi_2 = 22.3032$, which means that the previous results are not valid in our case. Moreover, the results of this chapter are based on extended Halanay inequality introduced in Lemma 3.2 which is valid for all $\Pi_1 \in \mathbb{R}$ and $\Pi_2 \geq 0$. Therefore, the results in this chapter are better as compared to those obtained in [60, 61, 62, 63, 64].

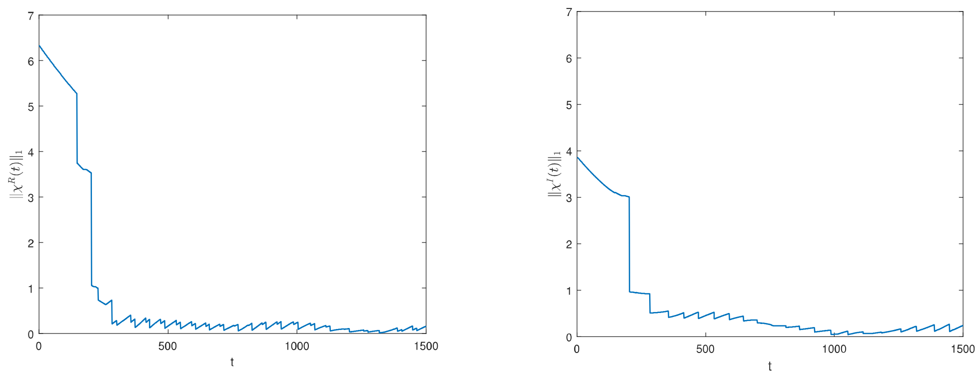
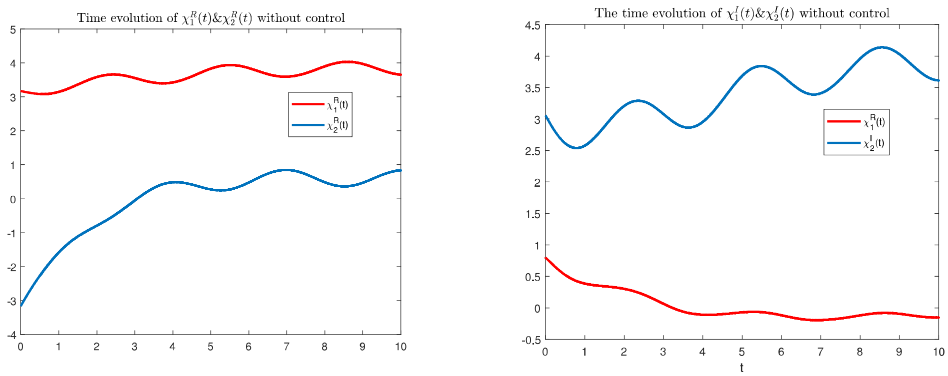


FIGURE 3.4: The time evolutions of real and imaginary parts of the error system (3.8) with activation functions (3.47) and impulsive control for $T_a = 0.098$ and $\varpi = 1.0035$.

Example 3.2. *In order to verify the results of Theorem 3.7, let us consider the same CVNN systems as given in Example 3.1 except the activation functions whose real and imaginary parts depend only on w^R and w^I, respectively.*

$$f_j(w) = g_j(w) = h_j(w) = \frac{1 - e^{-w_j^R}}{1 + e^{-w_j^R}} + i\frac{1}{1 + e^{-w_j^I}}. \tag{3.47}$$

In this example, we will verify the results for $q = 1$. By a simple calculation, we get $L_{qj}^{RR} = M_{qj}^{RR} = H_{qj}^{RR} = 0.5$, $L_{qj}^{II} = M_{qj}^{II} = H_{qj}^{II} = 0.25$ for $j = 1, 2$, $\mu_1(-\bar{C}) = -0.7$, $\mu_1(\hat{A}^*\Gamma) = 0.3750$, $\|\bar{A}_1\|_1 = \|\bar{A}_2\|_1 = 0.6$, $\|\bar{B}_1\|_1 = \|\bar{B}_2\|_1 = 0.7$, $\|\bar{D}_1\|_1 = \|\bar{D}_2\|_1 = 1.4$. Using these values, one can obtain $\Phi_1 = 0.5$, $\Phi_2 = 0.25$, $\Omega_1 = 0.5$, $\Omega_2 = 0.25$, $\widehat{\Pi_1} = 0.5750$, and $\widehat{\Pi_2} = 3.15$. Set $\theta = 3$. Then the maximum impulsive interval will be $\kappa = 0.0985$ for which the inequality $\widehat{\Pi_1} + \widehat{\Pi_2}\theta - \frac{\ln\theta}{\kappa} = -1.1284 < 0$ holds. Using Remark 3.2.4, we can choose $\lambda_2 = 0.1 \in (0,\ 0.1016)$. For randomly chosen initial condition $\chi(s) = \begin{pmatrix} 3.17 - 3.16i \\ 0.80 + 3.06i \end{pmatrix} \in \mathbb{C}^2$ for all $s \in [-1, 0]$, the trajectories of the real and imaginary parts of the error system (3.8) without impulsive controller are depicted in Figure 3.5.



(a) (b)

FIGURE 3.5: The time evolution of real and imaginary parts of the error system (3.8) with activation functions (3.47) without impulsive control.

Case 1: Let the matrices of heterogeneous impulsive strengths at distinct points of the impulsive sequence $\mathcal{S}$ *is* $\bar{\Gamma}_1 = diag\{1.2,\ 0.95,\ 1.1,\ 0.8\}$ *and* $\bar{\Gamma}_2 = diag\{0.5,\ 0.4,\ 0.49,\ 0.5\}$*, i.e.,* $\varpi_1 = \|\bar{\Gamma}_1\|_2 = 1.2$ *and* $\varpi_2 = \|\bar{\Gamma}_2\|_2 = 0.5$*, then the corresponding AIG is* $\varpi = \frac{1.2+0.5}{2} = 0.85 < 1$*. Suppose that* $N_0 = 2,\ T_a = 0.0588,$ *, then* $\tilde{\beta}_1 = 0.01$*. One can calculate* $\|\chi(s)\|_1 = 7.6388$ *and* $\tilde{r}_2 = 0.99$*. From the inequality (3.13), we have*

$$\|\mathcal{E}(t)\|_1 \leq 7.6388 M_1 e^{-0.99t}, \forall t \geq 0. \tag{3.48}$$

From inequality (3.48), it is clear that the error system (3.8) converges globally exponentially at the convergence rate $\tilde{r}_2 = 0.0999$*, which are confirmed through the illustration of the convergence of real and imaginary parts of the system (3.8) depicted in Figures 3.6. That is, the slave system (3.2) under the hybrid impulsive control (3.7) with heterogeneous impulses is synchronizing globally exponentially to the master system (3.1) at the convergence rate* $\tilde{r}_2 = 0.0999$*.*

Case 2: Let $\bar{\Gamma}_1 = diag\{1.001,\ 0.3,\ 1.005,\ 1.003\}$ *and* $\bar{\Gamma}_2 = diag\{0.2,\ 1.002,\ 1.001,\ 0.2\}$*, i.e.,* $\varpi_1 = \|\bar{\Gamma}_1\|_2 = 1.005$ *and* $\varpi_2 = \|\bar{\Gamma}_2\|_2 = 1.002$*, then the corresponding average impulsive gain is* $\varpi = \frac{1.005+1.002}{2} = 1.0035 > 1$*. Suppose that the average impulsive*

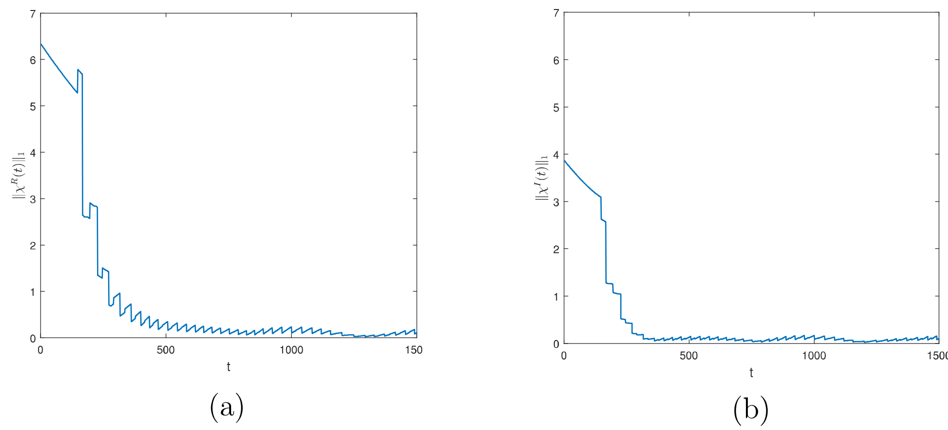


FIGURE 3.6: The time evolution of real and imaginary parts of the error system (3.8) with activation functions (3.47) and impulsive control for $T_a = 0.0588$ and $\varpi = 0.85$.

interval $T_a = 0.098,\ N_0 = 2$, *then we have* $\tilde{\beta}_2 = 0.0357$ *and* $\bar{r}_2 = 0.0643$. *From the inequality* (3.14), *we can write*

$$\|\mathcal{E}(t)\|_1 \leq 7.6388 M_2 e^{-0.0643t}, \quad \forall t \geq 0. \tag{3.49}$$

Inequality (3.49) *states that the error system* (3.8) *converges globally exponentially at convergence rate* $\bar{r}_2 = 0.0643$. *The exponential convergence of the real and imaginary parts of the error system is depicted through Figure 3.4. Hence, Theorem 3.7 is verified.*

3.5 Conclusions

In this chapter, the heterogeneous impulsive effect on synchronization problem of master-slave CVNNs with mixed time-varying delays has been investigated. The heterogeneous impulses are composite form of time-varying and non-identical impulses, i.e., impulsive strengths are time-varying and non-identical for each neuron of the neural network. In addition, the time-varying synchronizing and desynchronizing impulses can be taken into account in heterogeneous impulsive effect. In order to deal with two types of heterogeneous impulses in one impulsive sequence and to remove upper or lower bound of the impulsive sequence, the concepts of AIG and AII, respectively have been applied. Assuming that the CVAFs can be separated into real and imaginary parts, the master-slave CVNNs are transformed into two RVNNs. Based on the novel concept of matrix measure and the generalized Halanay inequality for impulsive systems, sufficient criteria involving the terms of AIG and AII have been derived to ascertain the exponential synchronization of CVNNs with mixed time varying delays under heterogeneous impulses. The dependency of convergence rates on AIG and AII is explicitly discussed. The efficiency of the theoretical results has been numerically verified through two numerical examples of complex valued neural networks.

The following enhancements to the theoretical results could be considered in the future work. Based on matrix measure method, the synchronization criteria of CVNNs without separating the activation functions into real and imaginary parts can be derived. The generalized Halanay inequality applied in this chapter is valid only if the bias parameters of master-slave CVNNs are equal. Therefore generalization of the impulsive inequality in Lemma 3.2 will be an important task to study the master-slave synchronization of CVNNs with distinct bias parameters.

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