CHAPTER 2

ON MGT THERMOELASTICITY: DOMAIN OF INFLUENCE RESULTS

2.1 A Domain of Influence Theorem under MGT Thermoelasticity Theory

2.1.1 Introduction[1]

The present chapter of the thesis aims to present some qualitative analysis on the recently proposed thermoelasticity theory, the Moore-Gibson-Thompson thermoelasticity theory. It is worth to recall that the domain of influence (DOI) theorem is a very effective tool to analyze the deformation of a material body under any thermomechanical effects. From this theorem, it can be concluded that a solution of given system vanishes outside a bounded domain at a finite time and in terms of data specified in a bounded support. This theorem verifies the hyperbolicity of the model and defines the finite propagation of thermoelastic disturbances. Due to the infinite behavior of thermal disturbances, the classical coupled thermoelasticity theory is unable to achieve this type of theorem. Nunziato and Cowin (1979) first introduced the idea of domain of influence theorem for elastic materials with voids. Ignaczak (1978b) introduced the domain of influence theorem in view of the linear thermoelasticity. Further, Ignaczak (1979a) also

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discussed the domain of influence theorem under the asymmetric elastodynamics. In linear elastodynamics, the DOI theorem with energy inequalities was reported by Carbonaro and Russo (1984). Ignaczak (1991) and further Hetnarski and Ignaczak (1999) presented the DOI theorems under the generalized thermoelasticity theories of LS type and also of GL type. For linear thermoelasticity with internal variable and thermal relaxation, the DOI results were validated by Cimmelli and Rogolino (2002). Later on, the concepts of domain of influence were described by Ignaczak and Ostoja-Starzewski (2010) in detail. Mukhopadhyay et al. (2011) presented the domain of influence theorem under the DPL model. Further, Kumar and Kumar (2015) investigated the domain of influence theorem under the TPL model. The domain of influence theorems under the type-II thermoelasticity theory were reported by Kumari and Mukhopadhyay (2016; 2017a). For the MGT theory for dipolar medium, Marin et al. (2020c) provided the DOI results.

The purpose of this subchapter is to extend the concept of domain of influence for potential-temperature disturbance under the MGT thermoelasticity theory which has been introduced in literature recently. We start with summarizing the field equations for a homogeneous and isotropic material under consideration of the MGT model. Further, an identity regarding the potential of the displacement and temperature is established in the context of a mixed initial-boundary value problem. Lastly, the domain of influence theorem is derived based on this identity by considering certain conditions to be satisfied by the material parameters. This theorem proves that under MGT theory, the pair of potential with temperature generates the thermoelastic disturbance vanishing outside the bounded domain for a bounded support of thermomechanical load and for a finite time. It has been shown that the given bounded domain is very much dependent on the support of the load and the result for the present context reduces to the domain of influence result derived in generalized thermoelasticity theory of Lord and Shulman under some special conditions.

2.1.2 Basic Equations and Problem Formulation

An isotropic and homogeneous thermoelastic material is considered which is enclosed by the closure of a bounded, open, and connected set. Let $\overline{B}$ denote the closure of that domain. The boundary and interior of $\overline{B}$ are classified as ∂B and B, respectively. The governing equations defined in $B \times [0, \infty)$ under the MGT model can be written in the following way:

Equation of motion:

$$\mu u_{i,jj} + (\mu + \lambda)\, u_{j,ji} - \rho \ddot{u}_i - \gamma \theta_{,i} = -F_i. \tag{2.1.1}$$

Heat conduction equation:

$$\left(K^* + K\frac{\partial}{\partial t}\right) \nabla^2 \theta = \left(\frac{\partial}{\partial t} + \tau_q \frac{\partial^2}{\partial t^2}\right) \left(\gamma T_0 \dot{u}_{i,i} + \rho C_E \dot{\theta}\right). \tag{2.1.2}$$

It is assumed that the material constants satisfy the following conditions:

$$\begin{aligned} &\mu > 0,\ \ \lambda > 0,\ \ C_E > 0,\ \ \rho > 0,\ \ 3\lambda + 2\mu > 0,\ \ T_0 > 0, \\ &K > 0,\ \ K^* > 0,\ \ \tau_q > 0. \end{aligned} \tag{2.1.3}$$

Now, assuming the absence of any body force in the system, F_i is considered to be zero and displacement is taken as the gradient of a scalar field. Consequently, we take $u_i = \varphi_{,i}$, where φ is used to represent a scalar field defined on $\overline{B} \times [0, \infty)$. Then from Eqs. (2.1.1) and (2.1.2), it is obtained that

$$\nabla^2 \varphi - \frac{\rho}{2\mu + \lambda} \ddot{\varphi} - \frac{\gamma}{2\mu + \lambda} \theta = 0 \ \text{ on } B \times [0, \infty), \tag{2.1.4}$$

$$\left(K^* + K\frac{\partial}{\partial t}\right) \nabla^2 \theta = \left(\frac{\partial}{\partial t} + \tau_q \frac{\partial^2}{\partial t^2}\right) \left(\gamma T_0 \nabla^2 \dot{\varphi} + \rho C_E \dot{\theta}\right) \ \text{ on } B \times [0, \infty). \tag{2.1.5}$$

For convenience of mathematical formulation in the above equations, the dimensionless variables and notations are introduced in the following way:

$$t' = \frac{C_E c_1^2 \rho}{K} t, \;\; x' = \frac{C_E c_1 \rho}{K} x, \;\; \theta' = \frac{\theta}{T_0}, \;\; \tau_q' = \frac{C_E c_1^2 \rho}{K} \tau_q,$$
$$\varphi' = \frac{(\lambda + 2\mu)\, C_E^2 c_1^2 \rho^2}{\gamma T_0 K^2} \varphi, \;\; b_0 = \frac{K^*}{(2\mu + \lambda)\, C_E}, \;\; c_1^2 = \frac{(\lambda + 2\mu)}{\rho}.$$

It is worth mentioning that Quintanilla (2019) has shown the stability condition of the solutions depending on the material parameters that the solution of the system in the context of MGT thermoelasticity is exponentially stable if the relation $K^*\tau_q < K$ is considered. Hence, in view of this, the relation $K^*\tau_q < K$ is transformed to the following dimensionless form by using above dimensionless variables and notations:

The relation $K^*\tau_q < K$ implies that

$$\frac{K^* \tau_q' K}{(2\mu + \lambda)\, C_E} < K.$$

Therefore

$$b_0 < \frac{1}{\tau_q'}.$$

After dropping the prime for simplicity, the following relation is obtained:

$$b_0 < \frac{1}{\tau_q}. \tag{2.1.6}$$

In view of the above, Eqs. (2.1.4) and (2.1.5) are transformed to the following dimensionless forms after dropping the primes for simplicity:

$$\nabla^2 \varphi - \ddot{\varphi} - \theta = 0 \;\text{ on } B \times [0, \infty), \tag{2.1.7}$$
$$\left(b_0 + \frac{\partial}{\partial t}\right) \nabla^2 \theta = \left(\frac{\partial}{\partial t} + \tau_q \frac{\partial^2}{\partial t^2}\right) \left(\varepsilon \nabla^2 \dot{\varphi} + \dot{\theta}\right) \;\text{ on } B \times [0, \infty), \tag{2.1.8}$$

where $\varepsilon = \frac{\gamma^2 T_0}{\rho^2 c_1^2 C_E}$ is the thermoelastic coupling constant.

Now, the initial conditions to the above Eqs. (2.1.7) and (2.1.8) for $x \in B$ are considered as

$$\begin{aligned}
\theta(x,0) &= \theta_0, \\
\dot{\theta}(x,0) &= \dot{\theta}_0, \\
\ddot{\theta}(x,0) &= \ddot{\theta}_0, \\
\varphi(x,0) &= \varphi_0, \\
\dot{\varphi}(x,0) &= \dot{\varphi}_0, \\
\ddot{\varphi}(x,0) &= \ddot{\varphi}_0,
\end{aligned} \tag{2.1.9}$$

and the following boundary conditions are adjoined:

$$\begin{aligned}
\theta &= \theta'(x,t), \\
\varphi_{,i} n_i &= g(x,t) \ \text{ on } \partial B \times (0,\infty).
\end{aligned} \tag{2.1.10}$$

2.1.3 Main Results

Firstly, the following two sets are introduced:

$$D_0(t) = \left\{ x \in \overline{B} : \begin{array}{c} (1)\,\text{for } x \in B,\ \varphi_0 \neq 0 \ \text{ or } \ \dot{\varphi}_0 \neq 0 \ \text{ or } \ \ddot{\varphi}_0 \neq 0 \ \text{ or } \ \theta_0 \neq 0 \\ \text{or } \ \dot{\theta}_0 \neq 0 \ \text{ or } \ \ddot{\theta}_0 \neq 0 \\ (2)\,\text{for } \ (x,\tau) \in \partial B \times [0,t],\ \theta'(x,\tau) \neq 0 \ \text{ or } \ g(x,\tau) \neq 0 \end{array} \right. \tag{2.1.11}$$

and

$$D(t) = \left\{ x \in \overline{B} : \overline{\Omega(x, vt)} \cap D_0(t) \neq \emptyset \right\}. \tag{2.1.12}$$

As described by Ignaczak and Starzewski (2010), one can explain that for a fixed time $t \in (0, \infty)$, the set $D_0(t)$ defines the support of thermomechanical load for the present system (2.1.7)−(2.1.10) under the MGT thermoelasticity theory. Further, in view of the above thermomechanical load, the set $D(t)$ represents the domain of influence where v is any real parameter and $\Omega(x, vt)$ is an open ball with center x and radius vt. Hence, $D(t)$ is a set of all the points of $\overline{B}$ that can be reached by those thermomechanical disturbances that start from the set $D_0(t)$ and whose propagation speed does not exceed v.

Now, an attempt is made in the next theorem to derive an identity in the context of MGT thermoelasticity theory that will be used to establish the final result.

Theorem-2.1.3.1: Let (φ, θ) be a solution to the mixed problem (2.1.7)−(2.1.10) and $l(x) \in C^1(\overline{B})$ represents a scalar field in such a way that the set

$$\mathscr{F}_0 = \left\{ x \in \overline{B} : l(x) > 0 \right\} \tag{2.1.13}$$

be bounded. Then

$$\begin{aligned}
&\frac{1}{2}\int_B \left\{ L_0(x, l(x)) - \left[l(x)\dot{L}_0(x,0) + L_0(x,0) \right] \right\} dB \\
&\quad + \frac{1}{2}\int_B \left\{ \int_0^{l(x)} L_1(x,t)\,dt - l(x)\,L_1(x,0) \right\} dB \\
&\quad + \int_B \left\{ \int_0^{l(x)} [l(x) - t]\, M(x,t)\,dt \right\} dB + \int_B \left\{ \int_0^{l(x)} N_i(x,t)\, l_{,i}(x)\,dt \right\} dB \\
&\quad = \int_{\partial B} \left\{ \int_0^{l(x)} [l(x) - t]\, N_i(x,t)\, n_i(x)\,dt \right\} dA,
\end{aligned} \tag{2.1.14}$$

where

$$L_0(x,t) = b_0\tau_q\theta_{,i}\theta_{,i}, \tag{2.1.15}$$

$$L_1(x,t) = \hat{\theta}^2 + \tau_q\dot{\theta}_{,i}\dot{\theta}_{,i} + b_0\theta_{,i}\theta_{,i} + \varepsilon(\nabla^2\hat{\varphi})^2 + \varepsilon\dot{\hat{\varphi}}_{,i}\dot{\hat{\varphi}}_{,i}, \tag{2.1.16}$$

$$M(x,t) = (1 - b_0\tau_q)\,\dot{\theta}_{,i}\dot{\theta}_{,i}, \tag{2.1.17}$$

$$N_i(x,t) = \hat{\theta}(b_0\theta_{,i} + \dot{\theta}_{,i} - \varepsilon\dot{\hat{\varphi}}_{,i}) + \varepsilon\dot{\hat{\varphi}}_{,i}\nabla^2\hat{\varphi} \quad \forall\ (x,t) \in \overline{B} \times [0,\infty). \tag{2.1.18}$$

The hat notation for any function $g = g(x,t)$ is used here as follows:

$$\hat{g} = \left(\frac{\partial}{\partial t} + \tau_q\frac{\partial^2}{\partial t^2}\right) g \ \text{ on } \overline{B} \times [0,\infty). \tag{2.1.19}$$

Proof: The hat operator is applied to Eq. (2.1.7) and the gradient is taken to get

$$(\nabla^2\hat{\varphi})_{,i} - \hat{\theta}_{,i} - \ddot{\hat{\varphi}}_{,i} = 0 \ \text{ on } \overline{B} \times [0,\infty). \tag{2.1.20}$$

We will employ the following identities:

$$\dot{\hat{\varphi}}_{,i}\ddot{\hat{\varphi}}_{,i} = \frac{1}{2}\frac{\partial}{\partial t}(\dot{\hat{\varphi}}_{,i}\dot{\hat{\varphi}}_{,i}),$$
$$\dot{\hat{\varphi}}_{,i}(\nabla^2\hat{\varphi})_{,i} = (\dot{\hat{\varphi}}_{,i}\nabla^2\hat{\varphi})_{,i} - \frac{1}{2}\frac{\partial}{\partial t}(\nabla^2\hat{\varphi})^2.$$

Then, taking multiplication of Eq. (2.1.20) with $\varepsilon\dot{\hat{\varphi}}_{,i}$ and using above identities, the following equation is acquired:

$$\frac{\varepsilon}{2}\frac{\partial}{\partial t}\left\{\dot{\hat{\varphi}}_{,i}\dot{\hat{\varphi}}_{,i} + (\nabla^2\hat{\varphi})^2\right\} + \varepsilon\dot{\hat{\varphi}}_{,i}\hat{\theta}_{,i} = \varepsilon(\dot{\hat{\varphi}}_{,i}\nabla^2\hat{\varphi})_{,i}. \tag{2.1.21}$$

Further, taking multiplication of Eq. (2.1.8) with $\hat{\theta}$ and making use of the following identities:

$$\hat{\theta}\nabla^2\theta = (\hat{\theta}\theta_{,i})_{,i} - \hat{\theta}_{,i}\theta_{,i},$$
$$\hat{\theta}\nabla^2\dot{\theta} = (\hat{\theta}\dot{\theta}_{,i})_{,i} - \hat{\theta}_{,i}\dot{\theta}_{,i},$$
$$\hat{\theta}\nabla^2\dot{\varphi} = (\hat{\theta}\dot{\varphi}_{,i})_{,i} - \hat{\theta}_{,i}\dot{\varphi}_{,i}.$$

We further find

$$\frac{1}{2}\frac{\partial}{\partial t}\left\{\hat{\theta}^2 + \tau_q\dot{\theta}_{,i}\dot{\theta}_{,i} + 2b_0\tau_q\dot{\theta}_{,i}\theta_{,i} + b_0\theta_{,i}\theta_{,i}\right\} - \varepsilon\dot{\varphi}_{,i}\hat{\theta}_{,i} + (1-b_0\tau_q)\,\dot{\theta}_{,i}\dot{\theta}_{,i} = \left[\hat{\theta}\left(\dot{\theta}_{,i} + b_0\theta_{,i} - \varepsilon\dot{\varphi}_{,i}\right)\right]_{,i}. \quad (2.1.22)$$

The following equality is obtained after summation of Eqs. (2.1.21) and (2.1.22):

$$\frac{1}{2}\frac{\partial}{\partial t}\left\{L_1(x,t) + \frac{\partial}{\partial t}L_0(x,t)\right\} + M(x,t) = N_{i,i}(x,t), \quad (2.1.23)$$

where L_0, L_1, M and N_i are given in Eqs. (2.1.15), (2.1.16), (2.1.17) and (2.1.18), respectively.

To solve further, the following relation is employed:

$$\int_0^{l(x)} N_{i,i}\left[l(x) - t\right]dt = \int_0^{l(x)}\left\{\left[N_i(x,t)\left[l(x)-t\right]\right]_{,i} - N_i(x,t)\,l_{,i}(x)\right\}dt = \left[\int_0^{l(x)} N_i(x,t)\left[l(x)-t\right]dt\right]_{,i} - \int_0^{l(x)} N_i(x,t)\,l_{,i}(x)\,dt. \quad (2.1.24)$$

Now, integrating two times of Eq. (2.1.23) from $t = 0$ to $t = l(x)$ over t and employing Eq. (2.1.24), we obtain

$$\frac{1}{2}\left[L_0\left(x,l\left(x\right)\right)-L_0\left(x,0\right)-l\left(x\right)\dot{L}_0\left(x,0\right)\right]+\frac{1}{2}\left[\int_0^{l(x)}L_1\left(x,t\right)dt-l\left(x\right)L_1\left(x,0\right)\right]$$
$$+\int_0^{l(x)}\left[l\left(x\right)-t\right]M\left(x,t\right)dt+\int_0^{l(x)}N_i\left(x,t\right)l_{,i}\left(x\right)dt=\left[\int_0^{l(x)}N_i\left(x,t\right)\left[l\left(x\right)-t\right]dt\right]_{,i}. \tag{2.1.25}$$

Since the Eq. (2.1.13) implies that the set $\mathscr{F}_0$ is bounded, each term in Eq. (2.1.25) is bounded. Thus, taking the integration of Eq. (2.1.25) over B and making use of the divergence theorem in the right-hand side, Eq. (2.1.14) is acquired. Thus, the proof of Theorem-2.1.3.1 is complete.

In the next theorem, a domain of influence theorem is derived on the basis of the identity (2.1.14) under the MGT thermoelasticity theory.

Theorem-2.1.3.2: If (φ,θ) is a smooth solution of problem (2.1.7) – (2.1.10) and if the set $D\left(t\right)$ given by Eq. (2.1.12) defines the domain of influence at time t and for thermomechanical load $D_0\left(t\right)$ given by Eq. (2.1.11), then

$$\varphi=\theta=0 \text{ on } \left\{\overline{B}-D\left(t\right)\right\}\times\left[0,t\right], \tag{2.1.26}$$

where $v>0$ satisfies the following identity:

$$v\geq\max\left\{1+\varepsilon,2,b_0,\frac{1}{\tau_q}\right\}. \tag{2.1.27}$$

Proof: Let

$$\Lambda=\overline{B}\cap\overline{\Omega\left(w,v\tau\right)} \tag{2.1.28}$$

and

$$l_\tau(x) = \begin{cases} \tau - \frac{1}{v}|x-w| & \text{for } x \in \Lambda \\ 0 & \text{for } x \notin \Lambda \end{cases}, \tag{2.1.29}$$

where $(w,\tau) \in \{B - D(t)\} \times (0,t)$ is a fixed point and v is a parameter defined by Eq. (2.1.27).

Now, since $\tau < t$ and from the definitions of Λ and domain $D(t)$ represented by Eqs. (2.1.28) and (2.1.12), respectively, the following relation is acquired:

$$D_0(t) \cap \Lambda = \emptyset. \tag{2.1.30}$$

Therefore, we obtain

$$\theta = 0, \quad \varphi_{,i} n_i = 0 \ \text{ on } \ (\Lambda \cap \partial B) \times [0,t] \tag{2.1.31}$$

and

$$\hat{\theta} = 0, \quad \dot{\hat{\varphi}}_{,i} n_i = 0 \ \text{ on } \ (\Lambda \cap \partial B) \times [0,t]. \tag{2.1.32}$$

Also

$$\begin{aligned} &\varphi(x,0) = \dot{\varphi}(x,0) = \ddot{\varphi}(x,0) = 0 \ \text{ on } \Lambda, \\ &\theta(x,0) = \dot{\theta}(x,0) = \ddot{\theta}(x,0) = 0 \ \text{ on } \Lambda. \end{aligned} \tag{2.1.33}$$

From Eqs. (2.1.18), (2.1.29) and (2.1.32), the following relation is achieved:

$$\int_{\partial B} \left\{ \int_0^{l_\tau(x)} [l_\tau(x) - t]\, N_i(x,t)\, n_i(x)\, dt \right\} dA = 0. \tag{2.1.34}$$

Moreover, Eqs. (2.1.33) and (2.1.20) yield

$$\hat{\theta}(x,0) = \nabla^2 \hat{\varphi}(x,0) = 0 \text{ on } \Lambda, \tag{2.1.35}$$

$$\theta_{,i}(x,0) = \dot{\hat{\varphi}}_{,i}(x,0) = 0 \text{ on } \Lambda, \tag{2.1.36}$$

$$\dot{\theta}_{,i}(x,0) = 0 \text{ on } \Lambda. \tag{2.1.37}$$

Hence, definitions of $L_0(x,t)$, $L_1(x,t)$ and $l_\tau(x)$ imply that

$$L_0(x, l_\tau(x)) - L_0(x,0) - l_\tau(x)\dot{L}_0(x,0) = \begin{cases} L_0(x, l_\tau(x)) & \text{for } x \in \Lambda \\ 0 & \text{for } x \notin \Lambda \end{cases} \tag{2.1.38}$$

and

$$\int_0^{l_\tau(x)} L_1(x,t)\,dt - l_\tau(x) L_1(x,0) = \begin{cases} \int_0^{l_\tau(x)} L_1(x,t)\,dt & \text{for } x \in \Lambda \\ 0 & \text{for } x \notin \Lambda \end{cases}. \tag{2.1.39}$$

Now, after substituting $l_\tau(x)$ into Eq. (2.1.14) and with the help of Eqs. (2.1.34), (2.1.38) and (2.1.39), one can find the following relation:

$$\frac{1}{2}\int_\Lambda L_0(x, l_\tau(x))\,dB + \frac{1}{2}\int_\Lambda \int_0^{l_\tau(x)} L_1(x,t)\,dtdB + \int_\Lambda \left\{ \int_0^{l_\tau(x)} [l_\tau(x) - t]\, M(x,t)\,dt \right\} dB$$
$$= -\int_\Lambda \left\{ \int_0^{l_\tau(x)} N_i(x,t)\, l_{\tau,i}(x)\,dt \right\} dB. \tag{2.1.40}$$

In view of Eq. (2.1.6), it is obtained that $M \geq 0$ on Λ. Therefore, the following inequality is found from Eqs. (2.1.29) and (2.1.40):

$$\frac{1}{2}\int_{\Lambda} L_0\left(x, l_\tau\left(x\right)\right) dB + \frac{1}{2}\int_{\Lambda}\int_0^{l_\tau(x)} L_1\left(x,t\right) dtdB \leq \frac{1}{v}\int_{\Lambda}\int_0^{l_\tau(x)} \left|N_i\left(x,t\right) dt\right| dB. \tag{2.1.41}$$

From Eq. (2.1.18), the following inequality is obtained:

$$\begin{aligned}\left|N_i\left(x,t\right)\right| \leq& \varepsilon|\dot{\hat{\varphi}}_{,i}||\nabla^2\hat{\varphi}| + |\hat{\theta}|\left(|b_0\theta_{,i} + \dot{\theta}_{,i}| + \varepsilon|\dot{\hat{\varphi}}_{,i}|\right)\\ \leq& \varepsilon\left[\frac{2\dot{\hat{\varphi}}_{,i}\dot{\hat{\varphi}}_{,i} + (\nabla^2\hat{\varphi})^2 + (\hat{\theta})^2}{2}\right] + \left[\frac{\dot{\theta}_{,i}\dot{\theta}_{,i} + b_0^2\theta_{,i}\theta_{,i} + (\hat{\theta})^2}{2}\right] + \frac{b_0}{2}\frac{d}{dt}\left(\theta_{,i}\theta_{,i}\right).\end{aligned} \tag{2.1.42}$$

Thus, in the contexts of $L_0\left(x,t\right)$ and $L_1\left(x,t\right)$ given by Eqs. (2.1.15) and (2.1.16), respectively and from the inequalities (2.1.41) and (2.1.42), it is acquired that

$$\begin{aligned}&\frac{b_0}{2}\left(\tau_q - \frac{1}{v}\right)\int_{\Lambda}\theta_{,i}\theta_{,i}dB + \frac{1}{2}\left(\tau_q - \frac{1}{v}\right)\int_{\Lambda}\int_0^{l_\tau(x)}\dot{\theta}_{,i}\dot{\theta}_{,i}dtdB\\ &+\frac{1}{2}\left(1 - \frac{1+\varepsilon}{v}\right)\int_{\Lambda}\int_0^{l_\tau(x)}(\hat{\theta})^2 dtdB + \frac{\varepsilon}{2}\left(1 - \frac{2}{v}\right)\int_{\Lambda}\int_0^{l_\tau(x)}\dot{\hat{\varphi}}_{,i}\dot{\hat{\varphi}}_{,i}dtdB\\ &+\frac{\varepsilon}{2}\left(1 - \frac{1}{v}\right)\int_{\Lambda}\int_0^{l_\tau(x)}(\nabla^2\hat{\varphi})^2 dtdB + \frac{b_0}{2}\left(1 - \frac{b_0}{v}\right)\int_{\Lambda}\int_0^{l_\tau(x)}\theta_{,i}\theta_{,i}dtdB \leq 0.\end{aligned} \tag{2.1.43}$$

Now, Eq. (2.1.43) has all non-negative integrals since all the coefficients in Eq. (2.1.43) are non-negative, provided $v \geq \max\left\{1+\varepsilon, 2, b_0, \frac{1}{\tau_q}\right\}$. Hence, all the integrals are non-negative and the equality sign holds in this equation which imply that each term in Λ vanishes.

Particularly, we take

$$\hat{\theta}\left(x, l_\tau\left(x\right)\right) = 0, \quad \nabla^2\hat{\varphi}\left(x, l_\tau\left(x\right)\right) = 0 \text{ on } \Lambda. \tag{2.1.44}$$

Since (φ, θ) is sufficiently smooth and from the definition of $l_\tau(x)$, the following is obtained:

$$\left.\begin{array}{r}\hat{\theta}(x, l_\tau(x)) \to \hat{\theta}(w, \tau) \\ \nabla^2 \hat{\varphi}(x, l_\tau(x)) \to \nabla^2 \hat{\varphi}(w, \tau)\end{array}\right\} \text{as } x \to w. \tag{2.1.45}$$

Consequently, taking the limit $x \to w$ in Eq. (2.1.44) and using Eq. (2.1.29), the following equation is found:

$$\nabla^2 \hat{\varphi}(w, \tau) = 0, \;\; \hat{\theta}(w, \tau) = 0. \tag{2.1.46}$$

In view of an arbitrary point (w, τ) of $\{B - D(t)\} \times (0, t)$ and with the smoothness property of (φ, θ), we get

$$\hat{\theta}(w, \tau) = 0, \;\; \nabla^2 \hat{\varphi}(w, \tau) = 0 \;\text{ on }\; \left\{\overline{B} - D(t)\right\} \times [0, t]. \tag{2.1.47}$$

Therefore, from Eqs. (2.1.47) and (2.1.7), the following is achieved:

$$\hat{\theta}(w, \tau) = \ddot{\hat{\varphi}}(w, \tau) = 0 \;\text{ on }\; \left\{\overline{B} - D(t)\right\} \times [0, t]. \tag{2.1.48}$$

Now, Eq. (2.1.48) implies the following results for $(x, \tau) \in \left\{\overline{B} - D(t)\right\} \times [0, t]$:

$$\ddot{\varphi}(w, \tau) = \ddot{\varphi}(w, 0) + \tau_q \dddot{\varphi}(w, 0)\left\{1 - e^{-\tau/\tau_q}\right\} \tag{2.1.49}$$

and

$$\theta(w, \tau) = \theta(w, 0) + \tau_q \dot{\theta}(w, 0)\left\{1 - e^{-\tau/\tau_q}\right\}. \tag{2.1.50}$$

Definition of $D(t)$ combining with Eq. (2.1.7) yields

$$\ddot{\varphi}(w, 0) = \dddot{\varphi}(w, 0) = \theta(w, 0) = \dot{\theta}(w, 0) = 0 \;\text{ on }\; \left\{\overline{B} - D(t)\right\}. \tag{2.1.51}$$

Hence, in view of the Eqs. (2.1.49) and (2.1.50), we find

$$\ddot{\varphi} = \theta = 0 \text{ on } \left\{\overline{B} - D(t)\right\} \times [0, t]. \tag{2.1.52}$$

Finally, from the Eq. (2.1.52) and the definition of $D(t)$, it is concluded that

$$\varphi = \theta = 0 \text{ on } \left\{\overline{B} - D(t)\right\} \times [0, t]. \tag{2.1.53}$$

This completes the proof of Theorem-2.1.3.2.

2.1.4 Conclusion

From the theorem 2.1.3.2 as established in previous subsection, it is observed that if the condition (2.1.6) holds then the pair (φ, θ), which satisfies the mixed-problem (2.1.7) – (2.1.10) under the MGT model, generates the thermoelastic disturbance vanishing outside the bounded set $D(t)$ for a finite time t and for a bounded support of thermomechanical loading. The bounded domain is observed to be dependent on the given support of thermomechanical loading. Moreover, in view of taking the relation (2.1.6), it is concluded that the thermoelastic disturbance propagates with finite speed less than or equal to v represented by the Eq. (2.1.27). Thus, it is clear that the maximum speed v depends on the phase lag τ_q, thermoelastic coupling constant ε, and b_0. We also observe that the propagation speed of thermoelastic disturbance generated from (φ, θ) is infinite if the phase-lag parameter τ_q vanishes. This is an important finding of the present subchapter.

Furthermore, it is worth recalling the fact as shown by Quintanilla (2019) that the solution of MGT theory decays exponentially provided the condition $K^*\tau_q < K$ holds, otherwise it leads to unstable solution of the problem. Here, in view of this condition it implies that $b_0 < \frac{1}{\tau_q}$ (see derivation of Eq. (2.1.6)). Hence, it is interesting to see that the inequality $v \geq \max\left\{1 + \varepsilon, 2, b_0, \frac{1}{\tau_q}\right\}$ reduces to $v \geq \max\left\{1 + \varepsilon, 2, \frac{1}{\tau_q}\right\}$ as obtained by Ignaczak and Starzewski (2010) under the LS thermoelasticity theory. Therefore, if the condition (2.1.6) is considered then the hyperbolicity of the MGT thermoelastic model is proved and the upper bound of propagation speed of disturbance under the MGT model is dependent on material parameters similar to the LS thermoelasticity theory. This is an important observation of the present work.

2.2 A Domain of Influence Theorem for a Natural Stress-Heat-Flux Problem in the Moore-Gibson-Thompson Thermoelasticity Theory

2.2.1 Introduction[2]

In the previous subchapter, the domain of influence theorem in terms of temperature and potential is established under the MGT thermoelasticity. Now, this subchapter is motivated to prove the domain of influence theorem corresponding to the natural stress-heat-flux problem for the theory of MGT thermoelasticity. Following is the organization of the work. The fundamental equations in terms of stress and heat-flux pair concerning an isotropic and homogeneous medium under the MGT model are summarized in Subsection 2.2.2. Further, Subsection 2.2.3 discusses some definitions related to the support of thermomechanical load and the set of the domain of influence. In Subsection 2.2.4, an energy identity is first presented in the present context and then the domain of influence theorem regarding this identity has been established. Lastly, it has been concluded that the pair of stress with heat-flux generates the stress-heat-flux disturbance vanishing outside the bounded domain for a finite time and for a support of prescribed thermomechanical load which is bounded. It is also found that the domain of influence depends on the thermoelastic coupling constant and other material parameters.

2.2.2 Basic Equations and Problem Formulation

In this subsection, the governing equations under the MGT thermoelastic model are formulated involving the pair of stress and heat-flux. The basic governing equations and

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the constitutive relations in the same context for isotropic and homogeneous medium are therefore considered in the following way:

Stress equation of motion:

$$\sigma_{ij,j} + F_i = \rho \ddot{u}_i. \tag{2.2.1}$$

Energy equation:

$$-q_{i,i} + \rho H = C_S \dot{\theta} + T_0 \alpha_t \dot{\sigma}_{kk}. \tag{2.2.2}$$

Stress-strain relation:

$$e_{ij} = \frac{1}{2\mu}\left(\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{kk}\delta_{ij}\right) + \alpha_t \theta \delta_{ij}. \tag{2.2.3}$$

Heat conduction equation:

$$\left(1 + \tau_q \frac{\partial}{\partial t}\right)\dot{q}_i = -\left(K^* + K\frac{\partial}{\partial t}\right)\theta_{,i}. \tag{2.2.4}$$

Strain-displacement relation:

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = u_{(i,j)}. \tag{2.2.5}$$

Here, C_S denotes the specific heat at constant stress.

Now, with the help of Eqs. (2.2.1) – (2.2.5), a problem on natural stress-heat-flux involving the pair (σ_{ij}, q_i) is considered that satisfies the field equations as follows:

$$\rho^{-1}\sigma_{(ik,kj)} - \left[\frac{1}{2\mu}\left(\ddot{\sigma}_{ij} - \frac{\lambda}{3\lambda + 2\mu}\ddot{\sigma}_{kk}\delta_{ij}\right) - \frac{T_0 \alpha_t^2}{C_S}\ddot{\sigma}_{kk}\delta_{ij}\right] + \frac{\alpha_t}{C_S}\dot{q}_{k,k}\delta_{ij} = \frac{\rho}{C_S}\alpha_t \dot{H}\delta_{ij} - \rho^{-1}F_{(i,j)}, \tag{2.2.6}$$

$$\left(K^* + K\frac{\partial}{\partial t}\right)\frac{1}{C_S}\left(q_{k,k} + \alpha_t T_0 \dot{\sigma}_{kk}\right)_{,i} - \left(1 + \tau_q \frac{\partial}{\partial t}\right)\ddot{q}_i = \left(K^* + K\frac{\partial}{\partial t}\right)\frac{\rho}{C_S}H_{,i}, \quad (2.2.7)$$

where, the notation $F_{(i,j)} = \frac{1}{2}\left(F_{i,j} + F_{j,i}\right)$ is used.

Now, we must recall that Eq. (2.2.6) is a generalization of the Ignaczak equation (Ignaczak (1991) and Ostoja-Starzewski (2019)).

The material parameters satisfying the following conditions are assumed:

$$\begin{aligned} &\rho > 0,\ \ \mu > 0,\ \ \lambda > 0,\ \ C_S > 0,\ \ \alpha_t > 0,\ \ T_0 > 0, \\ &\tau_q > 0,\ \ K^* > 0,\ \ 3\lambda + 2\mu > 0. \end{aligned} \quad (2.2.8)$$

Further, as discussed in previous subchapter, the following relation is considered:

$$K > K^* \tau_q. \quad (2.2.9)$$

Now, we set the following notations in Eqs. (2.2.6) and (2.2.7):

$$G_{(ij)} = \rho^{-1} F_{(i,j)} - \frac{\rho}{C_S}\alpha_t \dot{H}\delta_{ij}, \quad (2.2.10)$$

$$m_i = -\left(K^* + K\frac{\partial}{\partial t}\right)\frac{\rho}{C_S}H_{,i}. \quad (2.2.11)$$

Then, from Eqs. (2.2.6) and (2.2.7), it is obtained that

$$\rho^{-1}\sigma_{(ik,kj)} - \left[\frac{1}{2\mu}\left(\ddot{\sigma}_{ij} - \frac{\lambda}{3\lambda + 2\mu}\ddot{\sigma}_{kk}\delta_{ij}\right) - \frac{T_0 \alpha_t^2}{C_S}\ddot{\sigma}_{kk}\delta_{ij}\right] + \frac{\alpha_t}{C_S}\dot{q}_{k,k}\delta_{ij} = -G_{(ij)}, \quad (2.2.12)$$

$$\left(K^* + K\frac{\partial}{\partial t}\right)\frac{1}{C_S}\left(q_{k,k} + \alpha_t T_0 \dot{\sigma}_{kk}\right)_{,i} - \left(1 + \tau_q \frac{\partial}{\partial t}\right)\ddot{q}_i = -m_i. \quad (2.2.13)$$

The initial conditions on $x \in B$ to the above Eqs. (2.2.12) and (2.2.13) are assumed in the following way:

$$\begin{aligned} \sigma_{ij}(x,0) &= \sigma_{ij}^0, \\ \dot{\sigma}_{ij}(x,0) &= \dot{\sigma}_{ij}^0, \\ q_i(x,0) &= q_i^0, \\ \dot{q}_i(x,0) &= \dot{q}_i^0, \\ \ddot{q}_i(x,0) &= \ddot{q}_i^0, \end{aligned} \tag{2.2.14}$$

and boundary conditions on $\partial B \times [0,\infty[$ are taken as

$$\begin{aligned} \sigma_{ij} n_j &= \sigma_i', \\ q_i n_i &= q'. \end{aligned} \tag{2.2.15}$$

The present problem which is based on the pair of stress and heat-flux is known as a natural stress-heat-flux problem (Ignaczak and Ostoja-Starzewski (2010)). Therefore, a solution to the natural stress-heat-flux problem is called a stress-heat-flux disturbance.

2.2.3 Some Definitions

Now, in the present context, we discuss the concepts of the support of thermomechanical load and the set of the domain of influence before moving on to the main results.

Definition 1. It is considered that $t \in (0,\infty)$ is a fixed time. Then, the set

$$D_0(t) = \begin{cases} & (1)\,\text{for } x \in B,\ \sigma_{ij}^0 \neq 0 \text{ or } \dot{\sigma}_{ij}^0 \neq 0 \text{ or } q_i^0 \neq 0 \text{ or } \dot{q}_i^0 \neq 0 \text{ or } \ddot{q}_i^0 \neq 0 \\ x \in \overline{B}: & (2)\,\text{for } (x,\tau) \in \partial B \times [0,t],\ \sigma_i' \neq 0 \text{ or } q' \neq 0 \\ & (3)\,\text{for } (x,\tau) \in \partial B \times [0,t],\ G_{(ij)} \neq 0 \text{ or } m_i \neq 0 \end{cases} \tag{2.2.16}$$

is said to be the support of thermomechanical load of the present system (2.2.12) – (2.2.15) at time t.

Definition 2. Let $\Omega\left(x, vt\right)$ is an open ball with center x and radius vt, where v is any real parameter. For the above thermomechanical load $D_0(t)$, the set

$$D(t) = \left\{x \in \overline{B} : \overline{\Omega\left(x, vt\right)} \cap D_0\left(t\right) \neq \emptyset\right\} \tag{2.2.17}$$

defines the domain of influence. Thus, the set $D(t)$ defines a set of all the points of $\overline{B}$ which can be accessed by the thermomechanical disturbances propagating from $D_0(t)$ with a finite speed not exceeding v (Ignaczak and Ostoja-Starzewski (2010)).

2.2.4 Main Results

Now, as in previous subchapter, we proceed to formulate an identity regarding the present context. For this we assume that the following inequality is satisfied by v:

$$v \geq \max(v_1, v_2, v_3, v_4), \tag{2.2.18}$$

where

$$v_1 = \left(\frac{2\mu}{\rho}\right)^{\frac{1}{2}}, \tag{2.2.19}$$

$$v_2 = \left\{\frac{K}{\tau_q C_S}\left[1 + \frac{C_S}{C_E}\left(1 - \frac{C_E}{C_S}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}}, \tag{2.2.20}$$

$$v_3 = \left\{\frac{K^*}{C_S}\left[1 + \frac{C_S}{C_E}\left(1 - \frac{C_E}{C_S}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}}, \tag{2.2.21}$$

$$v_4 = \left\{\frac{(3\lambda + 2\mu)}{\rho}\frac{C_S}{C_E}\left[1 - \left(1 - \frac{C_E}{C_S}\right)^{\frac{1}{2}}\right]^{-1}\right\}^{\frac{1}{2}}, \tag{2.2.22}$$

where C_E represents the specific heat at constant strain satisfying the following relation with C_S :

$$C_S = C_E + 3\alpha_t^2 (3\lambda + 2\mu) T_0. \tag{2.2.23}$$

Now, the following energy identity is derived for the present context in view of the above definitions.

Theorem-2.2.4.1: Let (σ_{ij}, q_i) satisfies the mixed problem (2.2.12) – (2.2.15) with smoothness property and $e(x) \in C^1(\overline{B})$ is considered to be a scalar field in such a way that the set

$$J_0 = \left\{x \in \overline{B} : \, e(x) > 0\right\} \tag{2.2.24}$$

is bounded. Then

$$\begin{aligned}
&\frac{1}{2}\int_B \left\{E_0\left(x, e\left(x\right)\right) - \left[e\left(x\right)\dot{E}_0\left(x, 0\right) + E_0\left(x, 0\right)\right]\right\} dB \\
&\quad + \frac{1}{2}\int_B \left\{\int_0^{e(x)} E_1\left(x, t\right) dt - e\left(x\right) E_1\left(x, 0\right)\right\} dB \\
&\quad + \int_B \left\{\int_0^{e(x)} \left[e\left(x\right) - t\right] L\left(x, t\right) dt\right\} dB + \int_B \left\{\int_0^{e(x)} M_i\left(x, t\right) e_{,i}\left(x\right) dt\right\} dB \\
&= \int_{\partial B} \left\{\int_0^{e(x)} \left[e\left(x\right) - t\right] M_i\left(x, t\right) n_i\left(x\right) dt\right\} dA + \int_B \left\{\int_0^{e(x)} \left[e\left(x\right) - t\right] N\left(x, t\right) dt\right\} dB,
\end{aligned} \tag{2.2.25}$$

where

$$E_0\left(x, t\right) = \frac{K^* \tau_q}{T_0} \left(\dot{q}_i\right)^2, \tag{2.2.26}$$

$$\begin{aligned}
E_1\left(x, t\right) =& \rho^{-1} \hat{\sigma}_{ik,k} \hat{\sigma}_{ij,j} + \frac{1}{2\mu}\left(\dot{\hat{\sigma}}_{ij}\dot{\hat{\sigma}}_{ij} - \frac{\lambda}{3\lambda + 2\mu}\left(\dot{\hat{\sigma}}_{kk}\right)^2\right) - \frac{\alpha_t^2 T_0}{C_S}\left(\dot{\hat{\sigma}}_{kk}\right)^2 \\
&+ \frac{1}{C_S T_0}\left(\hat{q}_{k,k}\right)^2 + \frac{K^*}{T_0}\left(\dot{q}_i\right)^2 + \frac{K\tau_q}{T_0}\left(\ddot{q}_i\right)^2,
\end{aligned} \tag{2.2.27}$$

$$L(x,t) = \frac{(K - K^* \tau_q)}{T_0} (\ddot{q}_i)^2, \tag{2.2.28}$$

$$M_i(x,t) = \rho^{-1} \dot{\hat{\sigma}}_{ij} \hat{\sigma}_{jk,k} + \frac{1}{C_S} \left(\alpha_t \dot{\hat{\sigma}}_{kk} + \frac{1}{T_0} \hat{q}_{k,k} \right) \left(K^* + K \frac{\partial}{\partial t} \right) \dot{q}_i, \tag{2.2.29}$$

$$N(x,t) = G_{(ij)} \dot{\hat{\sigma}}_{ij} + \frac{1}{\theta_0} m_i \dot{\hat{q}}_i, \tag{2.2.30}$$

where for any function $g = g(x,t)$ defined on $x \in \overline{B} \times [0, \infty[$, $\hat{g}(\,)$ is denoted in the following way:

$$\hat{g} = \left(K^* + K \frac{\partial}{\partial t} \right) g. \tag{2.2.31}$$

Proof: First, we apply the hat operator as represented by Eq. (2.2.31) on Eq. (2.2.12) and multiply by $\dot{\hat{\sigma}}_{ij}$ to both sides of Eq. (2.2.12), then we obtain

$$\rho^{-1} \hat{\sigma}_{(ik,kj)} \dot{\hat{\sigma}}_{ij} - \left[\frac{1}{2\mu} \left(\ddot{\hat{\sigma}}_{ij} \dot{\hat{\sigma}}_{ij} - \frac{\lambda}{3\lambda + 2\mu} \ddot{\hat{\sigma}}_{kk} \dot{\hat{\sigma}}_{kk} \right) - \frac{T_0 \alpha_t^2}{C_S} \ddot{\hat{\sigma}}_{kk} \dot{\hat{\sigma}}_{kk} \right] + \frac{\alpha_t}{C_S} \dot{\hat{q}}_{k,k} \dot{\hat{\sigma}}_{kk} = -G_{(ij)} \dot{\hat{\sigma}}_{ij}. \tag{2.2.32}$$

Now, we multiply both sides of Eq. (2.2.13) through $T_0^{-1} \dot{\hat{q}}_i$ and get

$$\left\{ \left(K^* + K \frac{\partial}{\partial t} \right) \frac{1}{C_S} \left(q_{k,k} + \alpha_t T_0 \dot{\sigma}_{kk} \right)_{,i} \right\} T_0^{-1} \dot{\hat{q}}_i - \left\{ \left(1 + \tau_q \frac{\partial}{\partial t} \right) \ddot{q}_i \right\} T_0^{-1} \dot{\hat{q}}_i = -m_i T_0^{-1} \dot{\hat{q}}_i. \tag{2.2.33}$$

Adding Eqs. (2.2.32) and (2.2.33) and after some straight-forward manipulations, the following equation is obtained:

$$\frac{1}{2} \frac{\partial}{\partial t} \left\{ \frac{\partial}{\partial t} E_0(x,t) + E_1(x,t) \right\} + L(x,t) = M_{i,i}(x,t) + N(x,t), \tag{2.2.34}$$

where E_0, E_1, L, M_i and N are given by Eqs. (2.2.26) – (2.2.30), respectively.

Since the following relation holds:

$$\int_0^{e(x)} M_{i,i}\left[e\left(x\right)-t\right]dt = \int_0^{e(x)}\Big\{\left[M_i\left(x,t\right)\left[e\left(x\right)-t\right]\right]_{,i} - M_i\left(x,t\right)e_{,i}\left(x\right)\Big\}\,dt$$
$$= \left[\int_0^{e(x)} M_i\left(x,t\right)\left[e\left(x\right)-t\right]dt\right]_{,i} - \int_0^{e(x)} M_i\left(x,t\right)e_{,i}\left(x\right)dt. \tag{2.2.35}$$

Therefore, taking double integration of Eq. (2.2.34) from $t=0$ to $t=e\left(x\right)$ over t and making use of Eq. (2.2.35), we get

$$\frac{1}{2}\left[E_0\left(x,e\left(x\right)\right)-E_0\left(x,0\right)-e\left(x\right)\dot{E}_0\left(x,0\right)\right]+\frac{1}{2}\left[\int_0^{e(x)} E_1\left(x,t\right)dt-e\left(x\right)E_1\left(x,0\right)\right]$$
$$+\int_0^{e(x)}\left[e\left(x\right)-t\right]L\left(x,t\right)dt+\int_0^{e(x)} M_i\left(x,t\right)e_{,i}\left(x\right)dt$$
$$=\left[\int_0^{e(x)} M_i\left(x,t\right)\left[e\left(x\right)-t\right]dt\right]_{,i}+\int_0^{e(x)}\left[e\left(x\right)-t\right]N\left(x,t\right)dt. \tag{2.2.36}$$

The set J_0 is bounded from the Eq. (2.2.24). So, each term in Eq. (2.2.36) is considered to be bounded. Therefore, taking the integration of Eq. (2.2.36) over B and employing the divergence theorem in the RHS, the Eq. (2.2.25) is obtained. This proves the Theorem-2.2.4.1.

Now, the domain of influence theorem for a natural stress-heat-flux problem is derived under the MGT thermoelasticity theory.

Theorem-2.2.4.2: Let v denotes a real parameter satisfying the inequality defined by the Eq. (2.2.18). Then, if the domain of influence is defined by the set $D\left(t\right)$ at time t and for thermomechanical load $D_0\left(t\right)$ and if the pair $\left(\sigma_{ij},q_i\right)$ satisfies the problem (2.2.12) – (2.2.15) with smoothness property, then

$$\sigma_{ij}=0,\quad q_i=0\ \text{ on }\ \left\{\overline{B}-D(t)\right\}\times\left[0,t\right]. \tag{2.2.37}$$

Proof. We consider a fixed point $(w,\tau)\in\{B-D(t)\}\times(0,t)$. Let

$$\Lambda=\overline{B}\cap\overline{\Omega(w,v\tau)} \tag{2.2.38}$$

and

$$e_{\tau}(x)=\begin{cases}\tau-\frac{1}{v}|x-w| & \text{for } x\in\Lambda\\ 0 & \text{for } x\notin\Lambda\end{cases}, \tag{2.2.39}$$

where v is a parameter given by the Eq. (2.2.18).

Now, from the definitions of domain $D(t)$ and Λ defined by the Eqs. (2.2.17) and (2.2.38), respectively and from the inequality $\tau<t$, we find

$$D_0(t)\cap\Lambda=\emptyset. \tag{2.2.40}$$

Therefore, it is obtained that

$$\sigma_{ij}n_j=0,\ \ q_in_i=0 \ \text{ on } \ (\Lambda\cap\partial B)\times[0,t] \tag{2.2.41}$$

and

$$\dot{\sigma}_{ij}n_j=0,\ \ \dot{q}_in_i=0,\ \ \ddot{q}_in_i=0 \ \text{ on } \ (\Lambda\cap\partial B)\times[0,t] \tag{2.2.42}$$

$$G_{(ij)}=0,\ \ m_i=0 \ \text{ on } \Lambda\times(0,t). \tag{2.2.43}$$

Further, the following is acquired:

$$\sigma_{ij}(x,0)=\dot{\sigma}_{ij}(x,0)=q_i(x,0)=\dot{q}_i(x,0)=\ddot{q}_i(x,0) \ \text{ on } \Lambda. \tag{2.2.44}$$

Now, in view of Eqs. (2.2.29), (2.2.39) and (2.2.42), we obtain

$$\int_{\partial B}\left\{\int_0^{e_\tau(x)}\left[e_\tau(x)-t\right]M_i(x,t)\,n_i(x)\,dt\right\}dA=0. \tag{2.2.45}$$

Similarly, in view of Eqs. (2.2.30), (2.2.39), and (2.2.43), it is found that

$$\int_{B}\left\{\int_0^{e_\tau(x)}\left[e_\tau(x)-t\right]N(x,t)\,dt\right\}dB=0. \tag{2.2.46}$$

Clearly, applying the definitions of $E_0(x,t)$, $E_1(x,t)$ and $e_\tau(x)$, the following results are achieved after some straight-forward manipulations:

$$E_0(x,e_\tau(x))-E_0(x,0)-e_\tau(x)\,\dot{E}_0(x,0)=\begin{cases}E_0(x,e_\tau(x)) & \text{for } x\in\Lambda\\ 0 & \text{for } x\notin\Lambda\end{cases} \tag{2.2.47}$$

and

$$\int_0^{e_\tau(x)}E_1(x,t)\,dt-e_\tau(x)\,E_1(x,0)=\begin{cases}\int_0^{e_\tau(x)}E_1(x,t)\,dt & \text{for } x\in\Lambda\\ 0 & \text{for } x\notin\Lambda\end{cases}. \tag{2.2.48}$$

Now, substituting $e_\tau(x)$ into the Eq. (2.2.25) and making use of the Eqs. (2.2.45) – (2.2.48), we achieve

$$\frac{1}{2}\int_\Lambda E_0(x,e_\tau(x))\,dB+\frac{1}{2}\int_\Lambda\int_0^{e_\tau(x)}E_1(x,t)\,dtdB+\int_\Lambda\left\{\int_0^{e_\tau(x)}\left[e_\tau(x)-t\right]N(x,t)\,dt\right\}dB$$
$$=-\int_\Lambda\left\{\int_0^{e_\tau(x)}M_i(x,t)\,e_{\tau,i}(x)\,dt\right\}dB. \tag{2.2.49}$$

Since Eq. (2.2.9) imply that $N\geq 0$ on Λ. Therefore, from the Eqs. (2.2.39) and (2.2.49), the inequality is found as follows:

$$\frac{1}{2}\int_{\Lambda} E_0\left(x, e_\tau\left(x\right)\right) dB + \frac{1}{2}\int_{\Lambda}\int_0^{e_\tau(x)} E_1\left(x,t\right) dtdB \leq \frac{1}{v}\int_{\Lambda}\int_0^{e_\tau(x)} \left|M_i\left(x,t\right) dt\right| dB. \tag{2.2.50}$$

Now,

$$\begin{aligned}\frac{1}{v}\left|M_i\right| \leq & \rho^{-1}\left|\frac{\dot{\hat{\sigma}}_{ij}}{v}\hat{\sigma}_{jk,k}\right| + \frac{1}{C_S}\left|\alpha_t\dot{\hat{\sigma}}_{kk} + \frac{1}{T_0}\hat{q}_{k,k}\right|\left|\left(K^* + K\frac{\partial}{\partial t}\right)\frac{\dot{q}_i}{v}\right| \\ \leq & \rho^{-1}\left|\frac{\dot{\hat{\sigma}}_{ij}}{v}\right|\left|\hat{\sigma}_{jk,k}\right| + \frac{\left|\alpha_t\right|}{C_S}\left|\dot{\hat{\sigma}}_{kk}\right|\left|\frac{K^*\dot{q}_i + K\ddot{q}_i}{v}\right| + \frac{1}{C_S T_0}\left|\hat{q}_{k,k}\right|\left|\frac{K^*\dot{q}_i + K\ddot{q}_i}{v}\right|.\end{aligned} \tag{2.2.51}$$

To compute each term of the RHS of Eq. (2.2.51) and simplify the Eq. (2.2.51), the following relation is used:

$$\sqrt{ab} \leq \frac{1}{2}\left(\varepsilon a + \varepsilon^{-1} b\right), \tag{2.2.52}$$

where ε denotes a dimensionless positive parameter and m and n are nonnegative physical fields with the equal dimension.

In order to compute the first term of Eq. (2.2.51), we employ $a = \left(\hat{\sigma}_{jk,k}\right)^2$, $b = \left(\frac{\dot{\hat{\sigma}}_{ij}}{v}\right)^2$ and $\varepsilon = 1$ in Eq. (2.2.52) and then obtain

$$\left|\hat{\sigma}_{jk,k}\right|\left|\frac{\dot{\hat{\sigma}}_{ij}}{v}\right| \leq \frac{1}{2}\left(\hat{\sigma}_{ij,j}\hat{\sigma}_{ik,k} + \frac{1}{v^2}\dot{\hat{\sigma}}_{ij}\dot{\hat{\sigma}}_{ij}\right). \tag{2.2.53}$$

In order to compute the second term of Eq. (2.2.51), the following relations are used:

$$a = \left(\dot{\hat{\sigma}}_{kk}\right)^2, \quad b = \frac{1}{v^2\left(\alpha_t T_0\right)^2}\left(K^*\dot{q}_i + K\ddot{q}_i\right)^2, \quad \varepsilon = \frac{C_E}{C_S}\left(1 - \frac{C_E}{C_S}\right)^{-\frac{1}{2}}. \tag{2.2.54}$$

Therefore, using Eq. (2.2.54) in Eq. (2.2.52), it is obtained that

$$
\begin{aligned}
\left|\dot{\hat{\sigma}}_{kk}\right| \frac{1}{v\left|\alpha_t\right| T_0}\left|K^*\dot{q}_i+K\ddot{q}_i\right| \leq & \frac{1}{2}\left\{\frac{C_E}{C_S}\left(1-\frac{C_E}{C_S}\right)^{-\frac{1}{2}}\left(\dot{\hat{\sigma}}_{kk}\right)^2\right. \\
& \left.+\frac{C_S}{C_E}\left(1-\frac{C_E}{C_S}\right)^{\frac{1}{2}} \frac{1}{v^2\alpha_t^2T_0^2}\left(K^*\dot{q}_i+K\ddot{q}_i\right)^2\right\} \\
\leq & \frac{1}{2}\left\{\frac{C_E}{C_S}\left(1-\frac{C_E}{C_S}\right)^{-\frac{1}{2}}\left(\dot{\hat{\sigma}}_{kk}\right)^2\right. \\
& \left.+\frac{C_S}{C_E}\left(1-\frac{C_E}{C_S}\right)^{\frac{1}{2}} \frac{1}{v^2\alpha_t^2T_0^2}\left[\left(K^*\dot{q}_i\right)^2+\left(K\ddot{q}_i\right)^2\right]\right\} \\
& +\frac{K^*K}{2v^2\alpha_t^2T_0^2}\frac{C_S}{C_E}\left(1-\frac{C_E}{C_S}\right)^{\frac{1}{2}} \frac{d}{dt}\left(\dot{q}_i\right)^2 .
\end{aligned} \tag{2.2.55}
$$

Now, by fixing $a=\left(\hat{q}_{k,k}\right)^2$, $b=\left(\frac{K^*\dot{q}_i+K\ddot{q}_i}{v}\right)^2$ and $\varepsilon=1$ in Eq. (2.2.52), we get the last term of Eq. (2.2.51) as

$$
\begin{aligned}
\left|\hat{q}_{k,k}\right|\left|\frac{K^*\dot{q}_i+K\ddot{q}_i}{v}\right| \leq & \frac{1}{2}\left\{\left(\hat{q}_{k,k}\right)^2+\frac{1}{v^2}\left(K^*\dot{q}_i+K\ddot{q}_i\right)^2\right\} \\
\leq & \frac{1}{2}\left\{\left(\hat{q}_{k,k}\right)^2+\frac{1}{v^2}\left[\left(K^*\dot{q}_i\right)^2+\left(K\ddot{q}_i\right)^2\right]\right\}+\frac{K^*K}{2v^2}\frac{d}{dt}\left(\dot{q}_i\right)^2 .
\end{aligned} \tag{2.2.56}
$$

Thus, Eqs. (2.2.51), (2.2.53), (2.2.55), and (2.2.56) yield

$$
\begin{aligned}
\frac{1}{v}\left|M_i\right| \leq & \frac{\rho^{-1}}{2}\left(\hat{\sigma}_{ij,j}\hat{\sigma}_{ik,k}+\frac{1}{v^2}\dot{\hat{\sigma}}_{ij}\dot{\hat{\sigma}}_{ij}\right) \\
& +\frac{\alpha_t^2T_0}{2C_S}\left\{\frac{C_E}{C_S}\left(1-\frac{C_E}{C_S}\right)^{-\frac{1}{2}}\left(\dot{\hat{\sigma}}_{kk}\right)^2+\frac{C_S}{C_E}\left(1-\frac{C_E}{C_S}\right)^{\frac{1}{2}} \frac{1}{v^2\alpha_t^2T_0^2}\left[\left(K^*\dot{q}_i\right)^2+\left(K\ddot{q}_i\right)^2\right]\right\} \\
& +\frac{1}{2C_ST_0}\left\{\left(\hat{q}_{k,k}\right)^2+\frac{1}{v^2}\left[\left(K^*\dot{q}_i\right)^2+\left(K\ddot{q}_i\right)^2\right]\right\} \\
& +\frac{K^*K}{2v^2C_ST_0}\left\{1+\frac{C_S}{C_E}\left(1-\frac{C_E}{C_S}\right)^{\frac{1}{2}}\right\}\frac{d}{dt}\left(\dot{q}_i\right)^2 .
\end{aligned} \tag{2.2.57}
$$

Further, the relation (2.2.23) implies that

$$\frac{\alpha_t^2 T_0}{C_S} = \frac{1}{3\,(3\lambda + 2\mu)} \left(1 - \frac{C_E}{C_S}\right). \tag{2.2.58}$$

Therefore, in view of the definitions of E_0 and E_1 given by Eqs. (2.2.26) and (2.2.27), respectively and using Eqs. (2.2.50), (2.2.57) and relation (2.2.58), we arrive at

$$\begin{aligned}
&\left(\frac{1}{2\mu} - \frac{1}{\rho v^2}\right) \int_\Lambda \int_0^{e_\tau(x)} \left(\dot{\hat{\sigma}}_{ij} - \frac{1}{3}\dot{\hat{\sigma}}_{kk}\delta_{ij}\right)\left(\dot{\hat{\sigma}}_{ij} - \frac{1}{3}\dot{\hat{\sigma}}_{kk}\delta_{ij}\right) dtdB \\
&+\frac{K^*}{T_0}\left[1 - \frac{K^*}{C_S v^2}\left\{1 + \frac{C_S}{C_E}\left(1 - \frac{C_E}{C_S}\right)^{\frac{1}{2}}\right\}\right] \int_\Lambda \int_0^{e_\tau(x)} (\dot{q}_i)^2 dtdB \\
&+\frac{K}{T_0}\left[\tau_q - \frac{K}{C_S v^2}\left\{1 + \frac{C_S}{C_E}\left(1 - \frac{C_E}{C_S}\right)^{\frac{1}{2}}\right\}\right] \int_\Lambda \int_0^{e_\tau(x)} (\ddot{q}_i)^2 dtdB \\
&+\frac{1}{3}\left[\frac{1}{3\lambda + 2\mu}\frac{C_E}{C_S}\left\{1 - \left(1 - \frac{C_E}{C_S}\right)^{\frac{1}{2}}\right\} - \frac{1}{\rho v^2}\right] \int_\Lambda \int_0^{e_\tau(x)} \left(\dot{\hat{\sigma}}_{kk}\right)^2 dtdB \\
&\qquad +\frac{K^*}{T_0}\left[\tau_q - \frac{K}{C_S v^2}\left\{1 + \frac{C_S}{C_E}\left(1 - \frac{C_E}{C_S}\right)^{\frac{1}{2}}\right\}\right] \int_\Lambda (\dot{q}_i)^2 dB \le 0.
\end{aligned} \tag{2.2.59}$$

From Eq. (2.2.18), it is concluded that the coefficients of all the integrals in Eq. (2.2.59) are nonnegative. Thus, the non-negativity of all integrals with the equality sign in this equation implies the vanishing of each term of Eq. (2.2.59) on Λ.

Particularly, we have

$$\dot{\hat{\sigma}}_{ij}\left(x, e_\tau(x)\right) = 0, \;\; \dot{q}_i\left(x, e_\tau(x)\right) = 0 \;\text{ on } \Lambda. \tag{2.2.60}$$

Since (σ_{ij}, q_i) is sufficiently smooth and from the definition of $e_\tau\left(x\right)$, it is obtained that

$$\left.\begin{aligned}
\dot{\hat{\sigma}}_{ij}\left(x, e_\tau(x)\right) &\to \dot{\hat{\sigma}}_{ij}\left(w, \tau\right) \\
\dot{q}_i\left(x, e_\tau(x)\right) &\to \dot{q}_i\left(w, \tau\right)
\end{aligned}\right\} \text{as } x \to w. \tag{2.2.61}$$

Consequently, the limit $x \to w$ is taken in Eq. (2.2.60) and the following equation is found from Eq. (2.2.39):

$$\dot{\dot{\sigma}}_{ij}(w,\tau) = 0, \;\; \dot{q}_i(w,\tau) = 0 \;\text{ on }\; \left\{\overline{B} - D(t)\right\} \times [0,t]\,. \tag{2.2.62}$$

In view of an arbitrary point (w,τ) of $\left\{\overline{B} - D(t)\right\} \times (0,t)$ and from the smoothness property of (σ_{ij}, q_i) in $\overline{B} \times [0,\infty)$, we find that

$$\dot{\dot{\sigma}}_{ij} = 0, \;\; \dot{q}_i = 0 \;\text{ on }\; \left\{\overline{B} - D(t)\right\} \times [0,t]\,. \tag{2.2.63}$$

Now, with regard to $(x,\tau) \in \left\{\overline{B} - D(t)\right\} \times [0,t]$, Eq. (2.2.63) implies the following results:

$$\sigma_{ij}(x,\tau) = \sigma_{ij}(x,0) + \left\{1 - e^{-\frac{K^*}{K}\tau}\right\} \frac{K}{K^*}\dot{\sigma}_{ij}(x,0) \tag{2.2.64}$$

and

$$q_i(x,\tau) = q_i(x,0)\,. \tag{2.2.65}$$

Since the definition of $D(t)$ yields

$$\sigma_{ij}(x,0) = \dot{\sigma}_{ij}(x,0) = q_i(x,0) = 0 \;\;\text{ on }\; \left\{\overline{B} - D(t)\right\}. \tag{2.2.66}$$

Hence, combining Eqs. (2.2.64) and (2.2.65) with Eq. (2.2.66), we finally obtain

$$\sigma_{ij} = 0, \;\;\; q_i = 0 \;\;\text{ on }\; \left\{\overline{B} - D(t)\right\} \times [0,t]\,. \tag{2.2.67}$$

Thus, this proves the Theorem-2.2.4.2.

2.2.5 Conclusion

A domain of influence theorem is presented in this subchapter for a thermoelastic process described by stress and heat flux as constitutive variables. This theorem indicates that the pair (σ_{ij}, q_i) satisfying the system (2.2.12) – (2.2.15) under the MGT model generates the stress-heat-flux disturbance vanishing outside the bounded set $D(t)$ for a prescribed bounded support of thermomechanical load and for a finite time t, provided the condition (2.2.9) holds. Furthermore, this theorem implies that if the relation given by Eq. (2.2.9) is considered then we find that the stress-heat-flux disturbance propagates with finite speed not exceeding v defined by the Eq. (2.2.18). Clearly, v is observed to be dependent on the phase lag τ_q, K, K^*, and some other thermoelastic parameters. It is also concluded that for a given load of the boundary, the associated domain of influence is specified with a boundary layer of thickness vt. The upper bound of the speed of stress-heat-flux disturbances can also be found from this theorem. It must be mentioned that the condition given by (2.2.9) is considered here in view of the fact as analyzed by Quintanilla (2019) that if the condition $K^*\tau_q < K$ holds then the solution under the MGT theory is exponentially stable, otherwise leads to instability of solution. Therefore, it can be concluded that the inequality $v \geq \max\{v_1, v_2, v_3, v_4\}$ reduces to $v \geq \max\{v_1, v_2, v_4\}$ which is interestingly the same inequality as obtained by Ignaczak and Ostoja-Starzewski (2010) for the case of generalized thermoelasticity theory of Lord and Shulman. Hence, we can conclude that the hyperbolicity of the MGT thermoelastic model is established if condition (2.2.9) is satisfied and the maximum speed of propagation of disturbance of stress-heat-flux under the MGT model depends on material parameters in a similar way like Lord-Shulman theory.