Chapter 2

Generalized-Hukuhara Subdifferential Analysis and Its Application in Nonconvex Composite Interval Optimization Problems

2.1 Introduction

Nonconvex and nonsmooth optimization problems are of great interest while modeling the problems in applied mathematics and operations research. A vast majority of machine learning algorithms train their models by solving optimization problems in which the designed objective is nonconvex and nonsmooth. The calculus of subgradients and subdifferentials of nonsmooth IVFs plays a key role in the analysis of nonconvex and nonsmooth composite IOPs, and the minimization of such functions can model a variety of imaging tasks.

2.2 Motivation and Contribution

The existing ideas on the calculus of nonsmooth IVFs are inadequate to derive the optimality conditions of nonsmooth IOPs. Several theories on optimal solutions for nondifferentiable interval-programming problems (see [25, 99]), the KKT theory for IOPs (see [48,50,100]), and Fritz-John theory [101] for IOPs do not hold for certain IVFs (see Note 2.1 and Note 2.2). In this chapter, we first briefly propose gH-subdifferential calculus for IVFs. We report a result on the directional derivative of the maximum of a finite number of convex IVFs. Next, we define the weak efficient solution of IOPs that observes the comparability of functions. By using this idea of weak efficient solution to an IOP, we derive a Fermat-type, a Fritz-John-type, and a KKT-type optimality conditions for weak efficient solutions of nonsmooth IOPs that do not involve convexity assumption. Further, it is known that the nature of an arbitrary problem is nonconvex; due to this, a relation has been proposed to estimate the weak efficient solution of a nonconvex composite problem to an IOP with the help of gH-subdifferential calculus of IVFs. The entire study is supported by suitable illustrative examples.

2.3 gH-Subdifferential

In this section, we derive some results based on gH-subgradient and gH-subdifferential of IVFs. In the sequel, we propose the Fermat-type, Fritz-John-type and a KKT-type conditions for IOPs.

Definition 2.1 (Convex set of intervals). *Let $\boldsymbol{S}$ be a nonempty subset of $I(\mathbb{R})^n$. Then, $\boldsymbol{S}$ is said to be a convex set if for every $\widehat{\boldsymbol{Y}},\ \widehat{\boldsymbol{Z}} \in \boldsymbol{S}$,*

$$\beta_1 \odot \widehat{\boldsymbol{Y}} \oplus \beta_2 \odot \widehat{\boldsymbol{Z}} \in \boldsymbol{S} \text{ for all } \beta_1,\ \beta_2 \in [0,1] \text{ with } \beta_1 + \beta_2 = 1.$$

Definition 2.2 (Convex combination of intervals). *An interval $\widehat{\boldsymbol{X}} \in I(\mathbb{R})^n$ is said to* *be a convex combination of the intervals* $\widehat{\boldsymbol{X}}_1, \widehat{\boldsymbol{X}}_2, \ldots, \widehat{\boldsymbol{X}}_p \in I(\mathbb{R})^n$ *if*

$$\widehat{\boldsymbol{X}} = \bigoplus_{j=1}^{p} \beta_j \odot \widehat{\boldsymbol{X}}_j \text{ with } \beta_j \geq 0 \text{ and } \sum_{j=1}^{p} \beta_j = 1.$$

Definition 2.3 (Convex hull of a set of intervals). *For a nonempty set* $\boldsymbol{S} \subseteq I(\mathbb{R})^n$*, the convex hull of set* $\boldsymbol{S}$*,* $\text{co}(\mathbf{S})$*, is defined by*

$$\text{co}(\mathbf{S}) = \left\{ \widehat{\boldsymbol{X}} \in I(\mathbb{R})^n : \widehat{\boldsymbol{X}} = \bigoplus_{j=1}^{p} \beta_j \odot \widehat{\boldsymbol{X}}_j,\ \ \widehat{\boldsymbol{X}}_j \in \boldsymbol{S} \text{ with } \beta_j \geq 0 \text{ and } \sum_{j=1}^{p} \beta_j = 1 \right\}.$$

Next, we define the supremum and infimum of IVFs whose domain being a set of interval vectors.

Definition 2.4 (Supremum and infimum of an IVF). *Let* $\boldsymbol{S} \subseteq I(\mathbb{R})^n$*. Then, supremum and infimum of an extended IVF* $\mathbf{\Gamma} : \boldsymbol{S} \to \overline{I(\mathbb{R})}$ *are defined by*

$$\sup_{\boldsymbol{S}} \mathbf{\Gamma} = \sup\{\mathbf{\Gamma}(\widehat{\boldsymbol{X}}) : \widehat{\boldsymbol{X}} \in \boldsymbol{S}\} \text{ and } \inf_{\boldsymbol{S}} \mathbf{\Gamma} = \inf\{\mathbf{\Gamma}(\widehat{\boldsymbol{X}}) : \widehat{\boldsymbol{X}} \in \boldsymbol{S}\}, \text{ respectively.}$$

Lemma 2.1 *Let* $\boldsymbol{S} \subseteq I(\mathbb{R})^n$ *and* $\mathbf{\Gamma} : \boldsymbol{S} \to \overline{I(\mathbb{R})}$ *be an extended IVF. Then, for* $\boldsymbol{S}_1, \boldsymbol{S}_2 \subseteq \boldsymbol{S}$ *with* $\boldsymbol{S}_1 \subseteq \boldsymbol{S}_2$ *and* $\beta \geq 0$,

(i) $\inf\limits_{\boldsymbol{S}_2} \mathbf{\Gamma} \preceq \inf\limits_{\boldsymbol{S}_1} \mathbf{\Gamma}$,

(ii) $\sup\limits_{\boldsymbol{S}_1} \mathbf{\Gamma} \preceq \sup\limits_{\boldsymbol{S}_2} \mathbf{\Gamma}$,

(iii) $\inf\limits_{\boldsymbol{S}} (\beta \odot \mathbf{\Gamma}) = \beta \odot \inf\limits_{\boldsymbol{S}} \mathbf{\Gamma}$, *and*

(iv) $\sup\limits_{\boldsymbol{S}} (\beta \odot \mathbf{\Gamma}) = \beta \odot \sup\limits_{\boldsymbol{S}} \mathbf{\Gamma}$.

Proof: See Appendix B.1. □

Remark 2.1 *It is to note that '$\beta \geq 0$' is a necessary condition in (iii) and (iv) of Lemma 2.1. For instance, consider the set of interval vectors $\boldsymbol{S} = \{\widehat{\boldsymbol{X}} = ([1, n], [1, 2n]) : n \in \mathbb{N}\} \subset I(\mathbb{R})^2$ and the IVF $\boldsymbol{\Gamma} : \boldsymbol{S} \to \overline{I(\mathbb{R})}$ given by*

$$\boldsymbol{\Gamma}(\widehat{\boldsymbol{X}}) = \boldsymbol{\Gamma}(\boldsymbol{X}_1, \boldsymbol{X}_2) = \boldsymbol{X}_1 \oplus \boldsymbol{X}_2.$$

In view of Definition 2.4 and taking $\beta = -1$, we observe that

$$\begin{aligned}
\beta \odot \inf_{\boldsymbol{S}} \boldsymbol{\Gamma} = \beta \odot \inf\{\boldsymbol{\Gamma}(\widehat{\boldsymbol{X}}) : \widehat{\boldsymbol{X}} \in \boldsymbol{S}\} &= (-1) \odot \inf\{[1, n] \oplus [1, 2n] : n \in \mathbb{N}\} \\
&= (-1) \odot \inf\{[2, 3n] : n \in \mathbb{N}\} \\
&= (-1) \odot [2, 3] = [-3, -2] \\
\text{and } \inf_{\boldsymbol{S}}(\beta \odot \boldsymbol{\Gamma}) = \inf\{(\beta \odot \boldsymbol{\Gamma})(\widehat{\boldsymbol{X}}) : \widehat{\boldsymbol{X}} \in \boldsymbol{S}\} &= \inf\{(-1) \odot ([1, n] \oplus [1, 2n]) : n \in \mathbb{N}\} \\
&= \inf\{(-1) \odot [2, 3n] : n \in \mathbb{N}\} \\
&= \inf\{[-3n, -2] : n \in \mathbb{N}\}.
\end{aligned}$$

Since the set $\{-3, -6, -9, \ldots\}$ has no lower bound in $\mathbb{R}$, $(\beta \odot \boldsymbol{\Gamma})$ has no lower bound in $I(\mathbb{R})$. Thus, infimum of $(\beta \odot \boldsymbol{\Gamma})$ does not exist in $I(\mathbb{R})$. Hence, $\inf_{\boldsymbol{S}}(\beta \odot \boldsymbol{\Gamma})$ does not exist in $I(\mathbb{R})$.
Similarly, we see that

$$\begin{aligned}
\sup_{\boldsymbol{S}}(\beta \odot \boldsymbol{\Gamma}) = \sup\{(\beta \odot \boldsymbol{\Gamma})(\widehat{\boldsymbol{X}}) : \widehat{\boldsymbol{X}} \in \boldsymbol{S}\} &= \sup\{((-1) \odot ([1, n]) \oplus [1, 2n]) : n \in \mathbb{N}\} \\
&= \sup\{((-1) \odot [2, 3n]) : n \in \mathbb{N}\} \\
&= \sup\{[-3n, -2] : n \in \mathbb{N}\} = [-3, -2] \\
\text{and } \beta \odot \sup_{\boldsymbol{S}} \boldsymbol{\Gamma} = \beta \odot \sup\{\boldsymbol{\Gamma}(\widehat{\boldsymbol{X}}) : \widehat{\boldsymbol{X}} \in \boldsymbol{S}\} &= (-1) \odot \sup\{[1, n] \oplus [1, 2n] : n \in \mathbb{N}\} \\
&= (-1) \odot \sup\{[2, 3n] : n \in \mathbb{N}\}.
\end{aligned}$$

Since the set $\{3, 6, 9, \ldots\}$ *has no upper bound in* $\mathbb{R}$, $\mathbf{\Gamma}$ *has no upper bound in* $I(\mathbb{R})$. *Thus, supremum of* $\mathbf{\Gamma}$ *does not exist in* $I(\mathbb{R})$. *Hence,* $\beta \odot \sup_{\boldsymbol{S}} \mathbf{\Gamma}$ *does not exist in* $I(\mathbb{R})$. *Therefore, we see that (iii) and (iv) of Lemma 2.1 are not true if* $\beta = -1$.

Lemma 2.2 *Let* $\boldsymbol{S} \subseteq I(\mathbb{R})^n$ *and* $\mathbf{\Gamma}_1,\ \mathbf{\Gamma}_2 : \boldsymbol{S} \to \overline{I(\mathbb{R})}$ *be extended IVFs. Then,*

(i) $\inf_{\boldsymbol{S}} \mathbf{\Gamma}_1 \oplus \inf_{\boldsymbol{S}} \mathbf{\Gamma}_2 \preceq \inf_{\boldsymbol{S}} (\mathbf{\Gamma}_1 \oplus \mathbf{\Gamma}_2)$ *and*

(ii) $\sup_{\boldsymbol{S}} (\mathbf{\Gamma}_1 \oplus \mathbf{\Gamma}_2) \preceq \sup_{\boldsymbol{S}} \mathbf{\Gamma}_1 \oplus \sup_{\boldsymbol{S}} \mathbf{\Gamma}_2$.

Proof: See Appendix B.2. □

Definition 2.5 (Closed set in $I(\mathbb{R})^n$). *A set* $\boldsymbol{S} \subseteq I(\mathbb{R})^n$ *is called* closed *if for every convergent sequence* $\{\widehat{\boldsymbol{G}}_k\}$ *in* $\boldsymbol{S}$ *converging to* $\widehat{\boldsymbol{G}}$, $\widehat{\boldsymbol{G}}$ *must belong to* $\boldsymbol{S}$.

Definition 2.6 (Bounded set in $I(\mathbb{R})^n$). *Let* $\boldsymbol{S} \subseteq I(\mathbb{R})^n$. *An interval* $\widehat{\boldsymbol{A}} \in I(\mathbb{R})^n$ *is said to be a lower bound or an upper bound of* $\boldsymbol{S}$ *if*

$$\widehat{\boldsymbol{A}} \preceq \widehat{\boldsymbol{C}} \text{ or } \widehat{\boldsymbol{C}} \preceq \widehat{\boldsymbol{A}}, \text{ respectively for all } \widehat{\boldsymbol{C}} \in \boldsymbol{S}.$$

A nonempty set $\boldsymbol{S} \subseteq I(\mathbb{R})^n$, *which is bounded above and below, is said to be* bounded.

Theorem 2.1 (Finite union of closed sets in $I(\mathbb{R})^n$). *Let* $\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_p$ *be a finite collection of closed sets in* $I(\mathbb{R})^n$. *Then,* $\bigcup_{j=1}^{p} \boldsymbol{S}_j$ *is closed.*

Proof: Let $\{\widehat{\mathbf{G}}_k\}$ be an arbitrary sequence in $\bigcup_{j=1}^{p} \mathbf{S}_j$ that converges to $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$, where $\widehat{\mathbf{G}}_k = (\mathbf{G}_{1_k}, \mathbf{G}_{2_k}, \ldots, \mathbf{G}_{n_k})^\top$ and $\widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n)^\top$. Since $\{\widehat{\mathbf{G}}_k\}$ contains infinitely many terms and $\bigcup_{j=1}^{p} \mathbf{S}_j$ is union of finite number of sets, there exists at least one $\mathbf{S}_j$ that contains infinitely many terms of the sequence $\{\widehat{\mathbf{G}}_k\}$. Hence, by Lemma 1.9, we get a subsequence of $\{\widehat{\mathbf{G}}_k\}$ in $\mathbf{S}_j$ which converges to $\widehat{\mathbf{G}}$. Since $\mathbf{S}_j$ is closed, by Definition 2.5, $\widehat{\mathbf{G}} \in \mathbf{S}_j$, which implies that $\widehat{\mathbf{G}} \in \bigcup_{j=1}^{p} \mathbf{S}_j$. Hence, $\bigcup_{j=1}^{p} \mathbf{S}_j$ is closed. □

Theorem 2.2 (Finite union of bounded sets in $I(\mathbb{R})^n$). *Let $\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_p$ be a finite collection of bounded sets in $I(\mathbb{R})^n$. Then, $\bigcup_{j=1}^{p} \boldsymbol{S}_j$ is bounded.*

Proof: Since $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_p$ are bounded sets in $I(\mathbb{R})^n$, there exist $\widehat{\mathbf{X}}_1, \widehat{\mathbf{X}}_2, \ldots, \widehat{\mathbf{X}}_p$ and $\widehat{\mathbf{Y}}_1, \widehat{\mathbf{Y}}_2, \ldots, \widehat{\mathbf{Y}}_p$ in $I(\mathbb{R})^n$ such that

$$\widehat{\mathbf{X}}_j \preceq \widehat{\mathbf{C}}_j \preceq \widehat{\mathbf{Y}}_j \text{ for each } \widehat{\mathbf{C}}_j \text{ in } \mathbf{S}_j,\ j = 1, 2, \ldots, p. \tag{2.1}$$

where $\widehat{\mathbf{X}}_j = (\mathbf{X}_{1_j}, \mathbf{X}_{2_j}, \ldots, \mathbf{X}_{n_j})^\top$ and $\widehat{\mathbf{Y}}_j = (\mathbf{Y}_{1_j}, \mathbf{Y}_{2_j}, \ldots, \mathbf{Y}_{n_j})^\top$.
Let $\widehat{\mathbf{X}} = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n)^\top$ and $\widehat{\mathbf{Y}} = (\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n)^\top$, where $\mathbf{X}_\mu = [\min \underline{x}_{\mu_j}, \min \overline{x}_{\mu_j}]$ and $\mathbf{Y}_\mu = [\max \underline{y}_{\mu_j}, \max \overline{y}_{\mu_j}]$ for all $\mu = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, p$. Then,

$$\widehat{\mathbf{X}} \preceq \widehat{\mathbf{X}}_j \text{ and } \widehat{\mathbf{Y}}_j \preceq \widehat{\mathbf{Y}} \text{ for each } j = 1, 2, \ldots, p. \tag{2.2}$$

From (2.1) and (2.2), we obtain that for all $j = 1, 2, \ldots, p$,

$$\widehat{\mathbf{X}} \preceq \widehat{\mathbf{C}}_j \preceq \widehat{\mathbf{Y}}$$
$$\text{or, } \widehat{\mathbf{X}} \preceq \widehat{\mathbf{Z}} \preceq \widehat{\mathbf{Y}},\ \text{ where } \widehat{\mathbf{Z}} \in \bigcup_{j=1}^{p} \mathbf{S}_j.$$

Hence, $\bigcup_{j=1}^{p} \mathbf{S}_j$ is a bounded set. □

Theorem 2.3 (Closedness of the convex hull of a set in $I(\mathbb{R})^n$). *Let $\boldsymbol{S}$ be a nonempty closed set in $I(\mathbb{R})^n$. Then, the convex hull of $\boldsymbol{S}$ is a closed set.*

Proof: Let $\{\widehat{\mathbf{G}}_k\}$ be an arbitrary sequence in $\text{co}(\mathbf{S})$ that converges to $\widehat{\mathbf{G}} \in I(\mathbb{R})^n$, where $\widehat{\mathbf{G}}_k = (\mathbf{G}_{1_k}, \mathbf{G}_{2_k}, \ldots, \mathbf{G}_{n_k})^\top$ and $\widehat{\mathbf{G}} = (\mathbf{G}_1, \mathbf{G}_2, \ldots, \mathbf{G}_n)^\top$.
Since $\{\widehat{\mathbf{G}}_k\} \in \text{co}(\mathbf{S})$, there exists $\{\beta_j^k\} \subset \mathbb{R}_+$ and $\{\widehat{\mathbf{G}}_j^k\} \in \mathbf{S}$ such that

$$\widehat{\mathbf{G}}_k = \bigoplus_{j=1}^{n} \beta_j^k \odot \widehat{\mathbf{G}}_j^k \text{ for all } k \in \mathbb{N} \text{ with } \sum_{j=1}^{n} \beta_j^k = 1.$$

Also, $\beta_j^k \geq 0$ with $\sum_{j=1}^n \beta_j^k = 1$ gives that β_j^k is a bounded sequence. Thus, we have

$$\beta_j^k \to \beta_j \text{ such that } \beta_j \geq 0 \text{ with } \sum_{j=1}^n \beta_j = 1.$$

Since $\mathbf{S}$ is closed, $\{\widehat{\mathbf{G}}_j^k\} \in \mathbf{S}$ must have a convergent subsequence. Assume that $\widehat{\mathbf{G}}_j^k \to \widehat{\mathbf{G}}_j$. Therefore, by closedness of $\mathbf{S}$,

$$\widehat{\mathbf{G}}_j \in \mathbf{S} \text{ and } \widehat{\mathbf{G}} = \bigoplus_{j=1}^n \beta_j \odot \widehat{\mathbf{G}}_j \in \text{co}(\mathbf{S}).$$

Thus, $\widehat{\mathbf{G}}_k \to \widehat{\mathbf{G}} \in \text{co}(\mathbf{S})$. Hence, $\text{co}(\mathbf{S})$ is a closed set. □

Theorem 2.4 (Boundedness of convex hull of a set in $I(\mathbb{R})^n$). *Let* $\boldsymbol{S}$ *be a nonempty bounded set in* $I(\mathbb{R})^n$. *Then, the convex hull of* $\boldsymbol{S}$ *is bounded.*

Proof: Since $\mathbf{S}$ is bounded, there exist $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{Y}}$ in $I(\mathbb{R})^n$ such that

$$\widehat{\mathbf{X}} \preceq \widehat{\mathbf{C}}_j \preceq \widehat{\mathbf{Y}} \text{ for all } \widehat{\mathbf{C}}_j \text{ in } \mathbf{S}.$$

Therefore, for all $\widehat{\mathbf{C}}_j$ in $\mathbf{S}$, $\beta_j \geq 0$, and $\sum_{j=1}^p \beta_j = 1$, we obtain

$$\begin{aligned}
&\bigoplus_{j=1}^p \beta_j \odot \widehat{\mathbf{X}} \preceq \bigoplus_{j=1}^p \beta_j \odot \widehat{\mathbf{C}}_j \preceq \bigoplus_{j=1}^p \beta_j \odot \widehat{\mathbf{Y}} \\
\Longrightarrow &\bigoplus_{j=1}^p \beta_j \odot \widehat{\mathbf{X}} \preceq \widehat{\mathbf{Z}} \preceq \bigoplus_{j=1}^p \beta_j \odot \widehat{\mathbf{Y}} \\
\Longrightarrow &\widehat{\mathbf{X}} \preceq \widehat{\mathbf{Z}} \preceq \widehat{\mathbf{Y}},
\end{aligned}$$

where $\widehat{\mathbf{Z}} = \bigoplus_{j=1}^p \beta_j \odot \widehat{\mathbf{C}}_j \in \text{co}(\mathbf{S})$. Since $\widehat{\mathbf{Z}}$ is arbitrary, $\text{co}(\mathbf{S})$ is a bounded set. □

Example 2.1 *Let* $\mathcal{Y} \subseteq \mathbb{R}^n$ *be convex. Take the IVF* $\boldsymbol{F} : \mathcal{Y} \to I(\mathbb{R})$, *which is defined by*

$$\boldsymbol{F}(y) = \|y\|_0 \odot \boldsymbol{C}, \tag{2.3}$$

where $\|y\|_0$ is the number of nonzero components in y and $\boldsymbol{0} \prec \boldsymbol{C} \in I(\mathbb{R})$. Let us calculate the gH-subdifferential set of $\boldsymbol{F}$ at $\bar{y} \in \mathcal{Y}$. There arise the following cases:

Case 1: If $\bar{y} = 0$ and $\widehat{\boldsymbol{G}} \in \partial \boldsymbol{F}(\bar{y})$, then for all $y \in \mathbb{R}^n$

$$(y - \bar{y})^\top \odot \widehat{\boldsymbol{G}} \preceq \boldsymbol{F}(y) \ominus_{gH} \boldsymbol{F}(\bar{y}) \implies y^\top \odot \widehat{\boldsymbol{G}} \preceq \|y\|_0 \odot \boldsymbol{C}.$$

It is possible only for $\widehat{\boldsymbol{G}} = \boldsymbol{0}$. Therefore, $\partial \boldsymbol{F}(0) = \{\boldsymbol{0}\}$.

Case 2: If $\bar{y} \neq 0$ and $\widehat{\boldsymbol{G}} \in \partial \boldsymbol{F}(\bar{y})$, then

$$(y - \bar{y})^\top \odot \widehat{\boldsymbol{G}} \preceq \|y\|_0 \odot \boldsymbol{C} \ominus_{gH} \|\bar{y}\|_0 \odot \boldsymbol{C} \text{ for all } y \in \mathbb{R}^n. \tag{2.4}$$

Let $\|y\|_0 = p$ and $\|\bar{y}\|_0 = q$. Then, we have

$$\bigoplus_{i=1}^{n} (y_i - \bar{y}_i) \odot \boldsymbol{G}_i \preceq [\min\{(p-q)\underline{c},\ (p-q)\overline{c}\},\ \max\{(p-q)\underline{c},\ (p-q)\overline{c}\}]. \tag{2.5}$$

Without loss of generality, let the first m components of $(y - \bar{y})$ be nonnegative and the rest $(n - m)$ be negative. Then, from (2.5), we have

$$\bigoplus_{i=1}^{m} (y_i - \bar{y}_i) \odot \boldsymbol{G}_i \bigoplus_{j=m+1}^{n} (y_j - \bar{y}_j) \odot \boldsymbol{G}_j$$
$$\preceq [\min\{(p-q)\underline{c},\ (p-q)\overline{c}\},\ \max\{(p-q)\underline{c},\ (p-q)\overline{c}\}].$$

Therefore, we get

$$\sum_{i=1}^{m} (y_i - \bar{y}_i)\underline{g_i} + \sum_{j=m+1}^{n} (y_j - \bar{y}_j)\overline{g_j} \leq \min\{(p-q)\underline{c},\ (p-q)\overline{c}\} \text{ and}$$
$$\sum_{i=1}^{m} (y_i - \bar{y}_i)\overline{g_i} + \sum_{j=m+1}^{n} (y_j - \bar{y}_j)\underline{g_j} \leq \max\{(p-q)\underline{c},\ (p-q)\overline{c}\}.$$

The above inequalities hold only when $\underline{g_i}, \overline{g_i}, \underline{g_j}$ and $\overline{g_j}$ are all zeros and $q \leq p$. But

$q \leq p$ for all y holds only when $\bar{y} = 0$. Hence, we have a contradiction to $\bar{y} \neq 0$.

From Case 1: and Case 2:, it is clear that gH-subgradient of IVF (2.3) exists only at $\bar{y} = 0$ and $\partial \boldsymbol{F}(0) = \{\boldsymbol{0}\}$.

Remark 2.2 *The IVF (2.3) of Example 2.1 does not satisfy the property $\boldsymbol{F}(\beta y) = \beta \odot \boldsymbol{F}(y)$. For instance, consider $y = (1, 0, \ldots, 0)$ and $\beta = \frac{1}{2}$. Then,*

$$\boldsymbol{F}(\beta y) = \boldsymbol{C} \text{ and } \beta \odot \boldsymbol{F}(y) = \frac{1}{2} \odot \boldsymbol{C}.$$

Hence, $\boldsymbol{F}(\beta y) \neq \beta \odot \boldsymbol{F}(y)$.

Remark 2.3 *The IVF (2.3) of Example 2.1 is not convex. For instance, consider $y_1 = (1, 0, \ldots, 0)$ and $y_2 = (0, 1, \ldots, 0)$. Then, for $\beta_1 = \beta_2 = \frac{1}{2}$, we have*

$$\beta_1 \odot \boldsymbol{F}(y_1) \oplus \beta_2 \odot \boldsymbol{F}(y_2) = \boldsymbol{C} \prec 2 \odot \boldsymbol{C} = \boldsymbol{F}(\beta_1 y_1 + \beta_2 y_2).$$

Thus, the IVF $\boldsymbol{F}$ is not convex.

Lemma 2.3 *Let $\mathcal{Y} \subseteq \mathbb{R}^n$ and $\boldsymbol{F} : \mathcal{Y} \to I(\mathbb{R})$ be an IVF. Then, for any $\bar{y} \in \text{dom}(\boldsymbol{F})$ and $\beta \geq 0$*

$$\partial(\beta \odot \boldsymbol{F})(\bar{y}) = \beta \odot \partial \boldsymbol{F}(\bar{y}),$$

where $\text{dom}(\beta \odot \boldsymbol{F}) = \text{dom}(\boldsymbol{F})$.

Proof: Let $\widehat{\mathbf{G}} \in \partial \mathbf{F}(\bar{y})$. Then, for any $y \in \text{dom}(\mathbf{F})$,

$$\begin{aligned}
& (y - \bar{y})^\top \odot \widehat{\mathbf{G}} \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{y}) \\
\iff \; & \beta \odot ((y - \bar{y})^\top \odot \widehat{\mathbf{G}}) \preceq \beta \odot \big(\mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{y})\big) \text{ for } \beta \geq 0 \\
\iff \; & (y - \bar{y})^\top \odot (\beta \odot \widehat{\mathbf{G}}) \preceq \beta \odot (\mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{y})) \\
\iff \; & (y - \bar{y})^\top \odot (\beta \odot \widehat{\mathbf{G}}) \preceq (\beta \odot \mathbf{F})(y) \ominus_{gH} (\beta \odot \mathbf{F})(\bar{y}))
\end{aligned}$$

$$\iff \beta \odot \widehat{\mathbf{G}} \in \partial(\beta \odot \mathbf{F})(\bar{y}),$$

which is the required result. □

Remark 2.4 *It is to observe that in Lemma 2.3, the condition on β to be nonnegative is necessary. For instance, consider $\beta = -1$ and the IVF $\boldsymbol{F} : \mathbb{R} \to I(\mathbb{R})$ given by*

$$\boldsymbol{F}(y) = [1, 2] \odot |y|.$$

We observe from [6, Example 1] that

$$\partial \boldsymbol{F}(0) = \{\boldsymbol{G} \in I(\mathbb{R}) : [-2, -1] \preceq \boldsymbol{G} \preceq [1, 2]\}$$
$$\text{and } \beta \odot \partial \boldsymbol{F}(0) = \{(-1) \odot \boldsymbol{G} \in I(\mathbb{R}) : [-2, -1] \preceq \boldsymbol{G} \preceq [1, 2]\}.$$

Now, let us assume that there exists $\boldsymbol{G} \in I(\mathbb{R})$ such that $\boldsymbol{G} \in \partial(\beta \odot \boldsymbol{F})(0)$. Then, for any $y \in \mathbb{R}$, we must have

$$(y - 0) \odot \boldsymbol{G} \preceq (\beta \odot \boldsymbol{F})(y) \ominus_{gH} (\beta \odot \boldsymbol{F})(0)$$
$$\text{i.e., } y \odot \boldsymbol{G} \preceq [-2, -1] \odot |y|,$$

which requires the following cases:

Case 1: for $y < 0$,

$$y \odot \boldsymbol{G} \preceq [-2, -1] \odot (-y), \text{ i.e., } [1, 2] \preceq \boldsymbol{G}, \text{ and}$$

Case 2: for $y > 0$,

$$y \odot \boldsymbol{G} \preceq [-2, -1] \odot y, \quad \text{i.e., } \boldsymbol{G} \preceq [-2, -1].$$

As there exists no $\boldsymbol{G} \in I(\mathbb{R})$ that satisfies $[1, 2] \preceq \boldsymbol{G}$ and $\boldsymbol{G} \preceq [-2, -1]$, we conclude *that*

$$\partial(\beta \odot \boldsymbol{F})(0) = \emptyset \neq \beta \odot \partial \boldsymbol{F}(0).$$

Lemma 2.4 *Let $\boldsymbol{F}_1, \boldsymbol{F}_2 : \mathcal{Y} \to I(\mathbb{R})$ be two convex IVFs on the nonempty convex set $\mathcal{Y} \subseteq \mathbb{R}^n$. Then, $\boldsymbol{F}_1 \oplus \boldsymbol{F}_2$ is a convex IVF on $\mathcal{Y}$.*

Proof: Let $\mathbf{F}(y) = \mathbf{F}_1(y) \oplus \mathbf{F}_2(y)$. Then, for any $y_1, y_2 \in \mathcal{Y}$ and $\beta \in [0, 1]$,

$$\begin{aligned}
&\mathbf{F}(\beta y_1 + (1-\beta) y_2) \\
=&\mathbf{F}_1(\beta y_1 + (1-\beta) y_2) \oplus \mathbf{F}_2(\beta y_1 + (1-\beta) y_2) \\
\preceq&\beta \odot \mathbf{F}_1(y_1) \oplus (1-\beta) \odot \mathbf{F}_1(y_2) \oplus \beta \odot \mathbf{F}_2(y_1) \oplus (1-\beta) \odot \mathbf{F}_2(y_2) \\
=&\beta \odot (\mathbf{F}_1(y_1) \oplus \mathbf{F}_2(y_1)) \oplus (1-\beta) \odot (\mathbf{F}_1(y_2) \oplus \mathbf{F}_2(y_2)) \\
=&\beta \odot \mathbf{F}(y_1) \oplus (1-\beta) \odot \mathbf{F}(y_2).
\end{aligned}$$

Thus, $\mathbf{F}_1 \oplus \mathbf{F}_2$ is a convex IVF on $\mathcal{Y}$. □

Theorem 2.5 (gH-directional derivative of the maximum function). *Let $\mathcal{Y}$ be a nonempty convex subset of $\mathbb{R}^n$. Let A be any finite set of indices. For each $i \in A$, let $\boldsymbol{G}_i : \mathcal{Y} \to \overline{I(\mathbb{R})}$ be convex and gH-continuous IVFs such that $\boldsymbol{G}_{i_{\mathscr{D}}}(\bar{y}; d)$ exists for all $\bar{y} \in \mathcal{Y}$. Let for each $y \in \mathcal{Y}$, the set $\{\boldsymbol{G}_i(y) : i \in A\}$ is a set of comparable intervals and define*

$$\boldsymbol{G}(y) = \max_{i \in A} \ \boldsymbol{G}_i(y).$$

Then, for any $\bar{y} \in \mathcal{Y}$ and $d \in \mathcal{Y}$,

$$\boldsymbol{G}_{\mathscr{D}}(\bar{y})(d) = \max_{i \in I(\bar{y})} \ \boldsymbol{G}_{i_{\mathscr{D}}}(\bar{y}; d), \ \text{ where } I(\bar{y}) = \{i \in A : \boldsymbol{G}_i(\bar{y}) = \boldsymbol{G}(\bar{y})\}. \tag{2.6}$$

Proof: Let $\bar{y} \in \mathcal{Y}$ and $d \in \mathcal{Y}$ such that $\bar{y} + \beta d \in \mathcal{Y}$ for $\beta > 0$. Then,

$$\mathbf{G}_i(\bar{y} + \beta d) \preceq \mathbf{G}(\bar{y} + \beta d) \ \forall \ i \in A$$

$$\text{or, } \mathbf{G}_i(\bar{y}+\beta d) \ominus_{gH} \mathbf{G}(\bar{y}) \preceq \mathbf{G}(\bar{y}+\beta d) \ominus_{gH} \mathbf{G}(\bar{y}) \text{ by (iii) of Remark 1.2, } \forall\, i \in A$$
$$\text{or, } \mathbf{G}_i(\bar{y}+\beta d) \ominus_{gH} \mathbf{G}_i(\bar{y}) \preceq \mathbf{G}(\bar{y}+\beta d) \ominus_{gH} \mathbf{G}(\bar{y}) \;\forall\, i \in I(\bar{y})$$
$$\text{or, } \lim_{\beta\to 0+} \frac{1}{\beta} \odot (\mathbf{G}_i(\bar{y}+\beta d) \ominus_{gH} \mathbf{G}_i(\bar{y})) \preceq \lim_{\beta\to 0+} \frac{1}{\beta} \odot (\mathbf{G}(\bar{y}+\beta d) \ominus_{gH} \mathbf{G}(\bar{y})) \;\forall\, i \in I(\bar{y})$$
$$\text{or, } \max \mathbf{G}_{i_{\mathscr{D}}}(\bar{y}; d) \preceq \mathbf{G}_{\mathscr{D}}(\bar{y})(d) \text{ by Lemma 1.8, } \forall\, i \in I(\bar{y}). \tag{2.7}$$

To prove the converse, we assert that there exists a neighbourhood $\mathcal{N}(\bar{y})$ such that $I(y) \subset I(\bar{y})$ for all $y \in \mathcal{N}(\bar{y})$. Assume contrarily that there exists a sequence $\{y_k\}$ in $\mathcal{Y}$ with $y_k \to \bar{y}$ such that $I(y_k) \not\subset I(\bar{y})$. Choose $i_k \in I(y_k)$ but $i_k \notin I(\bar{y})$. Since $I(y_k)$ is closed, $i_k \to \bar{i} \in I(y_k)$. By gH-continuity of $\mathbf{G}_i$, we have

$$\mathbf{G}_{\bar{i}}(y_k) = \mathbf{G}(y_k) \implies \mathbf{G}_{\bar{i}}(\bar{y}) = \mathbf{G}(\bar{y}),$$

which is a contradiction to $i_k \notin I(\bar{y})$. Thus, $I(y) \subset I(\bar{y})$ for all $y \in \mathcal{N}(\bar{y})$.

Let us consider $\{\beta_k\} \subset \mathbb{R}_+$, $\beta_k \to 0$ and $\bar{y} + \beta_k d \in \mathcal{N}(\bar{y})$ for all $d \in \mathcal{Y}$. Then,

$$\mathbf{G}_i(\bar{y}) \preceq \mathbf{G}(\bar{y}) \;\forall\, i \in A$$
$$\text{or, } \mathbf{G}(\bar{y}+\beta_k d) \ominus_{gH} \mathbf{G}(\bar{y}) \preceq \mathbf{G}(\bar{y}+\beta_k d) \ominus_{gH} \mathbf{G}_i(\bar{x}) \text{ by (iii) of Remark 1.2, } \forall\, i \in A$$
$$\text{or, } \mathbf{G}(\bar{y}+\beta_k d) \ominus_{gH} \mathbf{G}(\bar{x}) \preceq \mathbf{G}_i(\bar{y}+\beta_k d) \ominus_{gH} \mathbf{G}_i(\bar{y}) \;\forall\, i \in I(\bar{y}+\beta_k d)$$
$$\text{or, } \lim_{k\to\infty} \frac{1}{\beta_k} \odot (\mathbf{G}(\bar{y}+\beta_k d) \ominus_{gH} \mathbf{G}(\bar{y})) \preceq \lim_{k\to\infty} \frac{1}{\beta_k} \odot (\mathbf{G}_i(\bar{y}+\beta_k d) \ominus_{gH} \mathbf{G}_i(\bar{y})) \;\forall\, i \in I(\bar{y})$$
$$\text{or, } \mathbf{G}_{\mathscr{D}}(\bar{y})(d) \preceq \max\ \mathbf{G}_{i_{\mathscr{D}}}(\bar{y}; d) \text{ by Lemma 1.8, } \forall\, i \in I(\bar{y}). \tag{2.8}$$

From (2.7) and (2.8), we obtain $\mathbf{G}_{\mathscr{D}}(\bar{y})(d) = \max_{i\in I(\bar{y})} \mathbf{G}_{i_{\mathscr{D}}}(\bar{y}; d)$. □

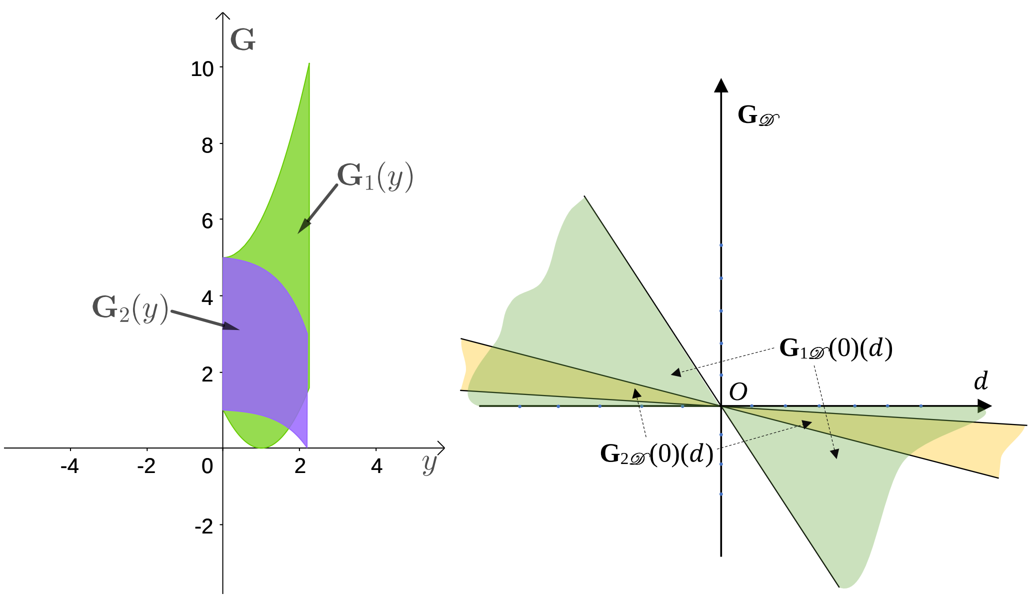
Remark 2.5 *It is to note that for each y in $\mathcal{Y}$, comparability of the intervals in $\{\boldsymbol{G}_i(y) : i \in A\}$ is a necessary condition in Theorem 2.5. For instance, consider $\mathcal{Y} = [0, 2]$ and the IVFs*

$$\boldsymbol{G}_1(y) = [y^2 - 2y + 1, y^2 + 5] \textit{ and } \boldsymbol{G}_2(y) = [1 - ye^{y-3}, 5 - 2ye^{y-3}], y \in \mathcal{Y}.$$

Here, $A = \{1, 2\}$. The IVFs $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are depicted in Figure 2.1(a). From Figure 2.1(a), notice that at several y's in $\mathcal{Y}$, the set $\{\boldsymbol{G}_i(y) : i \in A\} = \{\boldsymbol{G}_1(y), \boldsymbol{G}_2(y)\}$ consists noncomparable intervals. Below, we show that the relation (2.6) in Theorem 2.5 does not hold at $\bar{y} = 0$. Here, $I(\bar{y}) = \{1, 2\}$. The gH-directional derivative of $\boldsymbol{G}_i$'s for $i \in I(\bar{y})$ at $\bar{y} = 0$ along $d \in \mathbb{R}$ are given by

$$\begin{aligned}\boldsymbol{G}_{1_{\mathscr{D}}}(0)(d) &= \lim_{\beta \to 0+} \frac{1}{\beta} \odot (\boldsymbol{G}_1(\beta d) \ominus_{gH} \boldsymbol{G}_1(0)) \\ &= \lim_{\beta \to 0+} \frac{1}{\beta} \odot ([\beta^2 d^2 - 2\beta d + 1, \beta^2 d^2 + 5] \ominus_{gH} [1, 5]) = [-2, 0] \odot d, \text{ and}\end{aligned} \tag{2.9}$$

$$\begin{aligned}\boldsymbol{G}_{2_{\mathscr{D}}}(0)(d) &= \lim_{\beta \to 0+} \frac{1}{\beta} \odot (\boldsymbol{G}_2(\beta d) \ominus_{gH} \boldsymbol{G}_2(0)) \\ &= \lim_{\beta \to 0+} \frac{1}{\beta} \odot ([-\beta d e^{\beta d - 3} + 1, -2\beta d e^{\beta d - 3} + 5] \ominus_{gH} [1, 5]) = [-2, -1] \odot e^{-3} d.\end{aligned} \tag{2.10}$$



(a) The IVFs $\mathbf{G}_1$ and $\mathbf{G}_2$ of Remark 2.5

(b) The IVFs $\mathbf{G}_{1_{\mathscr{D}}}$ and $\mathbf{G}_{2_{\mathscr{D}}}$ of Remark 2.5

Figure 2.1: Geometrical illustration of IVFs of Remark 2.5

The functions $\boldsymbol{G}_{1_{\mathscr{D}}}(0)(d)$ and $\boldsymbol{G}_{2_{\mathscr{D}}}(0)(d)$ are drawn in Figure 2.1(b). We observe *that for any* $d \in \mathbb{R} \setminus \{0\}$, $\boldsymbol{G}_{1_{\mathscr{D}}}(0)(d) \subset \boldsymbol{G}_{2_{\mathscr{D}}}(0)(d)$. *Hence, the right side of the relation* (2.6) *does not exist for any* $d \in \mathbb{R} \setminus \{0\}$. *Accordingly, the question of holding* (2.6) *does not arise.*

From this example of $\boldsymbol{G}_1$ *and* $\boldsymbol{G}_2$, *we thus see that for each* y *in* $\mathcal{Y}$, *comparability of the intervals in* $\{\boldsymbol{G}_i(y) : i \in A\}$ *is a necessary condition to hold* (2.6).

Theorem 2.6 (Fermat-type efficient point condition). *Let* $\mathcal{Y}$ *be a nonempty convex subset of* $\mathbb{R}^n$ *and* $\boldsymbol{F} : \mathcal{Y} \to \overline{I(\mathbb{R})}$ *be a convex IVF. Then,* $\bar{y}$ *is a weak efficient solution of* $\inf_{y \in \mathcal{Y}} \boldsymbol{F}(y)$ *if and only if* $\widehat{\boldsymbol{0}} \in \partial \boldsymbol{F}(\bar{y})$.

Proof: If $\bar{y}$ is a weak efficient solution, then for all $y \in \mathcal{Y}$, we have

$$\begin{aligned} & \mathbf{0} \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{y}) \\ \iff & (y - \bar{y})^\top \odot \widehat{\mathbf{0}} \preceq \mathbf{F}(y) \ominus_{gH} \mathbf{F}(\bar{y}) \\ \iff & \widehat{\mathbf{0}} \in \partial \mathbf{F}(\bar{y}), \end{aligned}$$

which is the required relation. □

Lemma 2.5 *Consider the IVF* $\boldsymbol{F} : \mathcal{Y} \to I(\mathbb{R})$ *and real valued functions* $g_1, g_2, \ldots, g_p : \mathcal{Y} \to \mathbb{R}$. *Let for the IOP*

$$\inf_{y \in \mathcal{Y}} \boldsymbol{F}(y) \tag{2.11}$$
$$\text{subject to } g_j(y) \leq 0, \ j = 1, 2, \ldots, p,$$

the infimum value, denoted by $\boldsymbol{F}_{\inf} = [\underline{f}_{\inf}, \overline{f}_{\inf}]$, *be finite. Consider another IOP*

$$\inf_{y \in \mathcal{Y}} \boldsymbol{F}^*(y) \tag{2.12}$$

where $\boldsymbol{F}^*(y) = \sup\{\boldsymbol{F}(y) \ominus_{gH} \boldsymbol{F}_{\inf}, g_1(y), g_2(y), \ldots, g_p(y)\}$, *i.e.,* $\boldsymbol{F}^*(y) = [\underline{f}^*(y), \overline{f}^*(y)]$, *where*

$$\underline{f}^*(y) = \sup\{\min\{\underline{f}(y) - \underline{f}_{\text{inf}}, \overline{f}(y) - \overline{f}_{\text{inf}}\}, g_1(y), g_2(y), \ldots, g_p(y)\} \text{ and}$$
$$\overline{f}^*(y) = \sup\{\max\{\underline{f}(y) - \underline{f}_{\text{inf}}, \overline{f}(y) - \overline{f}_{\text{inf}}\}, g_1(y), g_2(y), \ldots, g_p(y)\}, \text{ respectively.}$$

Then, the sets of weak efficient points of (2.11) and (2.12) are identical.

Proof: Let the set of weak efficient points of (2.11) be Y'. To prove the required relation, we show that

(i) $\mathbf{F}^*(y) = \mathbf{0}$ for all $y \in Y'$ or

(ii) $\mathbf{0} \prec \mathbf{F}^*(y)$ for all $y \notin Y'$.

If $y \in Y'$, then $g_j(y) \leq 0$ for all $i = 1, 2, \ldots, p$ and $\mathbf{F}(y) = \mathbf{F}_{\text{inf}}$. This implies that $\mathbf{F}^*(y) = 0$.

If $y \notin Y'$, then y is either 'not feasible' or 'feasible but not weak efficient'.

Let y be not feasible. Then, for some j, $g_j(y) > 0$, which implies $\mathbf{0} \prec \mathbf{F}^*(x)$.

Let y be feasible but not weak efficient. Then, $\mathbf{F}_{\text{inf}} \prec \mathbf{F}(y)$, which implies that $\mathbf{0} \prec \mathbf{F}^*(y)$.

□

Theorem 2.7 (Fritz-John-type necessary condition for IOPs). *Let $\mathcal{Y}$ be a nonempty convex subset of $\mathbb{R}^n$. Consider the constrained IOP*

$$\begin{aligned} &\inf_{y \in \mathcal{Y}} && \boldsymbol{F}(y) \\ &\text{subject to} && g_j(y) \leq 0, \; j = 1, 2, \ldots, p, \end{aligned} \tag{2.13}$$

where $\boldsymbol{F} : \mathcal{Y} \to I(\mathbb{R})$ be a convex IVF and $g_1, g_2, \ldots, g_p : \mathcal{Y} \to \mathbb{R}$ be real-valued convex functions. If $\bar{y}$ is a weak efficient solution of (2.13), then there exist $\beta_j \geq 0, \; j =$

$0, 1, 2, \ldots, p$, *not all zeros, such that*

$$\widehat{\boldsymbol{0}} \in \beta_0 \odot \partial \boldsymbol{F}(\bar{y}) \oplus \sum_{j=1}^{p} \beta_j \partial g_j(\bar{y}) \tag{2.14}$$

$$\text{and } \beta_j g_j(\bar{y}) = 0, \; j = 1, 2, \ldots, p.$$

Proof: Let $\bar{y}$ be a weak efficient solution of (2.13). Denote $\mathbf{F}(\bar{y}) = \mathbf{F}_{\text{inf}}$. Then, by Lemma 2.5, $\bar{y}$ is a weak efficient solution of $\inf_{y \in \mathcal{Y}} \mathbf{F}^*(y)$,

$$\text{where } \mathbf{F}^*(y) = \sup\{\mathbf{F}(y) \ominus_{gH} \mathbf{F}_{\text{inf}}, g_1(y), g_2(y), \ldots, g_p(y)\}.$$

Since $\mathbf{F}^*(\bar{y}) = \mathbf{0}$, then by Theorem 2.6,

$$\widehat{\mathbf{0}} \in \partial \mathbf{F}^*(\bar{y}). \tag{2.15}$$

From [102, Theorem 3.50], we have

$$\partial \mathbf{F}^*(\bar{y}) = \beta_0 \odot \partial(\mathbf{F}(\bar{y}) \ominus_{gH} \mathbf{F}_{\text{inf}}) \oplus \sum_{j \in I(\bar{y})} \beta_j \partial g_j(\bar{y}), \tag{2.16}$$

where $\beta_j \geq 0$ such that $\sum_{j \in I(\bar{y})} \beta_j = 1$ and $I(\bar{y}) = \{j \in \{1, 2, \ldots, p\} : g_j(\bar{y}) = 0\}$.
In view of (2.15) and (2.16), we obtain

$$\widehat{\mathbf{0}} \in \beta_0 \odot \partial(\mathbf{F}(\bar{y}) \ominus_{gH} \mathbf{F}_{\text{inf}}) \oplus \sum_{j \in I(\bar{y})} \beta_j \partial g_j(\bar{y}).$$

Let $\mathbf{H}(\bar{y}) = \mathbf{F}(\bar{y}) \ominus_{gH} \mathbf{F}_{\text{inf}}$. Then, $\partial(\mathbf{H})(\bar{y}) = \partial \mathbf{F}(\bar{y})$. Now, by taking $\beta_j = 0$ for $j \notin I(\bar{y})$, we get the desired result. □

Note 2.1 *It is to be observed that the condition on $\bar{y}$ to be a weak efficient (instead of efficient) point of (2.13) is necessary. If $\bar{y}$ is an efficient solution of the IOP (2.13), then $\boldsymbol{F}(\bar{y})$ and $\boldsymbol{F}(y)$ can be noncomparable IVFs and (2.14) is not true at $\bar{y}$. Consider* *the IOP*

$$\inf\ \boldsymbol{F}(y) = [1,2] \odot y^2 \oplus [0,2] \odot y \oplus [2,5] \tag{2.17}$$
$$\textit{subject to}\ \ g_1(y) = y - 1 \leq 0,\ g_2(y) = 3y - 2 \leq 0,\ y \in \mathcal{Y} = [-2, 1].$$

Note that the set of feasible solutions of $\boldsymbol{F}$ *is* $\left[-2, \frac{2}{3}\right]$ *and* $\boldsymbol{F}$, *(refer to Figure 2.2)* g_1, g_2 *are convex on* $\mathcal{Y}$. *Also,* $\boldsymbol{F}$ *is gH-differentiable on* $\mathcal{Y}$ *as* $\underline{f}(y) = y^2 + 2y + 2$ *and* $\overline{f}(y) = 2y^2 + 5$ *are differentiable on* $\mathcal{Y}$. *Thus, from Theorem 1.4,*

$$\partial \boldsymbol{F}(y) = \{\nabla \boldsymbol{F}(y)\} = \{[2,4] \odot y \oplus [0,2]\}$$
$$\textit{and}\ \ \partial g_1(y) = \{\nabla g_1(y)\} = \{1\},\ \partial g_2(y) = \{\nabla g_2(y)\} = \{3\}\ \textit{for all}\ y \in \mathcal{Y}.$$

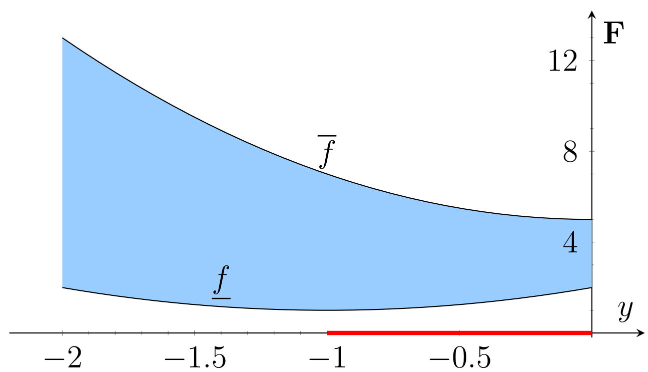


Figure 2.2: The IVF **F** of Note 2.1

All y's in the red line segment in Figure 2.2 are efficient solutions of (2.17). *Thus,* $\bar{y} = 0 \in [-1, 0]$ *is an efficient solution of the IOP (2.17). However, for all* $\beta_0, \beta_1, \beta_2 \geq 0$, *not all zeros,*

$$\beta_0 \odot \partial \boldsymbol{F}(\bar{y}) \oplus \beta_1 \partial g_1(\bar{y}) \oplus \beta_2 \partial g_2(\bar{y}) = \beta_0 \odot [0,2] \oplus \beta_1 \odot [1,1] \oplus \beta_2 \odot [3,3] \neq \boldsymbol{0}.$$

Hence, $\boldsymbol{0} \notin \beta_0 \odot \partial \boldsymbol{F}(\bar{y}) \oplus \beta_1 \partial g(\bar{y}) \oplus \beta_2 \partial g_2(\bar{y})$.

Next, we identify a KKT-type optimality condition based on the weak efficient solution of an IOP. To propose these conditions, we need to take an extra condition, which refers to *Slater's condition*:

$$\text{there exists } \bar{y} \in \mathcal{Y} \text{ such that } g_i(\bar{y}) < 0 \ \text{ for all } \ i = 1, 2, \ldots, p. \tag{2.18}$$

Theorem 2.8 (KKT-type necessary condition for IOPs). *Let $\mathcal{Y}$ be a nonempty convex subset of $\mathbb{R}^n$. Consider the constrained IOP*

$$\inf_{y \in \mathcal{Y}} \boldsymbol{F}(y) \tag{2.19}$$

$$\text{subject to } g_j(y) \leq 0, \ j = 1, 2, \ldots, p,$$

where $\boldsymbol{F} : \mathcal{Y} \to I(\mathbb{R})$ is a convex IVF and $g_1, g_2, \ldots, g_p : \mathcal{Y} \to \mathbb{R}$ are real-valued convex functions. Suppose the Slater's condition (2.18) is satisfied. Then, $\bar{y}$ is a weak efficient solution of (2.19) if and only if there exist $\beta_j \geq 0, \ j = 1, 2, \ldots, p$, such that

$$\widehat{\boldsymbol{0}} \in \partial \boldsymbol{F}(\bar{y}) \oplus \sum_{j=1}^{p} \beta_j \partial g_j(\bar{y}) \tag{2.20}$$

$$\textit{and } \beta_j g_j(\bar{y}) = 0, \ j = 1, 2, \ldots, p. \tag{2.21}$$

Proof: Let $\bar{y}$ be a weak efficient solution of (2.19). Then, from Theorem 2.7, there exist $\tilde{\beta}_i \geq 0$, not all zeros, such that

$$\widehat{\mathbf{0}} \in \tilde{\beta}_0 \odot \partial \mathbf{F}(\bar{y}) \oplus \sum_{j=1}^{p} \tilde{\beta}_j \partial g_j(\bar{y}) \text{ and} \tag{2.22}$$

$$\tilde{\beta}_j g_j(\bar{y}) = 0, \ j = 1, 2, \ldots, p. \tag{2.23}$$

If $\tilde{\beta}_0 \neq 0$, then the result is true by taking $\beta_j = \frac{\tilde{\beta}_j}{\tilde{\beta}_0}, \ j = 1, 2, \ldots, p$ in (2.22). Let us assume contrarily that $\tilde{\beta}_0 = 0$. Then,

$$\widehat{\mathbf{0}} \in \sum_{j=1}^{p} \tilde{\beta}_j \partial g_j(\bar{y}) \implies \widehat{\mathbf{0}} \in \sum_{j=1}^{p} \tilde{\beta}_j \partial g_j(\bar{y}).$$

From [102, Theorem 3.78], for a point x^* which satisfies the Slater's condition (2.18), we have $\sum_{j=1}^{p} \tilde{\beta}_j g_j(x^*) \geq 0$, which controverts the presumption that $\tilde{\beta}_j \geq 0$ and $g_j(x^*) < 0$ for some j and all $\tilde{\beta}_j$'s are not zero. Thus, $\tilde{\beta}_0 > 0$.

To prove the converse part, assume that $\bar{y}$ satisfies (2.20) and (2.21) for some $\beta_j \geq 0,\ j = 1, 2, \ldots, p$.

Define a convex IVF $\mathbf{H}$ by $\mathbf{H}(y) = \mathbf{F}(y) \oplus \sum_{j=1}^{p} \beta_j g_j(y)$. Then,

$$\partial \mathbf{H}(\bar{y}) = \partial \mathbf{F}(\bar{y}) \oplus \sum_{j=1}^{p} \beta_j \partial g_j(\bar{y}).$$

Since $\widehat{\mathbf{0}} \in \partial \mathbf{H}(\bar{y})$, $\bar{y}$ is a weak efficient point of $\mathbf{H}$. Therefore, by (2.23) and Theorem 2.6,

$$\mathbf{F}(\bar{y}) = \mathbf{F}(\bar{y}) \oplus \sum_{j=1}^{p} \beta_j g_j(\bar{y}) = \mathbf{H}(\bar{y}). \tag{2.24}$$

Let $\hat{y}$ be a feasible point of (2.19). Then,

$$\mathbf{H}(\hat{y}) = \mathbf{F}(\hat{y}) \oplus \sum_{j=1}^{p} \beta_j g_j(\hat{y}) \preceq \mathbf{F}(\hat{y}). \tag{2.25}$$

From (2.24) and (2.25), we obtain $\mathbf{F}(\bar{y}) \preceq \mathbf{F}(\hat{y})$, and therefore $\bar{y}$ is a weak efficient solution of (2.19). □

Example 2.2 *In this example, we provide an instance of verification of the result in Theorem 2.8. Consider the following IOP:*

$$\min\ \boldsymbol{F}(y_1, y_2) = [1, 2] \odot |y_1| \oplus [2, 3] \odot |y_2| \oplus [1, 3]$$

$$\text{subject to } g_1(y_1, y_2) = y_1 + y_2 - 1 \leq 0,\ \ g_2(y_1, y_2) = 3y_1 + y_2 - 2 \leq 0$$

$$(y_1, y_2) \in \mathcal{Y} \subseteq \mathbb{R}^2.$$

The IVF $\boldsymbol{F}$ *with* $\underline{f}(y_1, y_2) = |y_1| + 2|y_2| + 1$ *and* $\overline{f}(y_1, y_2) = 2|y_1| + 3|y_2| + 3$ *are shown with pink and green surfaces, respectively, in Figure 2.3. From Figure 2.3, it can be observed that* $(y_1, y_2) = (0, 0)$ *is a weak efficient solution of* $\boldsymbol{F}$.
Observing that

$$\boldsymbol{0} = y_1 \odot \boldsymbol{0} \oplus y_2 \odot \boldsymbol{0} \preceq [1, 2] \odot |y_1| \oplus [2, 3] \odot |y_2|,$$

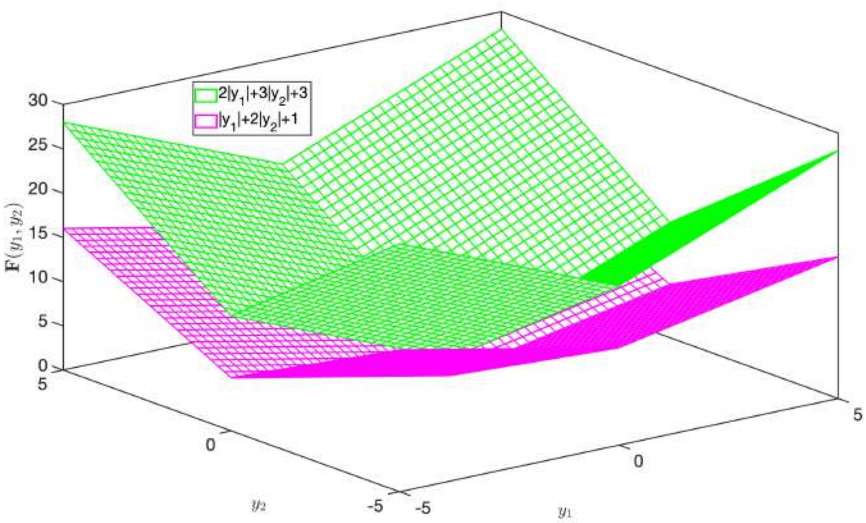


Figure 2.3: The IVF **F** of Example 2.2

we have $\widehat{\boldsymbol{0}} \in \partial \boldsymbol{F}(0, 0)$. *Note that the set of feasible points is* $\left(-\infty, \frac{1}{2}\right] \times \left(-\infty, \frac{1}{2}\right]$, *and* $\boldsymbol{F}$ *(refer to Figure 2.3),* g_1 *and* g_2 *are convex on* $\mathcal{Y}$. *Also,* g_1, g_2 *are differentiable on* $\mathcal{Y}$. *Therefore,*

$$\nabla g_1(y_1, y_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \nabla g_2(y_1, y_2) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Now, we observe that for $\beta_1 = 0, \beta_2 = 0,$ *we have*

$$\widehat{\boldsymbol{0}} \in \partial \boldsymbol{F}(0,0) \oplus \beta_1 \partial g_1(0,0) \oplus \beta_2 \partial g_2(0,0)$$
$$\textit{and } \beta_1 g_1(0,0) = 0, \ \beta_2 g_2(0,0) = 0,$$

and hence (2.20) and (2.21) are true for the problem under consideration.

Note 2.2 (Comparison with the existing Fritz-John and KKT optimality conditions for IOPs).

To the best of our knowledge, the existing theories on Fritz-John and KKT optimality conditions are in [48, 101, 103] and [32, 48, 100, 103, 104], respectively. We have made the comparison of our derived Fritz-John-type conditions and KKT-type conditions for IOPs with the existing theories based on

(i) the use of lower and upper functions, and

(ii) the restriction of results to smooth IOPs.

It is to be noted that in [32, 48, 100, 101, 103, 104], the Fritz-John and KKT conditions require explicit expression of the lower and upper functions (i.e., $\underline{f}$ *and* $\overline{f}$*) of the objective function. However, the task to find the expressions of* $\underline{f}(y)$ *and* $\overline{f}(y)$ *is not always easy. For instance, consider the IVF*

$$\boldsymbol{F}(y) = \frac{[-2,3] \odot e^y}{[-2,3] \odot \cos y \oplus [1,2] \odot y}, \ y \in \mathbb{R}.$$

In our results, we have used the IVF $\boldsymbol{F}$ *in which there is no need to have the expressions of* $\underline{f}$ *and* $\overline{f}$ *to derive the required results.*

Next, we observe that Fritz-John conditions and KKT conditions in [32, 48, 100, 101, 103, 104] are applicable for smooth IOPs. For instance, the theories in [32, 48, 100, 101, *103, 104] are not applicable for the following IOP:*

$$\inf \ \boldsymbol{F}(y_1, y_2) = [0, 2] \odot y_1^2 \oplus [1, 2] \odot |y_2| \tag{2.26}$$
$$\text{subject to} \ \ g_1(y_1, y_2) = y_1 + y_2 - 1 \leq 0, \ \ g_2(y_1, y_2) = 3y_1 + y_2 - 2 \leq 0$$
$$y_1, y_2 \in \mathcal{Y} \subseteq \mathbb{R}^2.$$

Due to the restriction of existing theories to smooth IOPs, we have derived our results for nonsmooth IOPs. Now, the lower and upper functions of $\boldsymbol{F}$ *in* (2.26) *are given by*

$$\underline{f}(y_1, y_2) = |y_2| \ \textit{and} \ \overline{f}(y_1, y_2) = 2y_1^2 + 2|y_2|.$$

Note that both the functions $\underline{f}$ *and* $\overline{f}$ *are not differentiable at* $(0, 0)$*. Thus, the Fritz-John and KKT conditions in [32, 48, 100, 101, 103, 104] cannot be applied.*
Now, observing that

$$\boldsymbol{0} = y_1 \odot \boldsymbol{0} \oplus y_2 \odot \boldsymbol{0} \preceq \boldsymbol{F}(y_1, y_2) \ominus_{gH} \boldsymbol{F}(0, 0),$$

we have $\widehat{\boldsymbol{0}} \in \partial \boldsymbol{F}(0, 0)$*. Note that the set of feasible points in* (2.26) *is* $(-\infty, \frac{1}{2}] \times (-\infty, \frac{1}{2}]$*, and* $\boldsymbol{F}$*,* g_1*,* g_2 *are convex on* $\mathcal{Y}$*. Also,* g_1, g_2 *are differentiable on* $\mathcal{Y}$ *and*

$$\nabla g_1(y_1, y_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ \textit{and} \ \nabla g_2(y_1, y_2) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

As we see that for any $\beta_0 > 0$*,* $\beta_1 = \beta_2 = 0$*,*

$$\widehat{\boldsymbol{0}} \in \beta_0 \partial \boldsymbol{F}(0, 0) \oplus \beta_1 \partial g_1(0, 0) \oplus \beta_2 \partial g_2(0, 0)$$
$$\textit{and} \ \widehat{\boldsymbol{0}} \in \partial \boldsymbol{F}(0, 0) \oplus \beta_1 \partial g_1(0, 0) \oplus \beta_2 \partial g_2(0, 0)$$
$$\textit{with} \ \beta_1 g_1(0, 0) = 0, \ \ \beta_2 g_2(0, 0) = 0,$$

the proposed Fritz-John-type condition (2.14) and KKT-type conditions (2.20) and (2.21) are applicable for the problem (2.26).

Further, since for very rarely smooth IVFs the gH-subdifferential sets get exactly equal to the zero vector, the proposed Fritz-John-type and KKT-type conditions are inclusion relations (see (2.14) and (2.20)) instead of equalities.

2.4 Application of gH-Subdifferentials in Nonconvex Composite Optimization

In this section, a necessary efficiency condition for nonconvex composite IOPs and a sufficient condition of convex IOPs are derived with the help of gH-subdifferentials.

Theorem 2.9 (Efficiency conditions for the composite model). *Let $\mathcal{Y}$ be a nonempty convex subset of $\mathbb{R}^n$. Let $\boldsymbol{F} : \mathcal{Y} \to I(\mathbb{R}) \cup \{+\infty\}$ be a proper IVF and let $\boldsymbol{H} : \mathcal{Y} \to I(\mathbb{R}) \cup \{+\infty\}$ be a proper convex IVF such that* $\text{dom}(\mathbf{H})$ *is a subset of the interior of* $\text{dom}(\mathbf{F})$*. Consider the IOP:*

$$\inf_{y \in \mathcal{Y}} \boldsymbol{P}(y) = \inf_{y \in \mathcal{Y}} \boldsymbol{F}(y) \oplus \boldsymbol{H}(y). \tag{2.27}$$

If $\bar{y} \in \text{dom}(\boldsymbol{H})$ is a weak efficient solution of (2.27) and $\boldsymbol{F}$ is gH-differentiable at $\bar{y}$, then

$$(-1) \odot \nabla \boldsymbol{F}(\bar{y}) \in \partial \boldsymbol{H}(\bar{y}). \tag{2.28}$$

The converse is true if $\boldsymbol{F}$ is convex on $\mathcal{Y}$.

Proof: Let $\bar{y}, y \in \text{dom}(\mathbf{H})$ and $\beta \in (0, 1)$ such that $y_\beta = (1-\beta)\bar{y} + \beta y \in \text{dom}(\mathbf{H})$. Since $\bar{y}$ is a weak efficient solution of (2.27),

$$\mathbf{P}(\bar{y}) \preceq \mathbf{P}(y_\beta)$$

$$\begin{aligned}
&\implies \mathbf{F}(\bar{y}) \oplus \mathbf{H}(\bar{y}) \preceq \mathbf{F}(y_\beta) \oplus \mathbf{H}(y_\beta)\\
&\implies \mathbf{F}(\bar{y}) \oplus \mathbf{H}(\bar{y}) \preceq \mathbf{F}((1-\beta)\bar{y} + \beta y) \oplus \mathbf{H}(\beta y + (1-\beta)\bar{y}).
\end{aligned}$$

Due to convexity of $\mathbf{H}$ and Lemma 1.4, we have

$$\begin{aligned}
&\mathbf{F}(\bar{y}) \oplus \mathbf{H}(\bar{y}) \preceq \mathbf{F}((1-\beta)\bar{y} + \beta y) \oplus \beta \odot \mathbf{H}(y) \oplus (1-\beta) \odot \mathbf{H}(\bar{y})\\
\implies &\mathbf{F}(\bar{y}) \oplus \beta \odot \mathbf{H}(\bar{y}) \preceq \mathbf{F}((1-\beta)\bar{y} + \beta y) \oplus \beta \odot \mathbf{H}(y) \text{ by (ii) of Remark 1.2}\\
\implies &\frac{1}{\beta} \odot \big(\mathbf{F}(\bar{y}) \ominus_{gH} \mathbf{F}((1-\beta)\bar{y} + \beta y)\big) \preceq \mathbf{H}(y) \ominus_{gH} \mathbf{H}(\bar{y}) \text{ by (i) of Lemma 1.4}\\
\implies &(-1) \odot \frac{1}{\beta} \odot \left(\mathbf{F}((1-\beta)\bar{y} + \beta y) \ominus_{gH} \mathbf{F}(\bar{y})\right) \preceq \mathbf{H}(y) \ominus_{gH} \mathbf{H}(\bar{y}).
\end{aligned}$$

Since $\mathbf{F}$ is gH-differentiable, as $\beta \to 0^+$, we have

$$\begin{aligned}
&(y-\bar{y})^\top \odot ((-1) \odot \nabla \mathbf{F}(\bar{y})) \preceq \mathbf{H}(y) \ominus_{gH} \mathbf{H}(\bar{y})\\
\implies &(-1) \odot \nabla \mathbf{F}(\bar{y}) \in \partial \mathbf{H}(\bar{y}).
\end{aligned}$$

To prove the latter part, we assume that $\mathbf{F}$ is convex on $\mathcal{Y}$ and $(-1) \odot \nabla \mathbf{F}(\bar{y}) \in \partial \mathbf{H}(\bar{y})$. Let $\bar{y}$ is not a weak efficient point of the problem (2.27). Then, there exists a point $y' \in \text{dom}(\mathbf{H})$ such that $\mathbf{P}(y') \preceq \mathbf{P}(\bar{y})$. Therefore, for any $\beta, \beta' \in (0,1)$ with $\beta + \beta' = 1$, we have

$$\begin{aligned}
&\beta \odot \mathbf{P}(y') \preceq \beta \odot \mathbf{P}(\bar{y})\\
\text{or, } &\beta \odot \mathbf{P}(y') \oplus \beta' \odot \mathbf{P}(\bar{y}) \preceq \beta \odot \mathbf{P}(\bar{y}) \oplus \beta' \odot \mathbf{P}(\bar{y})\\
\text{or, } &\beta \odot \mathbf{P}(y') \oplus \beta' \odot \mathbf{P}(\bar{y}) \preceq (\beta \oplus \beta') \odot \mathbf{P}(\bar{y}) = \mathbf{P}(\bar{y}).
\end{aligned}$$

Due to convexity of $\mathbf{F}$ and $\mathbf{H}$ on $\mathcal{Y}$, we have

$$\mathbf{F}(\beta y' + \beta' \bar{y}) \oplus \beta \odot \mathbf{H}(y') \oplus \beta' \odot \mathbf{H}(\bar{y})$$

$$\preceq \beta \odot \mathbf{F}(y') \oplus \beta' \odot \mathbf{F}(\bar{y}) \oplus \beta \odot \mathbf{H}(y') \oplus \beta' \odot \mathbf{H}(\bar{y})$$
$$\preceq \mathbf{F}(\bar{y}) \oplus \mathbf{H}(\bar{y}).$$

Thus, from Lemma 1.4, we obtain

$$\begin{aligned}
\mathbf{F}(\bar{y} + \beta(y' - \bar{y})) \oplus \beta \odot \mathbf{H}(y') &\preceq \mathbf{F}(\bar{y}) \oplus \beta \odot \mathbf{H}(\bar{y}) \\
\text{or, } \mathbf{H}(y') \ominus_{gH} \mathbf{H}(\bar{y}) &\preceq \frac{1}{\beta} \odot (\mathbf{F}(\bar{y}) \ominus_{gH} \mathbf{F}(\bar{y} + \beta(y' - \bar{y})) \\
\text{or, } \mathbf{H}(y') \ominus_{gH} \mathbf{H}(\bar{y}) &\preceq (-1) \odot \frac{1}{\beta} \odot (\mathbf{F}(\bar{y} + \beta(y' - \bar{y})).
\end{aligned}$$

This is a contradiction to the asuumption that $(-1) \odot \nabla \mathbf{F}(\bar{y}) \in \partial \mathbf{H}(\bar{y})$ for all $y \in \mathcal{Y}$. Hence, $\bar{y}$ is a weak efficient point of $\mathbf{F}$. □

Note 2.3 *For nonconvex $\boldsymbol{F}$, the converse of Theorem 2.9 is not true. For instance, consider the IOP*

$$\inf_{y \in [-5,5]} \boldsymbol{F}(y) \oplus \boldsymbol{H}(y), \tag{2.29}$$

where $\boldsymbol{F}, \boldsymbol{H} : [-5, 5] \to I(\mathbb{R})$ are defined by

$$\boldsymbol{F}(y) = [2, 4] \odot y^3 \oplus [1, 1] \text{ and } \boldsymbol{H}(y) = [3, 3].$$

Therefore,

$$\underline{f}(y) = \begin{cases} 2y^3 + 1, & \text{for } y \geq 0 \\ 4y^3 + 1, & \text{for } y < 0, \end{cases} \text{ and } \overline{f}(y) = \begin{cases} 4y^3 + 1, & \text{for } y \geq 0 \\ 2y^3 + 1, & \text{for } y < 0. \end{cases}$$

From Definition 1.7 of convex IVF, $\boldsymbol{F}$ is not convex as $\underline{f}$ and $\overline{f}$ are nonconvex (refer to Figure 2.4). Note that $\boldsymbol{F}$ and $\boldsymbol{H}$ are gH-differentiable and

$$\nabla \boldsymbol{F}(y) = [6, 12] \odot y^2 \text{ and } \nabla \boldsymbol{H}(y) = \boldsymbol{0}.$$

At $\bar{y} = 0$, $\partial \boldsymbol{H}(\bar{y}) = \{\nabla \boldsymbol{H}(\bar{y})\} = \{\boldsymbol{0}\}$, *and* $\nabla \boldsymbol{F}(\bar{y}) = \boldsymbol{0}$. *Hence,* $(-1) \odot \nabla \boldsymbol{F}(0) \in \partial \boldsymbol{H}(\bar{y})$. *However,* $\bar{y}$ *is not an efficient point of the IOP (2.29) (refer to Figure 2.4).*

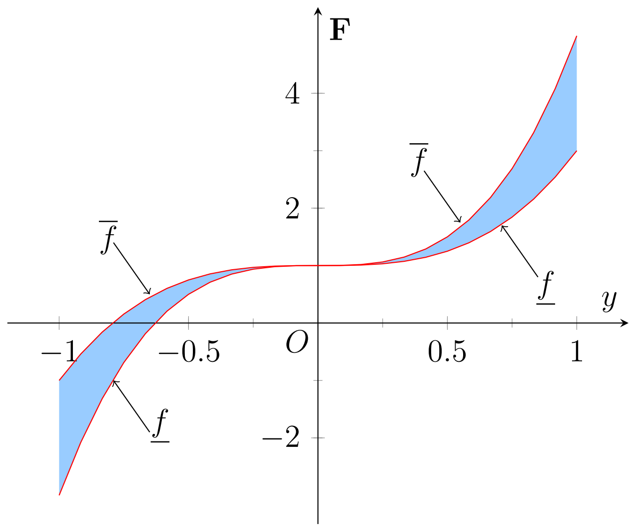


Figure 2.4: The IVF **F** of Note 2.3

In the below Examples 2.3 and 2.4, we exemplify Theorem 2.9 by considering some special cases for **H**.

Definition 2.7 (Indicator function for IVF). *Let* $\mathcal{Y}$ *be a subset of* $\mathbb{R}^n$. *Then, the indicator function* $\boldsymbol{\delta}_{\mathcal{Y}} : \mathbb{R}^n \to \overline{I(\mathbb{R})}$ *at a point* y, *is defined as*

$$\boldsymbol{\delta}_{\mathcal{Y}}(y) = \begin{cases} \boldsymbol{0}, & \text{if } y \in \mathcal{Y} \\ +\infty, & \text{if } y \notin \mathcal{Y}. \end{cases}$$

Example 2.3 (Convex constrained nonconvex programming problem for IVF). *Let* $\mathcal{Y}$ *be a nonempty convex subset of* $\mathbb{R}^n$. *Let* $\boldsymbol{F} : \mathcal{Y} \to I(\mathbb{R}) \cup \{+\infty\}$ *be a nonconvex gH-differentiable IVF and* $\boldsymbol{H} : \mathcal{Y} \to \overline{I(\mathbb{R})}$ *be a convex IVF which is defined by* $\boldsymbol{H}(y) = \boldsymbol{\delta}_{\mathcal{Y}}(y)$. *Consider the IOP*

$$\inf_{y \in \mathcal{Y}} \boldsymbol{F}(y) \oplus \boldsymbol{H}(y). \tag{2.30}$$

If $\bar{y} \in \mathcal{Y}$ is a weak efficient point of (2.30), then from Theorem 2.9, we have

$$(-1) \odot \nabla \boldsymbol{F}(\bar{y}) \in \partial \boldsymbol{H}(\bar{y}) \implies (-1) \odot \nabla \boldsymbol{F}(\bar{y}) \in \partial \boldsymbol{\delta}_{\mathcal{Y}}(\bar{y}).$$

We observe that if $\widehat{\boldsymbol{G}} \in I(\mathbb{R})^n$ is a subgradient of $\boldsymbol{\delta}_{\mathcal{Y}}$ at $\bar{y} \in \mathcal{Y}$, then for all $y \in \mathcal{Y}$

$$\begin{aligned} & (y-\bar{y})^\top \odot \widehat{\boldsymbol{G}} \preceq \boldsymbol{\delta}_{\mathcal{Y}}(y) \ominus_{gH} \boldsymbol{\delta}_{\mathcal{Y}}(\bar{y}) \\ \implies & (y-\bar{y})^\top \odot \widehat{\boldsymbol{G}} \preceq \boldsymbol{0}. \end{aligned}$$

Therefore, we have

$$(-1) \odot (y-\bar{y})^\top \odot \nabla \boldsymbol{F}(\bar{y}) \preceq \boldsymbol{0}.$$

Example 2.4 *Let the generic element of $\mathbb{R}^n$ be $y = (y_1, y_2, \ldots, y_n)^\top$ and $p \in \{1, 2, \ldots, n\}$. Let $\boldsymbol{F} : \mathbb{R}^n \to I(\mathbb{R}) \cup \{+\infty\}$ be a gH-differentiable IVF and $\boldsymbol{H} : \mathbb{R}^n \to I(\mathbb{R}) \cup \{+\infty\}$ be a convex IVF. Consider the IOP*

$$\inf_{y \in \mathcal{Y}} \ \boldsymbol{F}(y) \oplus \boldsymbol{H}(y) \tag{2.31}$$

where $\boldsymbol{0} \preceq \boldsymbol{C}$ and $\boldsymbol{H}(y) = \boldsymbol{C} \odot |y_p|$.

If $\bar{y} \in \mathcal{Y}$ is a weak efficient point of (2.31), then by Theorem 2.9, we have

$$(-1) \odot \nabla \boldsymbol{F}(\bar{y}) \in \partial \boldsymbol{H}(\bar{y}).$$

From [6], the gH-subdifferential set of $\boldsymbol{H}$ at any $\bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_{p-1}, 0, \bar{y}_{p+1}, \ldots, \bar{y}_n)^\top$ in the plane $y_p = 0$ is given by

$$\{(\boldsymbol{G}_1, \boldsymbol{G}_2, \ldots, \boldsymbol{G}_n)^\top \in I(\mathbb{R})^n : (-1) \odot \boldsymbol{C} \preceq \boldsymbol{G}_j \preceq \boldsymbol{C}, \text{ for all } j = 1, 2, \ldots, n\}.$$

Therefore,

$$\nabla \boldsymbol{F}(\bar{y}) = (D_1 \boldsymbol{F}(\bar{y}), D_2 \boldsymbol{F}(\bar{y}), \ldots, D_n \boldsymbol{F}(\bar{y}))^\top,$$

is given by

$$D_j\boldsymbol{F}(\bar{y}) = \begin{cases} (-1)\odot \boldsymbol{C} & \text{if } \bar{y}_j < 0, \\ \boldsymbol{C} & \text{if } \bar{y}_j > 0, \\ \boldsymbol{G}_j \in I(\mathbb{R}) : (-1)\odot \boldsymbol{C} \preceq \boldsymbol{G}_j \preceq \boldsymbol{C} & \text{if } \bar{y}_j = 0, \end{cases} \tag{2.32}$$

for each $j = 1, 2, \ldots, n$. Thus, (2.32) is a necessary condition for $\bar{y}$ to be a weak efficient point of the IOP (2.31).

Remark 2.6 *From Theorem 2.9, we can observe that there is no need to develop the separate calculus for solving a nonconvex composite problem. Also, Note 2.3 shows that the convexity of IVF $\boldsymbol{F}$ is a necessary condition in Theorem 2.9 for the converse to hold.*

Another application of Theorem 2.9 is given in the following Example 2.5 in estimating Lasso optimality conditions.

Example 2.5 (Lasso optimality condition for IOPs). *Let $\boldsymbol{F} : \mathbb{R}^p \to I(\mathbb{R}) \cup \{+\infty\}$ be a gH-differentiable IVF and $\boldsymbol{H} : \mathbb{R}^p \to I(\mathbb{R}) \cup \{+\infty\}$ be a convex IVF. Consider the IOP*

$$\min_{\beta\in\mathbb{R}^p} \; (\boldsymbol{F}(\beta) \oplus \boldsymbol{H}(\beta)), \tag{2.33}$$

where $\boldsymbol{F}(\beta) = \frac{1}{2}\odot \|y - Q\beta\|_2^2 \odot \boldsymbol{A}$ and $\boldsymbol{H}(\beta) = \lambda \odot |\beta_s| \odot \boldsymbol{B}, \beta = (\beta_1, \beta_2, \ldots, \beta_s, \ldots, \beta_p) \in \mathbb{R}^p$, $\boldsymbol{0} \preceq \boldsymbol{A}$, $\boldsymbol{0} \preceq \boldsymbol{B}$, $y \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n\times p}$, and $\lambda \geq 0$.

Therefore, we have

$$\underline{f}(\beta) = \frac{1}{2}\|y - Q\beta\|_2^2\underline{a} \text{ and } \overline{f}(\beta) = \frac{1}{2}\|y - Q\beta\|_2^2\overline{a}.$$

Since $\boldsymbol{F}$ *is a* gH*-differentiable IVF, therefore at* $\bar{\beta} \in \mathbb{R}^p$*, we have*

$$\nabla \boldsymbol{F}(\bar{\beta}) = (D_1\boldsymbol{F}(\bar{\beta}), D_2\boldsymbol{F}(\bar{\beta}), \ldots, D_n\boldsymbol{F}(\bar{\beta}))^\top = (-Q^\top(y - Q\bar{\beta})) \odot \boldsymbol{A}.$$

Also, from [6], the gH*-subdifferential set of* $\boldsymbol{H}$ *at any* $\bar{\beta} = (\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_s, \ldots, \bar{\beta}_n)^\top$ *is given by*

$$\partial \boldsymbol{H}(\bar{\beta}) = \begin{cases} (-1) \odot \boldsymbol{B} & \text{if } \bar{\beta}_s < 0, \\ \boldsymbol{B} & \text{if } \bar{\beta}_s > 0, \\ \{\boldsymbol{G}_j \in I(\mathbb{R}) : (-1) \odot \boldsymbol{B} \preceq \boldsymbol{G}_j \preceq \boldsymbol{B}\} & \text{if } \bar{\beta}_s = 0. \end{cases}$$

Therefore, from Theorem 2.9, an efficient solution $\bar{\beta}$ *of* (2.33) *is characterized by*

$$\begin{aligned} D_j\boldsymbol{F}(\bar{\beta}) &= Q_j^\top(y - Q\bar{\beta}) \odot \boldsymbol{A} \\ &= \begin{cases} (-1)\lambda \odot \boldsymbol{B} & \text{if } \bar{\beta}_s < 0, \\ \lambda \odot \boldsymbol{B} & \text{if } \bar{\beta}_s > 0, \\ \{\boldsymbol{G}_j \in I(\mathbb{R}) : (-1)\lambda \odot \boldsymbol{B} \preceq \boldsymbol{G}_j \preceq \lambda \odot \boldsymbol{B}\} & \text{if } \bar{\beta}_s = 0, \end{cases} \end{aligned}$$

for each $j = 1, 2, \ldots, n$*.*

2.5 Conclusion

In this chapter, several major results on nonsmooth IVFs and IOPs have been derived—gH-directional derivative of the maximum of IVFs (Theorem 2.5), Fritz-John-type necessary efficiency condition (Theorem 2.7), and KKT-type necessary and sufficient efficiency condition for IOPs (Theorem 2.8).

To derive these results, we have discussed the concepts of infimum and supremum (Definition 2.4) of an IVF on a set of interval vectors, followed by closedness (Definition 2.5), boundedness (Definition 2.6), and convex hull (Definition 2.3) in $I(\mathbb{R})$; also, we have derived some properties related to these concepts. One can trivially notice that in the degenerate case, Definitions 2.1, 2.2, 2.3, and 2.4, reduce to the corresponding conventional definitions for the real-valued functions (see [102,105]). Finally, using the proposed calculus for IVFs, a characterization of the efficient solutions of a nonconvex composite model with IVFs (Theorem 2.9) has been derived.

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