

Chapter 7

Generation of fractals by Φ -iterated tupling system

In this chapter, using the ideas of the cyclic map and m -tuple fixed point, we introduce Φ -tupling and discuss the existence of strong m -tuple fixed point on m closed subsets in the complete metric space. Also, we discuss an application of strong m -tuple fixed point theorem for constructing of strong m -tuple fractal through the Φ -iterated tupling system.

7.1 Introduction

In 1987, Guo and Lakshmikantham [45] introduced the idea of coupled fixed point theory. Subsequently, Gnana-Bhaskar and Lakshmikantham [25] have established some coupled fixed point results using “mixed monotone property” in the partial ordered metric space, and several results on coupled fixed point theory are implemented by Karapinar [56]. Later, several researchers have introduced the triple or more fixed points results in various metric spaces; we cite here some of the following contributions [6, 15]. Throughout this chapter, we symbolize the product space “ $X \times X \times \dots m$ times” by X^m and “ $C(X) \times C(X) \times \dots m$ times” by $C(X)^m$, where $C(X)$ is the set of all compact subsets of X . Also we use some notations as first N natural numbers by \mathbb{N}_N and for all i implies $i \in \mathbb{N}$ with $1 \leq i \leq m$.

In 2010, Samet and Vetro [88] have introduced m -tuple fixed point as follows:

Definition 7.1. (m -tuple fixed point)

A tuple $(x_1, x_2, \dots, x_m) \in X^m$ is said to be m -tuple fixed point of $F : X^m \rightarrow X$ if

$$\begin{cases} F(x_1, x_2, x_3, \dots, x_m) = x_1, \\ F(x_2, x_3, x_4, \dots, x_1) = x_2, \\ \vdots \\ F(x_m, x_1, x_2, \dots, x_{m-1}) = x_m. \end{cases}$$

Theorem 7.2. [85] *A mapping $F : X^m \rightarrow X$ has a m -tuple fixed point if and only if the mapping $\mathcal{F} : X^m \rightarrow X^m$ defined by*

$$\mathcal{F}(x_1, x_2, x_3, \dots, x_m) = \left(F(x_1, x_2, x_3, \dots, x_m), F(x_2, x_3, \dots, x_m, x_1), \dots, F(x_m, x_1, x_2, \dots, x_{m-1}) \right)$$

has a fixed point in X^m .

See, e.g., [50, 58] for related results and extensions on m -tuple fixed points.

In the following, we discuss a class of functions that was used by Boyd and Wong [29] in 1969 in their famous generalization of Banach's theorem.

Definition 7.3. (Function of Class Φ [29])

Let $\phi : [0, \infty) \rightarrow [0, \infty)$, then $\phi \in \Phi$, if it satisfies the following two conditions:

- (i) ϕ is upper semi-continuous from the right;
- (ii) $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$.

7.1.1 Delineation

The current chapter is structured as follows: Section 7.2 focuses on establishing the existence of a strong m -tuple fixed point theory. In Section 7.3, we discuss an application of this strong m -tuple fixed results to generate a strong m -tuple fractal through the Φ -iterated tupling system.

7.2 Strong m -tuple fixed point result

Definition 7.4. Let $F : X^m \rightarrow X$ be a map. An element $(x, x, x, \dots, x) \in X^m$ is said to be strong m -tuple fixed point of F if $F(x, x, x, \dots, x) = x$.

Definition 7.5. [1] Let $\{A_i : i \in \{1, 2, \dots, m\}\}$ be m subsets of X . A function $f : X \rightarrow X$ is said to be cyclic mapping with respect to A_i for all i if $f(A_1) \subseteq A_2, f(A_2) \subseteq A_3, \dots, f(A_m) \subseteq A_1$.

In the following, we have defined the Φ -tupling using the ideas of m -tuple mapping and the cyclic mapping.

Definition 7.6. A mapping $F : X^m \rightarrow X$ is said to be Φ -tupling with respect to $A_1, A_2, A_3, \dots, A_m$ if there exists $\phi \in \Phi$ and for $x_i \in A_i$ and $y_i \in A_{i+1}$, F and ϕ satisfy the following two conditions:

(i) $F(x_i, x_{i+1}, \dots, x_m, x_1, \dots, x_{i-1}) \in A_{i+1}$;

(ii) $d\left(F(x_i, x_{i+1}, \dots, x_{i-1}), F(y_i, y_{i+1}, \dots, y_{i-1})\right) \leq \phi\left(\frac{d(x_i, y_i) + d(x_{i+1}, y_{i+1}) + \dots + d(x_{i-1}, y_{i-1})}{m}\right)$.

Theorem 7.7. Let $A_1, A_2, A_3, \dots, A_m$ be non-empty closed subsets of the complete metric space (X, d) and $F : X^m \rightarrow X$ is Φ -tupling with respect to A_1, A_2, \dots, A_m for some $\phi \in \Phi$. Then $\bigcap_i A_i \neq \emptyset$ and there exists a unique $x \in \bigcap_i A_i$ such that $F(x, x, x, \dots, x) = x$.

Proof. First we have to show that $\bigcap_i A_i \neq \emptyset$ (non-empty set). Let $x_1^0 \in A_1, x_2^0 \in A_2, x_3^0 \in A_3, \dots, x_m^0 \in A_m$. We construct the sequences $\{x_1^n\}, \{x_2^n\}, \dots, \{x_m^n\} \in X$ as follows:

$$x_1^n = F(x_m^{n-1}, x_1^{n-1}, \dots, x_{m-1}^{n-1}), x_2^n = F(x_1^{n-1}, x_2^{n-1}, \dots, x_m^{n-1}), \dots, x_m^n = F(x_{m-1}^{n-1}, x_m^{n-1}, \dots, x_{m-2}^{n-1}).$$

Therefore, $x_1^n \in A_1, x_2^n \in A_2, \dots, x_m^n \in A_m$ for all $n \in \mathbb{N}$.

Now,

$$\begin{aligned} & d(x_i^{n+1}, x_{i+1}^{n+2}) + d(x_{i+1}^{n+1}, x_i^{n+2}) \\ &= d\left(F(x_{i-1}^n, x_i^n, \dots, x_{i-2}^n), F(x_i^{n+1}, x_{i+1}^{n+1}, \dots, x_{i-1}^{n+1})\right) \\ & \quad + d\left(F(x_i^n, x_{i+1}^n, \dots, x_{i-1}^n), F(x_{i-1}^{n+1}, x_i^{n+1}, \dots, x_{i-2}^{n+1})\right) \\ & \leq \phi\left(\frac{d(x_{i-1}^n, x_i^{n+1}) + d(x_i^n, x_{i+1}^{n+1}) + \dots + d(x_{i-2}^n, x_{i-1}^{n+1})}{m}\right) \end{aligned}$$

$$+ \phi \left(\frac{d(x_i^n, x_{i-1}^{n+1}) + d(x_{i+1}^n, x_i^{n+1}) + \cdots + d(x_{i-1}^n, x_{i-2}^{n+1})}{m} \right).$$

Now, taking summation over i and after using the properties of ϕ we obtain

$$\sum_i \{d(x_i^{n+1}, x_{i+1}^{n+2}) + d(x_{i+1}^{n+1}, x_i^{n+2})\} \leq \sum_i \{d(x_i^n, x_{i+1}^{n+1}) + d(x_{i+1}^n, x_i^{n+1})\}.$$

Therefore, the sequence $\sum_i \{d(x_i^n, x_{i+1}^{n+1}) + d(x_{i+1}^n, x_i^{n+1})\}$ is monotone decreasing and let it converges to r . Now, the upper semi-continuous property of ϕ gives,

$$\begin{aligned} r &= \limsup_{n \rightarrow \infty} \sum_i \{d(x_i^n, x_{i+1}^{n+1}) + d(x_{i+1}^n, x_i^{n+1})\} \\ &\leq \limsup_{n \rightarrow \infty} \phi \left(\sum_i \{d(x_i^n, x_{i+1}^{n+1}) + d(x_{i+1}^n, x_i^{n+1})\} \right) \leq \phi(r). \end{aligned}$$

Therefore, $r = 0$, i.e, $\lim_{n \rightarrow \infty} \{d(x_i^n, x_{i+1}^{n+1}) + d(x_{i+1}^n, x_i^{n+1})\} = 0$ for all $i \in \{1, 2, \dots, m\}$.

Hence, we must have

$$\lim_{n \rightarrow \infty} d(x_i^n, x_{i+1}^{n+1}) = 0 \text{ and } \lim_{n \rightarrow \infty} d(x_{i+1}^n, x_i^{n+1}) = 0. \quad (7.1)$$

Again,

$$\begin{aligned} &d(x_i^{n+1}, x_{i+1}^{n+1}) \\ &= d \left(F(x_{i-1}^n, x_i^n, \dots, x_{i-2}^n), F(x_i^n, x_{i+1}^n, \dots, x_{i-1}^n) \right) \\ &\leq \phi \left(\frac{d(x_{i-1}^n, x_i^n) + d(x_i^n, x_{i+1}^n) + \cdots + d(x_{i-2}^n, x_{i-1}^n)}{m} \right). \end{aligned}$$

Now, taking the summation over i and using the property $\phi(t) < t$ we get,

$$\sum_i d(x_i^{n+1}, x_{i+1}^{n+1}) \leq \sum_i d(x_i^n, x_{i+1}^n).$$

Therefore, the sequence $\{\sum_i d(x_i^n, x_{i+1}^n)\}$ is monotone decreasing, let it converges to s . Again, using the properties of ϕ , in a similar way we obtain

$$\lim_{n \rightarrow \infty} d(x_i^n, x_{i+1}^n) = 0 \text{ for all } i \in \{1, 2, \dots, m\}. \quad (7.2)$$

Now, using relations (7.1), (7.2), and the triangle inequality $d(x_i^n, x_i^{n+1}) \leq d(x_i^n, x_{i+1}^n) + d(x_{i+1}^n, x_i^{n+1})$, we obtain

$$\lim_{n \rightarrow \infty} d(x_i^n, x_i^{n+1}) = 0 \quad \text{for all } i \in \{1, 2, \dots, m\}. \quad (7.3)$$

Now, we will prove that the sequences $\{x_1^n\}, \{x_2^n\}, \dots, \{x_m^n\}$ are Cauchy. If possible, let at least one of them be not Cauchy. Then for a given $\epsilon > 0$, we can get sub-sequences $\{x_1^{p(l)}\}, \{x_1^{v(l)}\}$ of $\{x_1^n\}$ and $\{x_2^{p(l)}\}, \{x_2^{v(l)}\}$ of $\{x_2^n\}$ and similarly $\{x_m^{p(l)}\}, \{x_m^{v(l)}\}$ of $\{x_m^n\}$ with $p(l) > v(l) \geq l$ such that

$$\sum_i d(x_i^{p(l)}, x_i^{v(l)}) \geq \epsilon \quad \text{and} \quad (7.4)$$

$$\sum_i d(x_i^{p(l)-1}, x_i^{v(l)}) < \epsilon. \quad (7.5)$$

By using the triangle inequality in relation (7.4) we get

$$\begin{aligned} \epsilon &\leq \sum_i d(x_i^{p(l)}, x_i^{v(l)}) = r_l \quad (\text{say}) \\ &\leq \sum_i \{d(x_i^{p(l)}, x_i^{p(l)-1}) + d(x_i^{p(l)-1}, x_i^{v(l)})\} \leq \sum_i d(x_i^{p(l)}, x_i^{p(l)-1}) + \epsilon. \quad [\text{by (7.5)}] \end{aligned}$$

Again, applying triangle inequality in relation (7.4) we have

$$\begin{aligned} \epsilon \leq r_l &= \sum_i d(x_i^{p(l)}, x_i^{v(l)}) \leq \sum_i \{d(x_i^{p(l)}, x_i^{p(l)+1}) + d(x_i^{p(l)+1}, x_i^{v(l)+1}) + d(x_i^{v(l)+1}, x_i^{v(l)})\} \\ &\leq \delta_{p(l)} + \delta_{v(l)} + \sum_i d(x_i^{p(l)+1}, x_i^{v(l)+1}), \end{aligned}$$

where we assume $\delta_k = \sum_i d(x_i^k, x_i^{k+1})$. Note that $\lim_{k \rightarrow \infty} \delta_k = 0$. [by using the relation (7.3)]

$$\text{Therefore, we have, } \epsilon \leq \delta_{p(l)} + \delta_{v(l)} + r_{l+1}. \quad (7.6)$$

Now,

$$\begin{aligned} d(x_i^{p(l)+1}, x_i^{v(l)+1}) &= d\left(F(x_{i-1}^{p(l)}, x_i^{p(l)}, \dots, x_{i-2}^{p(l)}), F(x_{i-1}^{v(l)}, x_i^{v(l)}, \dots, x_{i-2}^{v(l)})\right) \\ &\leq \phi\left(\frac{d(x_{i-1}^{p(l)}, x_{i-1}^{v(l)}) + d(x_i^{p(l)}, x_i^{v(l)}) + \dots + d(x_{i-2}^{p(l)}, x_{i-2}^{v(l)})}{m}\right). \end{aligned}$$

Now taking summation over i and after using the properties of ϕ , we obtain $r_{l+1} \leq r_l$. Therefore, $\{r_l\}$ is a monotone decreasing sequence. Thus, $\lim_{l \rightarrow \infty} r_l = r$. Further, upper semi continuous of ϕ gives us $r = \lim_{l \rightarrow \infty} r_l = \limsup_{l \rightarrow \infty} \phi(r_l) \leq \phi(r)$. Therefore, $r = 0$ i.e, $\lim_{l \rightarrow \infty} r_l = \sum_i d(x_i^{p(l)}, x_i^{v(l)}) = 0$. Now, taking the limit as $l \rightarrow \infty$ in relation (7.6), we get $\epsilon \leq 0$, which is a contradiction. Therefore, $\{x_1^n\}, \{x_2^n\}, \dots, \{x_m^n\}$ all are Cauchy sequences. Since X is complete and A_1, A_2, \dots, A_m are closed subsets of X , so there exists $x_i \in A_i$ for all i such that $\lim_{n \rightarrow \infty} x_i^n = x_i$ for all $i \in \{1, 2, \dots, m\}$.

Also, we have

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} d(x_1^n, x_2^n) = d(x_1, x_2) \implies x_1 = x_2 = x, \quad (\text{let}) \\ 0 &= \lim_{n \rightarrow \infty} d(x_2^n, x_3^n) = d(x_2, x_3) \implies x_2 = x_3 = x, \\ &\vdots \\ 0 &= \lim_{n \rightarrow \infty} d(x_m^n, x_1^n) = d(x_m, x_1) \implies x_m = x_1 = x. \end{aligned}$$

Thus,

$$\lim_{n \rightarrow \infty} x_1^n = \lim_{n \rightarrow \infty} x_2^n = \dots = \lim_{n \rightarrow \infty} x_m^n = x.$$

Therefore, $x \in \bigcap_i A_i$, and hence $\bigcap_i A_i \neq \emptyset$.

Now we will prove that (x, x, \dots, x) is a strong m -tuple fixed point of F .

Now,

$$\begin{aligned} d\left(x, F(x, x, \dots, x)\right) &\leq d(x, x_1^{n+1}) + d\left(x_1^{n+1}, F(x, x, x, \dots, x)\right) \\ &= d(x, x_1^{n+1}) + d\left(F(x_m^n, x_1^n, \dots, x_{m-1}^n), F(x, x, x, \dots, x)\right) \\ &\leq d(x, x_1^{n+1}) + \phi\left(\frac{d(x_m^n, x) + d(x_1^n, x) + \dots + d(x_{m-1}^n, x)}{m}\right) \\ &\leq d(x, x_1^{n+1}) + d(x_m^n, x) + d(x_1^n, x) + \dots + d(x_{m-1}^n, x). \end{aligned}$$

Therefore, $d(x, F(x, x, \dots, x)) = 0$ as $n \rightarrow \infty$. i.e. $F(x, x, \dots, x) = x$.

For the uniqueness of x , let $y \in \bigcap_i A_i$ be another point such that $y = F(y, y, \dots, y)$. Then,

$$\begin{aligned} d(x, y) &= d\left(F(x, x, x, \dots, x), F(y, y, \dots, y)\right) \\ &\leq \phi\left(\frac{d(x, y) + d(x, y) + \dots + d(x, y)}{m}\right) \end{aligned}$$

$$\leq \phi(d(x, y)),$$

which is absurd, therefore, $d(x, y) = 0 \implies x = y$.

Hence the desired result. \square

Remark 7.8. It may be noted that for any $x_1^0 \in A_1, x_2^0 \in A_2, \dots, x_m^0 \in A_m$. The iterations of these m -tuple $F(x_1^n, x_2^n, \dots, x_m^n)$ converge to the strong m -tuple fixed point.

Corollary 7.9. If $m = 2$, then the Theorem 7.7, reduces to strong coupled fixed point theorem (see [37] Theorem 2.8.).

Example 7.10. Assume that $X = \mathbb{R}$ and d is the usual metric on \mathbb{R} . Let $A_{2m-1} = [-\frac{1}{2^{2m-1}}, 0]$ and $A_{2m} = [0, \frac{1}{2^{2m}}]$ and $F : X^m \rightarrow X$ be defined by

$$F(x_1, x_2, \dots, x_m) = \frac{x_2 - x_1}{2^{2m-1}} - \frac{(x_2 - x_1)^2}{2^{2m}}.$$

Let $x_i \in A_i$ and $y_i \in A_{i+1}$. Therefore, $F(x_{i-1}, x_i, \dots, x_{i-2}) \in A_i$ for all $i \in \{1, 2, \dots, m\}$.

Now,

$$\begin{aligned} d\left(F(x_i, x_{i+1}, \dots, x_{i-1}), F(y_i, y_{i+1}, \dots, y_{i-1})\right) &= \left| \frac{x_{i+1} - x_i}{2^{2m-1}} - \frac{(x_{i+1} - x_i)^2}{2^{2m}} - \frac{y_{i+1} - y_i}{2^{2m-1}} + \frac{(y_{i+1} - y_i)^2}{2^{2m}} \right| \\ &\leq \sum_i \left\{ \frac{|x_i - y_i|}{2^{2m-1}} + \frac{(x_i - y_i)^2}{2^{2m}} \right\}. \end{aligned}$$

Since for any positive numbers a_i , we have the relation $\sum a_i^2 \leq (\sum a_i)^2$, therefore,

$$\sum_i \left\{ \frac{|x_i - y_i|}{2^{2m-1}} + \frac{(x_i - y_i)^2}{2^{2m}} \right\} \leq \phi\left(\frac{d(x_i, y_i) + d(x_{i+1}, y_{i+1}) + \dots + d(x_{i-1}, y_{i-1})}{m}\right).$$

Therefore,

$$d\left(F(x_i, x_{i+1}, \dots, x_{i-1}), F(y_i, y_{i+1}, \dots, y_{i-1})\right) \leq \phi\left(\frac{d(x_i, y_i) + d(x_{i+1}, y_{i+1}) + \dots + d(x_{i-1}, y_{i-1})}{m}\right).$$

$$\text{here, } \phi(t) = \begin{cases} \frac{mt}{2^{2m-1}} + \frac{m^2 t^2}{2^{2m}}, & \text{if } 0 \leq t \leq 1 \\ 0, & \text{if } t > 1, \end{cases}$$

which satisfies all the conditions of the function of class Φ .

Therefore, F is a Φ -tupling with respect to A_1, A_2, \dots, A_m and F satisfies all the conditions of Theorem 7.7. Here $0 \in \bigcap_i A_i$, precisely 0 is a strong m -tuple fixed point, $F(0, 0, 0, \dots, 0) = 0$.

7.3 Generation of strong m -tuple fractals

In this section, we have analyzed the idea of strong m -tuple fractal sets through Φ -iterated tupling systems constructed in the same way as that of IFS in Hutchinson-Barnsley theory [48].

Definition 7.11. A Φ -iterated tupling system contains a complete space (X, d) on m closed subsets A_1, A_2, \dots, A_m of X and a finite set of Φ tuplings $F_j : X^m \rightarrow X$ for some $\phi_j \in \Phi$ for $j \in \mathbb{N}_N$, where \mathbb{N}_N denotes the first N natural numbers.

It is denoted by $\left\langle (X, d); A_1, A_2, \dots, A_m; F_j, j \in \mathbb{N}_N \right\rangle$.

If each F_j is continuous then $\left\langle (X, d); A_1, A_2, \dots, A_m; F_j, j \in \mathbb{N}_N \right\rangle$ is said to be continuous Φ -iterated tupling system.

Definition 7.12. Let (X, d) be a metric space. The mapping $h : C(X) \times C(X) \rightarrow \mathbb{R}$, defined as

$$h(P, Q) = \max\{D(P, Q), D(Q, P)\}, \quad \text{where } D(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b),$$

is a metric on $C(X)$, called the Hausdorff metric induced by d .

Lemma 7.13. [10] *If $\{A_j : j \in \mathbb{N}_N\}$ and $\{B_j : j \in \mathbb{N}_N\}$ are two finite collection of $C(X)$, where $C(X)$ is the set of all compact subsets of X then,*

$$h\left(\bigcup_{1 \leq j \leq N} A_j, \bigcup_{1 \leq j \leq N} B_j\right) \leq \max_{1 \leq j \leq N} h(A_j, B_j).$$

Theorem 7.14. *Let A_1, A_2, \dots, A_m be non-empty subsets of X and $F : X^m \rightarrow X$ be a continuous Φ -tupling with respect to A_1, A_2, \dots, A_m for some monotone increasing function $\phi \in \Phi$. Then, \tilde{F} is a Φ -tupling on $C(A_1), C(A_2), \dots, C(A_m)$ with Hausdorff metric h induced by d .*

Proof. It may be noted that for all $B_i \in C(A_i)$ for $i = \{1, 2, 3, \dots, m\}$.

$$\tilde{F}(B_1, B_2, \dots, B_m) \in C(A_2), \tilde{F}(B_2, B_3, \dots, B_1) \in C(A_3), \dots, \tilde{F}(B_m, B_1, \dots, B_{m-1}) \in C(A_1).$$

Let $B_i^j \in C(A_i)$ for all $i \in \{1, 2, \dots, m\}$ and $j \in \mathbb{N}$.

Now,

$$\begin{aligned}
& D\left(\tilde{F}(B_1^1, B_2^1, \dots, B_m^1), \tilde{F}(B_2^2, B_3^2, \dots, B_1^2)\right) \\
&= D\left(\{F(x_1^1, x_2^1, \dots, x_m^1) : x_i^1 \in B_i^1\}, \{F(x_2^2, x_3^2, \dots, x_1^2) : x_i^2 \in B_i^2\}\right) \\
&= \sup_{x_i^1 \in B_i^1} \inf_{x_i^2 \in B_i^2} d\left(F(x_1^1, x_2^1, \dots, x_m^1), F(x_2^2, x_3^2, \dots, x_1^2)\right) \\
&\leq \sup_{x_i^1 \in B_i^1} \inf_{x_i^2 \in B_i^2} \phi\left(\frac{d(x_1^1, x_2^2) + d(x_2^1, x_3^2) + \dots + d(x_m^1, x_1^2)}{m}\right). \tag{7.7}
\end{aligned}$$

Since B_i^1, B_i^2 are compact for all $i \in \{1, 2, \dots, m\}$ so there exists $\tilde{x}_{i+1}^2 \in B_{i+1}^2$ such that

$$\inf_{x_{i+1}^2 \in B_{i+1}^2} d(x_i^1, x_{i+1}^2) = d(x_i^1, \tilde{x}_{i+1}^2).$$

Now,

$$\begin{aligned}
& \inf_{x_i^2 \in B_i^2} \phi\left(\frac{d(x_1^1, x_2^2) + d(x_2^1, x_3^2) + \dots + d(x_m^1, x_1^2)}{m}\right) \\
&\leq \phi\left(\frac{d(x_1^1, \tilde{x}_2^2) + d(x_2^1, \tilde{x}_3^2) + \dots + d(x_m^1, \tilde{x}_1^2)}{m}\right) \\
&= \phi\left(\frac{\inf_{x_2^2 \in B_2^2} d(x_1^1, x_2^2) + \inf_{x_3^2 \in B_3^2} d(x_2^1, x_3^2) + \dots + \inf_{x_1^2 \in B_1^2} d(x_m^1, x_1^2)}{m}\right).
\end{aligned}$$

Since ϕ is monotone increasing function we have

$$\begin{aligned}
& \phi\left(\frac{\inf_{x_2^2 \in B_2^2} d(x_1^1, x_2^2) + \inf_{x_3^2 \in B_3^2} d(x_2^1, x_3^2) + \dots + \inf_{x_1^2 \in B_1^2} d(x_m^1, x_1^2)}{m}\right) \\
&\leq \phi\left(\frac{\sup_{x_1^1 \in B_1^1} \inf_{x_2^2 \in B_2^2} d(x_1^1, x_2^2) + \sup_{x_2^1 \in B_2^1} \inf_{x_3^2 \in B_3^2} d(x_2^1, x_3^2) + \dots + \sup_{x_m^1 \in B_m^1} \inf_{x_1^2 \in B_1^2} d(x_m^1, x_1^2)}{m}\right) \\
&= \phi\left(\frac{D(B_1^1, B_2^2) + D(B_2^1, B_3^2) + \dots + D(B_m^1, B_1^2)}{m}\right) \\
&\leq \phi\left(\frac{h(B_1^1, B_2^2) + h(B_2^1, B_3^2) + \dots + h(B_m^1, B_1^2)}{m}\right).
\end{aligned}$$

Therefore,

$$\inf_{x_i^2 \in B_i^2} \phi\left(\frac{d(x_1^1, x_2^2) + d(x_2^1, x_3^2) + \dots + d(x_m^1, x_1^2)}{m}\right) \leq \phi\left(\frac{h(B_1^1, B_2^2) + h(B_2^1, B_3^2) + \dots + h(B_m^1, B_1^2)}{m}\right).$$

Now taking $\sup_{x_i^1 \in B_i^1}$ on both sides we get

$$\sup_{x_i^1 \in B_i^1} \inf_{x_i^2 \in B_i^2} \phi \left(\frac{d(x_1^1, x_2^2) + d(x_2^1, x_3^2) + \cdots + d(x_m^1, x_1^2)}{m} \right) \leq \phi \left(\frac{h(B_1^1, B_2^2) + h(B_2^1, B_3^2) + \cdots + h(B_m^1, B_1^2)}{m} \right).$$

Therefore, from the relation (7.7) we have

$$D \left(\tilde{F}(B_1^1, B_2^1, \dots, B_m^1), \tilde{F}(B_2^2, B_3^2, \dots, B_1^2) \right) \leq \phi \left(\frac{h(B_1^1, B_2^2) + h(B_2^1, B_3^2) + \cdots + h(B_m^1, B_1^2)}{m} \right). \quad (7.8)$$

Similarly, we can get

$$D \left(\tilde{F}(B_2^2, B_3^2, \dots, B_1^2), \tilde{F}(B_1^1, B_2^1, \dots, B_m^1) \right) \leq \phi \left(\frac{h(B_2^2, B_1^1) + h(B_3^2, B_2^1) + \cdots + h(B_1^2, B_m^1)}{m} \right). \quad (7.9)$$

From the above relations (7.8) and (7.9) we have

$$\begin{aligned} & h \left(\tilde{F}(B_1^1, B_2^1, \dots, B_m^1), \tilde{F}(B_2^2, B_3^2, \dots, B_1^2) \right) \\ &= \max \left\{ D \left(\tilde{F}(B_1^1, B_2^1, \dots, B_m^1), \tilde{F}(B_2^2, B_3^2, \dots, B_1^2) \right), D \left(\tilde{F}(B_2^2, B_3^2, \dots, B_1^2), \tilde{F}(B_1^1, B_2^1, \dots, B_m^1) \right) \right\} \\ &\leq \phi \left(\frac{h(B_1^1, B_2^2) + h(B_2^1, B_3^2) + \cdots + h(B_m^1, B_1^2)}{m} \right). \end{aligned}$$

Hence the desired result. \square

Theorem 7.15. Let $\{F_j : j \in \mathbb{N}_N\}$ be continuous and Φ -tupling family on X^m with respect to A_1, A_2, \dots, A_m for some $\phi_1, \phi_2, \dots, \phi_N \in \Phi$ respectively and each ϕ_j is monotone increasing function. Then the map $F : (C(X))^m \rightarrow C(X)$ defined by

$$F(B_1, B_2, \dots, B_m) = \cup_{j=1}^N \tilde{F}_j(B_1, B_2, \dots, B_m)$$

is a Φ -tupling with $C(A_1), C(A_2), \dots, C(A_m)$ for $\phi \in \Phi$, where $\phi(t) = \max_{1 \leq j \leq N} \phi_j(t)$.

Proof. One should observe that for all $B_i \in C(A_i)$ with $i = \{1, 2, 3, \dots, m\}$.

$$F(B_1, B_2, \dots, B_m) \in C(A_2), F(B_2, B_3, \dots, B_1) \in C(A_3), \dots, F(B_m, B_1, \dots, B_{m-1}) \in C(A_1).$$

Let $B_i^j \in C(A_i)$ for $j \in \mathbb{N}$, then

$$\begin{aligned}
& h\left(F(B_1^1, B_2^1, \dots, B_m^1), F(B_2^2, B_3^2, \dots, B_1^2)\right) \\
&= h\left(\cup_{j=1}^N \tilde{F}_j(B_1^1, B_2^1, \dots, B_m^1), \cup_{j=1}^N \tilde{F}_j(B_2^2, B_3^2, \dots, B_1^2)\right) \\
&\leq \max_{1 \leq j \leq N} \left\{ h\left(\tilde{F}_j(B_1^1, B_2^1, \dots, B_m^1), \tilde{F}_j(B_2^2, B_3^2, \dots, B_1^2)\right) \right\} \\
&\leq \max_{1 \leq j \leq N} \phi_j \left(\frac{h(B_1^1, B_2^2) + h(B_2^1, B_3^2) + \dots + h(B_m^1, B_1^2)}{m} \right) \\
&= \phi \left(\frac{h(B_1^1, B_2^2) + h(B_2^1, B_3^2) + \dots + h(B_m^1, B_1^2)}{m} \right).
\end{aligned}$$

Hence the theorem. \square

The following theorem provides a positive answer to the question of Adrian and Gabriela [73] concerning the generation of multiple fractals.

Theorem 7.16. Let $\left\langle (X, d); A_1, A_2, \dots, A_m; F_j, j \in \mathbb{N}_N \right\rangle$ be a continuous Φ -iterated tupling system on compact metric space (X, d) , where F_j is Φ -tupling on X^m with closed subsets A_1, A_2, \dots, A_m of X . Let $\phi_j \in \Phi$ and each ϕ_j is monotone increasing. Let $F : (C(X))^m \rightarrow C(X)$ be defined as

$$F(B_1, B_2, \dots, B_m) = \cup_{j=1}^N \tilde{F}_j(B_1, B_2, \dots, B_m).$$

Then there exists a strong m -tuple fractal of F , that means there exists $\tilde{S} \in C(A_1) \cap C(A_2) \cap \dots \cap C(A_m)$ such that $F(\tilde{S}, \tilde{S}, \dots, \tilde{S}) = \tilde{S}$.

Proof. From the above Theorem 7.15, we have F is a Φ -tupling for $\phi(t) = \max_{1 \leq j \leq N} \phi_j(t)$. Again $(C(X), h)$ is compact metric space as (X, d) is compact. Here A_1, A_2, \dots, A_m are compact subsets of X as they are also closed. Therefore, $C(A_1), C(A_2), \dots, C(A_m)$ are also compact subsets of $C(X)$ and thereby, they are closed subsets of $C(X)$.

Therefore, by the Theorem 7.7, there exists a strong m -tuple fixed point \tilde{S} of F such that $\tilde{S} \in C(A_1) \cap C(A_2) \cap \dots \cap C(A_m)$. Thus, $F(\tilde{S}, \tilde{S}, \dots, \tilde{S}) = \tilde{S}$.

Hence the proof. \square

Remark 7.17. The iteration $F(A_1^n, A_2^n, \dots, A_m^n)$ converges to the strong m -tuple fractal for any arbitrary choice of $A_1^0 \in C(A_1), A_2^0 \in C(A_2), \dots, A_m^0 \in C(A_m)$.

Corollary 7.18. *If $m = 2$, then the Theorem 7.16 reduces to strong coupled fractal theorem (see [37] Theorem 3.13.).*

We now give a supporting example of the above theorem for generating a strong m -tuple fractal.

Example 7.19. Suppose $X = \mathbb{R}$ and d be the usual metric on \mathbb{R} . Let $F_1, F_2 : X^2 \rightarrow X$ be defined by

$$F_1(x_1, x_2) = \frac{x_2 - x_1}{16} \text{ and } F_2(x_1, x_2) = \frac{1 + x_2 - x_1}{16}.$$

Then, F_1, F_2 are Φ -tuplings with respect to $A_1 = [-\frac{1}{4}, \frac{3}{8}]$ and $A_2 = [-\frac{1}{8}, \frac{1}{2}]$ for $\phi(t) = \frac{t}{4}$.

Therefore, the Φ -iterated tupling system $\langle (X, d); A_1, A_2; F_1, F_2 \rangle$ generates a fractal. Here we have computed the first few iterations in the following:

Let $B_1 = [-\frac{1}{8}, \frac{3}{8}]$ be an arbitrary element in $C(A_i)$ for all $i = \{1, 2\}$.

Let us consider sets $B_{n+1} = F(B_n, B_n) = F_1(B_n, B_n) \cup F_2(B_n, B_n)$

$$B_2 = F(B_1, B_1) = F_1(B_1, B_1) \cup F_2(B_1, B_1) = [-\frac{1}{32}, \frac{3}{32}]$$

$$B_3 = F(B_2, B_2) = [-\frac{1}{128}, \frac{1}{128}] \cup [\frac{7}{128}, \frac{9}{128}]$$

$$B_4 = [-\frac{5}{1024}, -\frac{3}{1024}] \cup [-\frac{1}{1024}, \frac{1}{1024}] \cup [\frac{3}{1024}, \frac{5}{1024}] \cup [\frac{59}{1024}, \frac{61}{1024}] \cup [\frac{63}{1024}, \frac{65}{1024}] \cup [\frac{67}{1024}, \frac{69}{1024}].$$

In the Figure-7.1, the first three iterations are given. In the above example, this Φ -iterated tupling system generates the Cantor set as its unique fractal.

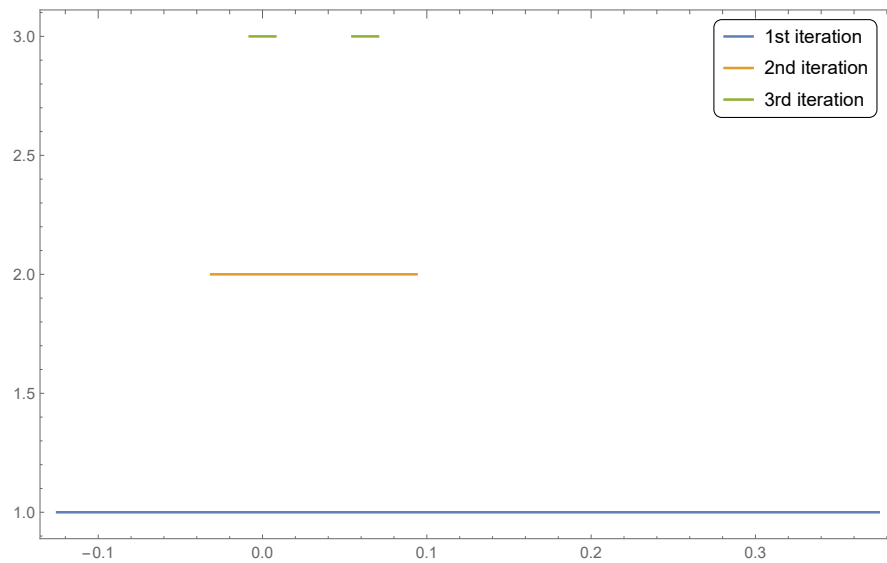


FIGURE 7.1: Graph for first three iterations of the Example 7.19

7.4 Concluding remarks

In this chapter, utilizing the concept of strong m -tuple fixed point method, m -tuple fractals are generated, which provides a positive answer to the question of Adrian and Gabriela [73] concerning the generation of multiple fractals.
