Chapter 5

A Class of Pseudo-Differential Operators Involving the Weinstein Transform

5.1 Introduction

Pseudo-differential operators are a generalization of partial differential operators that extend the concept of differentiation to functions that are not necessarily smooth or rapidly decreasing. They play a significant role in various branches of mathematics, particularly in the theory of partial differential equations (PDEs) like Sobolev spaces and harmonic analysis. The aforesaid theory provides a proper framework for analyzing and solving a wide range of problems involving differential equations in non-smooth settings.

Using Fourier transform theory, Kohn and Nirenberg [26] developed algebra and calculus of pseudo-differential operators, and they studied many important properties.

Later on, Hörmander [22, 23], Kato [25], Rhuzhansky and Turunen [54, 55], Treves [80], Wong [88], made important contributions in this area by taking the concept of the Fourier transform. In 1996, by using the Fourier transform theory Pathak [47] introduced generalized Sobolev spaces and pseudo-differential operators on the space of ultradistributions. Pathak and Pandey [48] developed a class of pseudo-differential operators associated with Bessel operators using Hankel transform theory. Using the same transform theory, Pathak and Upadhyay [51] considered pseudo-differential operators associated with the homogeneous class of symbol. Pathak and Upadhyay [52], proved the $L^p_\mu-$ boundedness result of pseudo-differential operators associated with the Bessel operator and applied these results in Sobolev type spaces. For the Weinstein transform concerns, Upadhyay and Sartaj, [84–86] presented many results regarding the theory of pseudo-differential operators.

In the present chapter, with the help of the Weinstein transform technique, the author introduces the concept of the pseudo-differential operators associated with the symbol $a(x,\xi)$ on the space $S_\omega(\mathbb{R}^{n+1}_+)$ by exploiting the theory of ultradistribution and studied many properties.

5.2 The Pseudo-differential Operator $A(x, D)$

In this section, definitions and properties regarding the pseudo-differential operators associated with the symbol $a(x,\xi)$ are obtained on the space $S_\omega(\mathbb{R}^{n+1}_+)$ by exploiting the theory of Weinstein transform.

Let $\{M_k\},\ k \in \mathbb{N}_0$ be the sequence of positive numbers that satisfies the following conditions: for some positive costants K and h,

$$(M.1) \qquad M_k^2 \leq M_{k-1}M_{k+1},\ k \in \mathbb{N}$$

$$(M.2) \qquad M_k \leq K\ h^p \min_{0\leq l\leq k} M_l\ M_{k-l},\ k \in \mathbb{N}_0$$

$$(M.3) \qquad \sum_{l=1}^{\infty} \frac{M_{l-1}}{M_l} < \infty.$$

We define functions $M(t)$ and $P(t),\ t > 0$, associated with the sequence $\{M_k\}$ as follows:

$$M(t) := \sup_k\ ln\ \big(t^k\ \frac{M_0}{M_k}\big). \tag{5.2.1}$$

$$P(t) := \sum_{k=0}^{\infty} t^k\ \frac{M_0}{M_k}. \tag{5.2.2}$$

For $x \in \mathbb{R}_+^{n+1}$, we write $M(x)$ for $M(|x|)$ when no confusion arrises.

Lemma 5.2.1. *Let $M(t)$ be the function associated with the sequence $\{M_k\}$, then from [24, p.8], we have*

$$M(Lt) \leq \frac{3}{2}\ L\ M(t)\ +\ C,\ t > 0,\ L \geq 1,\ C \in \mathbb{R}. \tag{5.2.3}$$

Definition 5.2.2. Let $\{M_p\}$ and $\{N_p\}$ be two sequences of positive real numbers satisfying the conditions $(M.1), (M.2)$ and $(M.3)$. Then $S(M_p, N_p; \mathbb{R}_+^{n+1})$ is defined to be the space of all complex valued infinitely differentiable functions ϕ on $\mathbb{R}_+^{n+1}$ satisfying

$$|x^k D^q \phi(x)| \leq C\ A^k\ B^q\ M_k\ N_q, \tag{5.2.4}$$

for certain positive constants $C,\ A$ and B depending on ϕ with

$$\mathcal{F}_w[S(M_p, N_p; \mathbb{R}_+^{n+1})] = S(N_p, M_p; \mathbb{R}_+^{n+1}). \tag{5.2.5}$$

Definition 5.2.3. Let $r \geq 0$. The symbol class S^r is the set of all complex-valued functions $a(x,\xi) : C^\infty(\mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1}) \longrightarrow \mathbb{C}$, even with respect to the last variable, such that for any two multi-indices α and β, there is a positive constant $C_{\alpha,\beta}$, depending α and β only, such that

$$\left|(D_x^\alpha D_\xi^\beta a(x,\xi))\right| \leq C_{\alpha,\beta}(1 + \|\xi\|^2)^{\frac{r-\beta}{2}}, \quad x, \xi \in \mathbb{R}_+^{n+1}.$$

Definition 5.2.4. Let $a(x,\xi) : C^\infty(\mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1}) \longrightarrow \mathbb{C}$ be a symbol. Then for $u \in S_\omega(\mathbb{R}_+^{n+1})$, the pseudo-differential operator $A(x,D)$ associated with the symbol $a(x,\xi)$ is defined by

$$(A(x,D)u)(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}) a(x,\xi)(\mathcal{F}_w u)(\xi)\, d\mu_\beta(\xi). \tag{5.2.6}$$

If $a(x,\xi)$ possesses a power series expansion in ξ with variable coefficients depending on x, i.e.,

$$a(x,\xi) = \sum_{|\alpha=0|}^{\infty} a_\alpha(x)(-1)^\alpha \|\xi\|^{2\alpha}. \tag{5.2.7}$$

Then from (5.2.6) and (5.2.7), we have

$$\begin{aligned}
&(A(x,D)u)(x) \\
&\quad = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}) \Big(\sum_{|\alpha=0|}^{\infty} a_\alpha(x)(-1)^\alpha \|\xi\|^{2\alpha}\Big)(\mathcal{F}_w u)(\xi)\, d\mu_\beta(\xi) \\
&\quad = \sum_{|\alpha=0|}^{\infty} a_\alpha(x) \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}) \Big((-1)^\alpha \|\xi\|^{2\alpha}\Big)(\mathcal{F}_w u)(\xi)\, d\mu_\beta(\xi) \\
&\quad = \sum_{|\alpha=0|}^{\infty} a_\alpha(x) \int_{\mathbb{R}_+^{n+1}} (-1)^\alpha \|\xi\|^{2\alpha} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})(\mathcal{F}_w u)(\xi)\, d\mu_\beta(\xi) \\
&\quad = \sum_{|\alpha|=0}^{\infty} a_\alpha(x) \mathcal{F}_w^{-1}\Big[(-1)^\alpha \|\xi\|^{2\alpha}(\mathcal{F}_w u)(t)\Big](x).
\end{aligned}$$

There In veiw of (1.4.5), we have

$$(A(x,D)u)(x) = \sum_{|\alpha|=0}^{\infty} a_\alpha(x)\mathcal{F}_w^{-1}\big(\mathcal{F}_w((\Delta_{n,\beta})_t^\alpha u)(\xi)\big)(x)$$
$$= \sum_{|\alpha|=0}^{\infty} a_\alpha(x)((\Delta_{n,\beta})^\alpha u)(x).$$

Therefore,

$$(A(x,D)u)(x) = \sum_{|\alpha|=0}^{\infty} a_\alpha(x)((\Delta_{n,\beta})_t^\alpha u)(x). \tag{5.2.8}$$

The above expression allows the pseudo-differential operator associated with symbol $a(x,\xi)$ of an infinite order differential operator.

If we take $x \in K$, where K is a compact subset of $\mathbb{R}_+^{n+1}$, and

$$|a_\alpha(x)| \leq C_K \frac{L^{|\alpha|}}{M_{|\alpha|}}. \tag{5.2.9}$$

Then, from (5.2.7), we have

$$|a(x,\xi)| \leq C_K \sum_{|\alpha|=0}^{\infty} \frac{\|\xi\|^{|\alpha|}\cdot(2L)^{|\alpha|}}{2^{|\alpha|}\cdot M_{|\alpha|}}$$
$$\leq C_K \sum_{|\alpha|=0}^{\infty} \frac{\|\xi\|^{|\alpha|}\cdot(2L)^{|\alpha|}}{2^{|\alpha|}\cdot M_{|\alpha|}}$$
$$\leq C'_K \sup_{\alpha}^{\infty} \frac{(2L\|\xi\|)^{|\alpha|}M_0}{M_{|\alpha|}} \cdot \Big(\sum_{|\alpha|=0}^{\infty} 2^{-|\alpha|}\Big)$$
$$\leq C''_K\ exp\ (M(2L\|\xi\|)),$$

where $C''_K = C'_K \sum_{|\alpha|=0}^{\infty} 2^{-|\alpha|}$.

This motivates us to consider a general class of symbols, which is denoted by S_ω^r, $r \geq 0$.

Definition 5.2.5. The function $a(x,\xi) : \mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1} \longrightarrow \mathbb{C}$ is said to belong to S_ω^r

if and only if $a(x,\xi) \in \mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1}$ and for all compact set $K \subset \mathbb{R}_+^{n+1}$ and for all $\alpha, \beta \in \mathbb{N}_0^{n+1}$ and a positive constant $C_K = C_{\alpha,\beta,r,K}$ such that the estimate

$$|D_\xi^\alpha D_x^\beta a(x,\xi)| \leq C_K e^{r\omega(\xi)}, \ \forall \ (x,\xi) \in K \times \mathbb{R}_+^{n+1} \tag{5.2.10}$$

holds true.

Theorem 5.2.6. *Let $a(x,\xi) \in S_\omega^r$. Then the pseudo-differential operator $A(x,D)$ is a well-defined mapping of $S_\omega(\mathbb{R}_+^{n+1})$ into $C^\infty(\mathbb{R}_+^{n+1})$.*

Proof: Now, the pseudo-differential operator $A(x,D)$ is associated with symbol $a(x,\xi)$ is defined by

$$(A(x,D)u)(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi\rangle} a(x,\xi)(\mathcal{F}_w u)(\xi)\, d\mu_\beta(\xi), \quad \forall \ u \in S_w(\mathbb{R}_+^{n+1}). \tag{5.2.11}$$

Let K be any compact set of $\mathbb{R}_+^{n+1}$, then from (5.2.10), we have

$$|a(x,\xi)| \leq C_{0,0,r,K} e^{r\omega(\xi)}, \quad \forall \ (x,\xi) \in K \times \mathbb{R}_+^{n+1}. \tag{5.2.12}$$

Now,

$$\begin{aligned}
&\left|e^{\langle x',\xi'\rangle} J_\beta(x_{n+1\xi_{n+1}}) a(x,\xi)(\mathcal{F}_w u)(\xi)\right| \\
&\quad \leq |e^{\langle x',\xi'\rangle} J_\beta(x_{n+1\xi_{n+1}})| \cdot |a(x,\,\xi)| \cdot |(\mathcal{F}_w u)(\xi)| \\
&\quad \leq |a(x,\,\xi)| \cdot |(\mathcal{F}_w u)(\xi)| \\
&\quad \leq C_{0,0,r,K} e^{r\omega(\xi)} |(\mathcal{F}_w u)(\xi)| \\
&\quad = C_{0,0,r,K} e^{r\omega(\xi)} e^{\lambda\omega(\xi)} |(\mathcal{F}_w u)(\xi)| e^{-\lambda\omega(\xi)}.
\end{aligned}$$

Also, since $u \in S_w(\mathbb{R}_+^{n+1})$, using (3.2.22), above expression becomes

$$\begin{aligned}&\left|e^{\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}) a(x,\xi)(\mathcal{F}_w u)(\xi)\right| \\ &\qquad \leq C_{0,0,r,K} e^{r\omega(\xi)} \cdot \pi_{0,\lambda}(u) e^{-\lambda\omega(\xi)} \\ &\qquad = C_{0,0,r,K} \cdot \pi_{0,\lambda}(u) e^{-(\lambda-r)\omega(\xi)} \in L^1(\mathbb{R}_+^{n+1}), \quad for\ \lambda > r.\end{aligned}$$

This demonstrates the existence of $(A(x,D)u)(x),\ \forall\ x \in \mathbb{R}_+^{n+1}$ as well as continuity in $\mathbb{R}_+^{n+1}$. Moreover, from (1.4.15), we have

$$\begin{aligned}&\left|(\Delta_{n,\beta})_x^\nu[e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}) a(x,\xi)(F_w u)(\xi)]\right| \\ &\quad = \left|(F_w u)(\xi)(\Delta_{n,\beta})_x^\nu[e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}) a(x,\xi)]\right| \\ &\quad = \Bigg|(F_w u)(\xi) \sum_{j=0}^{\nu} \sum_{m=1}^{2j} \sum_{q=0}^{m} \sum_{|\rho'|\leq 2(\nu-j)} \sum_{\delta_1,\delta_2,\ldots,\delta_n\geq 0} \binom{\nu}{j}\binom{m}{q}\binom{\nu-j}{\delta_1\delta_2\ldots.\delta_n} \\ &\quad\ \times \frac{1}{\rho'!} E'_{\beta,m} x_{n+1}^{m-\nu}\left(D_x^{\rho'+q} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})\right)\left(D_x^{\rho'+2\delta'+m-q} a(x,\xi)\right)\Bigg|.\end{aligned}$$

In view of (1.3.30) and (5.2.10), we obtain

$$\begin{aligned}&\left|(\Delta_{n,\beta})_x^\nu[e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\eta_{n+1}) a(x,\xi)(F_w u)(\xi)]\right| \\ &\quad \leq \sum_{j=0}^{\nu} \sum_{m=1}^{2j} \sum_{q=0}^{m} \sum_{|\rho'|\leq 2(\nu-j)} \sum_{\delta_1,\delta_2,\ldots,\delta_n\geq 0} \binom{\nu}{j}\binom{m}{q}\binom{\nu-j}{\delta_1\delta_2\ldots.\delta_n} \\ &\quad\ \times \frac{1}{\rho'!} E'_{\beta,m} |x_{n+1}|^{m-\nu} \|\xi\|^{\rho'+q} C_{(0,\rho'+2\delta'+m-q,r,K)} e^{r\omega(\xi)} |(F_w u)(\xi)|.\end{aligned}$$

So that we have

$$
\begin{aligned}
&\left|(\Delta_{n,\beta})_x^{\nu}[e^{i\langle x',\xi'\rangle}J_\beta(x_{n+1}\xi_{n+1})a(x,\xi)(F_w u)(\xi)]\right| \\
&\leq \sum_{j=0}^{\nu}\sum_{m=1}^{2j}\sum_{q=0}^{m}\sum_{|\rho'|\leq 2(\nu-j)}\binom{\nu}{j}\binom{m}{q}\frac{1}{\rho'!}E'_{\beta,m}|x_{n+1}|^{m-\nu}\|\xi\|^{\rho'+q} \\
&\times C_{(0,\rho'+m-q,r,K)}e^{r\omega(\xi)}|(F_w u)(\xi)| \\
&\leq \sum_{j=0}^{\nu}\sum_{m=1}^{2j}\sum_{q=0}^{m}\binom{\nu}{j}\binom{m}{q}\frac{1}{2(\nu-j)!}E'_{\beta,m}|x_{n+1}|^{m-\nu}\|\xi\|^{2(\nu-j)+q} \\
&\times C_{(0,2(\nu-j)+m-q,r,K)}e^{r\omega(\xi)}|(F_w u)(\xi)| \\
&\leq \sum_{j=0}^{\nu}\sum_{m=1}^{2j}\binom{\nu}{j}\frac{1}{2(\nu-j)!}E'_{\beta,m}|x_{n+1}|^{m-\nu}\|\xi\|^{2(\nu-j)+m} \\
&\times C_{(0,2(\nu-j),r,K)}e^{r\omega(\xi)}|(F_w u)(\xi)| \\
&\leq \sum_{j=0}^{\nu}\binom{\nu}{j}\frac{1}{2(\nu-j)!}E'_{\beta,j}|x_{n+1}|^{2j-\nu}\|\xi\|^{2\nu}C_{(0,2(\nu-j),r,K)}e^{r\omega(\xi)}|(F_w u)(\xi)| \\
&\leq E'_{\beta,\nu}\mid x_{n+1}\mid^{\nu}(1+\|\xi\|)^{2\nu}C_{(0,0,r,K)}e^{r\omega(\xi)}e^{-\lambda\omega(\xi)}\cdot\sup_{\xi\in\mathbb{R}_+^{n+1}}e^{\lambda\omega(\xi)}|(F_w u)(\xi)| \\
&\leq E'_{\beta,\nu}\mid x_{n+1}\mid^{\nu}(1+\|\xi\|)^{2\nu}C_{(0,0,r,K)}e^{r\omega(\xi)}e^{-\lambda\omega(\xi)}\cdot\pi_{0,\lambda}(u) \\
&\leq E'_{\beta,\nu}(1+\mid x\mid)^{\nu}e^{\omega(\xi)}C_{(0,0,r,K)}e^{-(\lambda-r)\omega(\xi)}\cdot\pi_{0,\lambda}(u).
\end{aligned}
$$

Therefore, above implies

$$\left|(\Delta_{n,\beta})_x^{\nu}[e^{i\langle x',\xi'\rangle}J_\beta(x_{n+1}\eta_{n+1})a(x,\xi)(F_w u)(\xi)]\right|\leq E'_{\beta,\nu}(1+\mid x\mid)^{\nu}C_{(0,0,r,K)}e^{-(\lambda-r-1)\omega(\xi)}\pi_{0,\lambda}(u).$$

This proves that $(A(x,D)u)(x)\in C^{\infty}(\mathbb{R}_+^{n+1})$.

Lemma 5.2.7. *Let $\mathcal{F}_w u\in S(M_p,N_p;\mathbb{R}_+^{n+1})$. Then*

$$|(F_w u)(\eta)|\leq De^{-M(\frac{\eta}{A})}, \tag{5.2.13}$$

where D and A are positive constants.

Proof: As $\mathcal{F}_w u \in S(M_p, N_p; \mathbb{R}_+^{n+1})$, then from (5.2.4) we have

$$|\eta^k D^q (\mathcal{F}_w u)(\eta)| \leq C\ A^k\ B^q\ M_k\ N_q, \tag{5.2.14}$$

where A, B and C are positive constants depending on the function $(\mathcal{F}_w u)$.
For $q = 0$ in (5.2.14), we obtain

$$|\eta^k (\mathcal{F}_w u)(\eta)| \leq C\ A^k\ M_k\ N_0.$$

So that we have

$$\begin{aligned} |(\mathcal{F}_w u)(\eta)| &\leq C\ N_0 M_0 \left(\frac{A^k}{\eta^k}\right) . \left(\frac{M_k}{M_0}\right) \\ &= D \left(\frac{\eta}{A}\right)^{-k} . \left(\frac{M_0}{M_k}\right)^{-1}, \end{aligned}$$

where $D = C\ N_0 M_0$, Now, we have

$$\begin{aligned} |(\mathcal{F}_w u)(\eta)| &= D e^{ln\left(\frac{\eta}{A}\right)^{-k}} \cdot e^{ln\left(\frac{M_0}{M_k}\right)^{-1}} \\ &= D e^{\left[ln\left(\frac{\eta}{A}\right)^{-k} + ln\left(\frac{M_0}{M_k}\right)^{-1}\right]} \\ &= D e^{ln\left[\left(\frac{\eta}{A}\right)^{-k} . \left(\frac{M_0}{M_k}\right)^{-1}\right]} \\ &\leq D \sup_k\ e^{-ln\left[\left(\frac{\eta}{A}\right)^{k} . \left(\frac{M_0}{M_k}\right)\right]} \\ &= D e^{-M(\frac{\eta}{A})}. \end{aligned}$$

Therefore,

$$|(F_w u)(\eta)| \leq D e^{-M(\frac{\eta}{A})}.$$

5.3 The Pseudo-differential Operator With Symbols of Exponential $M-$Growth And ω-Growth

In this section, we consider the symbol with growth governed by indicatrix $M(t)$ and discuss the representation of the pseudo-differential operator $A(x, D)$ is obtained by the Theorem 5.3.6.

We now consider the symbol to belong to the symbol class $S^{r,l}_{(M)}$ defined below:

Definition 5.3.1. The function $a(x,\xi) : \mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+ \longrightarrow \mathbb{C}$ belongs to $S^{r,l}_{(M)}$ if and only if $a(x,\xi) \in C^\infty(\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+)$ for $r \in \mathbb{R}, l > 0,\ \forall\ \alpha, \beta \in \mathbb{N}^{n+1}$ there exists positive constants $L > 0$ and $C_{\alpha,l,r} > 0$ such that

$$|(\frac{\partial}{\partial\xi})^\alpha(\frac{\partial}{\partial x})^\beta a(x,\xi)| \leq C_{\alpha,l,r}L^{|\beta|}M_{|\beta|}e^{[rM(\xi)-lM(x)]}. \tag{5.3.1}$$

Lemma 5.3.2. *Let the symbol $a(x,\xi) \in S^{r,l}_{(M)}$. Then*

$$\big|(\Delta_{n,\beta})^\alpha_x[e^{i\langle x',\eta'\rangle}J_\beta(x_{n+1}\eta_{n+1})a(x,\eta)]\big| \leq E''_{\beta,\alpha}(1+\|\eta\|^2)^\alpha e^{rM(\eta)}\|x\|^\alpha e^{-lM(x)}, \tag{5.3.2}$$

for $E''_{\beta,\alpha}$ is a positive constant.

Proof: From (1.4.15), we find that

$$\begin{aligned}&(\Delta_{n,\beta})^\alpha_x[e^{i\langle x',\eta'\rangle}J_\beta(x_{n+1}\eta_{n+1})a(x,\eta)]\\ &\quad=\sum_{j=0}^{\alpha}\sum_{m=1}^{2j}\sum_{q=0}^{m}\sum_{|\rho'|\leq 2(\alpha-j)}\sum_{\delta_1,\delta_2,\ldots,\delta_n\geq 0}\binom{\alpha}{j}\binom{m}{q}\binom{\alpha-j}{\delta_1,\delta_2,\ldots.,\delta_n}\frac{1}{\rho'!}E'_{\beta,m}\\ &\quad\times x^{m-\alpha}_{n+1}\Big(D^{\rho'+q}_x e^{i\langle x',\eta'\rangle}J_\beta(x_{n+1}\eta_{n+1})\Big)\Big(D^{\rho'+2\delta'+m-q}_x a(x,\eta)\Big),\end{aligned}$$

where $E'_{\beta,m}$ is a constant for $m \in \{0, 1, ..., \alpha\}$ which depends on β.

Therefore,

$$\begin{aligned}
&\left|(\Delta_{n,\beta})_x^\alpha [e^{i\langle x',\eta'\rangle} J_\beta(x_{n+1}\eta_{n+1}) a(x,\eta)]\right| \\
&\quad = \left| \sum_{j=0}^{\alpha} \sum_{m=1}^{2j} \sum_{q=0}^{m} \sum_{|\rho'|\le 2(\alpha-j)} \sum_{\delta_1,\delta_2,\ldots,\delta_n \ge 0} \binom{\alpha}{j} \binom{m}{q} \binom{\alpha-j}{\delta_1,\delta_2,....,\delta_n} \frac{1}{\rho'!} E'_{\beta,m} \right. \\
&\quad\quad \left. \times\, x_{n+1}^{m-\alpha} \left(D_x^{\rho'+q} e^{i\langle x',\eta'\rangle} J_\beta(x_{n+1}\eta_{n+1}) \right) \left(D_x^{\rho'+2\delta'+m-q} a(x,\eta) \right) \right|.
\end{aligned}$$

Using (1.3.30) and (5.3.1), above yeilds

$$\begin{aligned}
&\left|(\Delta_{n,\beta})_x^\alpha [e^{i\langle x',\eta'\rangle} J_\beta(x_{n+1}\eta_{n+1}) a(x,\eta)]\right| \\
&\quad \le \sum_{j=0}^{\alpha} \sum_{m=1}^{2j} \sum_{q=0}^{m} \sum_{|\rho'|\le 2(\alpha-j)} \sum_{\delta_1,\delta_2,\ldots,\delta_n \ge 0} \binom{\alpha}{j} \binom{m}{q} \binom{\alpha-j}{\delta_1,\delta_2,....,\delta_n} \frac{1}{\rho'!} E'_{\beta,m} \\
&\quad\quad \times \left|x_{n+1}\right|^{m-\alpha} \|\eta\|^{|\rho'+q|} \left| C_{0,l,m} L^{|\rho'+2\delta'+m-q|} M_{|\rho'+2\delta'+m-q|} \right| e^{rM(\eta)-lM(x)} \\
&\quad \le \sum_{j=0}^{\alpha} \sum_{m=1}^{2j} \sum_{q=0}^{m} \sum_{|\rho'|\le 2(\alpha-j)} \binom{\alpha}{j} \binom{m}{q} \frac{1}{\rho'!} E'_{\beta,m} \left| C_{0,l,m} L^{|\rho'+m-q|} M_{|\rho'+m-q|} \right| \\
&\quad\quad \times \left|x_{n+1}\right|^{m-\alpha} \|\eta\|^{|\rho'+q|} e^{rM(\eta)-lM(x)} \\
&\quad \le \sum_{j=0}^{\alpha} \sum_{m=1}^{2j} \sum_{q=0}^{m} \binom{\alpha}{j} \binom{m}{q} \frac{1}{2(\alpha-j)!} E'_{\beta,m} \left| C_{0,l,m} L^{|2(\alpha-j)+m-q|} M_{|2(\alpha-j)+m-q|} \right| \\
&\quad\quad \times \left|x_{n+1}\right|^{m-\alpha} \|\eta\|^{|2(\alpha-j)+q|} e^{rM(\eta)-lM(x)} \\
&\quad \le \sum_{j=0}^{\alpha} \sum_{m=1}^{2j} \sum_{q=0}^{m} \binom{\alpha}{j} \binom{m}{q} \frac{1}{2(\alpha-j)!} E'_{\beta,m} \left| C_{0,l,m} L^{|2(\alpha-j)+m-q|} M_{|2(\alpha-j)+m-q|} \right| \\
&\quad\quad \times \left|x_{n+1}\right|^{m-\alpha} \|\eta\|^{|2(\alpha-j)+q|} e^{rM(\eta)-lM(x)} \\
&\quad \le \sum_{j=0}^{\alpha} \sum_{m=1}^{2j} \binom{\alpha}{j} \frac{1}{2(\alpha-j)!} E'_{\beta,m} \left| C_{0,l,j} L^{|2(\alpha-j)|} M_{|2(\alpha-j)|} \right| \left|x_{n+1}\right|^{m-\alpha} \\
&\quad\quad \times \|\eta\|^{|2(\alpha-j)+m|} e^{rM(\eta)-lM(x)} \\
&\quad \le \sum_{j=0}^{\alpha} \binom{\alpha}{j} \frac{1}{2(\alpha-j)!} E'_{\beta,j} \left| C_{0,l,j} L^{|2(\alpha-j)|} M_{|2(\alpha-j)|} \right| \left|x_{n+1}\right|^{2j-\alpha} \\
&\quad\quad \times \|\eta\|^{|2\alpha|} e^{rM(\eta)-lM(x)}
\end{aligned}$$

Therefore, we get

$$\begin{aligned}&\big|(\Delta_{n,\beta})_x^\alpha[e^{i\langle x',\eta'\rangle}J_\beta(x_{n+1}\eta_{n+1})a(x,\eta)]\big|\\&\quad\le E'_{\beta,\alpha}\big|C_{0,l,\alpha}L^0M_0\big|\big|x_{n+1}\big|^\alpha\|\eta\|^{|2\alpha}le^{rM(\eta)-lM(x)}\\&\quad\le E''_{\beta,\alpha}\|x\|^\alpha(1+\|\eta\|^2)^\alpha e^{rM(\eta)-lM(x)},\end{aligned}$$

where $E''_{\beta,\alpha}=E'_{\beta,\alpha}\big|C_{0,l,r}L^0M_0\big|$, is a positive constants depending on $\alpha,\beta,0,l,r$. Now we conclude that

$$\big|(\Delta_{n,\beta})_x^\alpha[e^{i\langle x',\eta'\rangle}J_\beta(x_{n+1}\eta_{n+1})a(x,\eta)]\big|\le E''_{\beta,\alpha}(1+\|\eta\|^2)^\alpha e^{rM(\eta)}\|x\|^\alpha e^{-lM(x)}.$$

Let $a(x,\xi)\in S^{r,l}_{(M)}$. Then the function $a_\eta(\xi)$ associated with the symbol $a(x,\xi)$ can be represented by

$$a_\eta(\xi)=\int_{\mathbb{R}^{n+1}_+}e^{-i\langle x',\xi'\rangle}J_\beta(x_{n+1}\xi_{n+1})[e^{i\langle x',\eta'\rangle}J_\beta(x_{n+1}\eta_{n+1})a(x,\eta)]d\mu_\beta(x).\qquad(5.3.3)$$

An estimate for $a_\eta(\xi)$ is given by Theorem 5.3.3.

Theorem 5.3.3. *Let the symbol $a(x,\xi)\in S^{r,l}_{(M)}$. Then the function $a_\eta(\xi)$ which is defined by (5.3.3), satisfies the following inequality*

$$|a_\eta(\xi)|\le E'''_{\beta,p}(1+\|\eta\|^2)^pe^{rM(\eta)}(1+\|\xi\|^2)^{-p}.\qquad(5.3.4)$$

Proof: Using the binomial expansion, we have

$$\begin{aligned}
&(1+\|\xi\|^2)^p a_\eta(\xi) \\
&\quad = \sum_{\alpha=0}^{p}\binom{p}{\alpha}\|\xi\|^{2\alpha}\int_{\mathbb{R}_+^{n+1}} e^{-i\langle x',\xi'\rangle}J_\beta(x_{n+1}\xi_{n+1})[e^{i\langle x',\eta'\rangle}J_\beta(x_{n+1}\eta_{n+1}) \\
&\qquad \times a(x,\eta)]d\mu_\beta(x) \\
&\quad = \sum_{\alpha=0}^{p}\binom{p}{\alpha}\int_{\mathbb{R}_+^{n+1}} \|\xi\|^{2\alpha}e^{-i\langle x',\xi'\rangle}J_\beta(x_{n+1}\xi_{n+1})[e^{i\langle x',\eta'\rangle}J_\beta(x_{n+1}\eta_{n+1}) \\
&\qquad \times a(x,\eta)]d\mu_\beta(x).
\end{aligned}$$

Using (1.4.5), we have

$$\begin{aligned}
(1+\|\xi\|^2)^p a_\eta(\xi) = {}& \sum_{\alpha=0}^{p}\binom{p}{\alpha}(-1)^\alpha\int_{\mathbb{R}_+^{n+1}} (\Delta_{n,\beta})_x^\alpha\big(e^{-i\langle x',\xi'\rangle}J_\beta(x_{n+1}\xi_{n+1})\big) \\
& \times [e^{i\langle x',\eta'\rangle}J_\beta(x_{n+1}\eta_{n+1})a(x,\eta)]d\mu_\beta(x).
\end{aligned}$$

Integrating by parts we get

$$\begin{aligned}
(1+\|\xi\|^2)^p a_\eta(\xi) \quad &= \sum_{\alpha=0}^{p}\binom{p}{\alpha}(-1)^\alpha\int_{\mathbb{R}_+^{n+1}} e^{-i\langle x',\xi'\rangle}J_\beta(x_{n+1}\xi_{n+1}) \\
& \quad \times (\Delta_{n,\beta})_x^\alpha[e^{i\langle x',\eta'\rangle}J_\beta(x_{n+1}\eta_{n+1})a(x,\eta)]d\mu_\beta(x).
\end{aligned}$$

Therefore,

$$(1+\|\xi\|^2)^p|a_\eta(\xi)| = \sum_{\alpha=0}^{p}\binom{p}{\alpha}\int_{\mathbb{R}_+^{n+1}} \big|(\Delta_{n,\beta})_x^\alpha[e^{i\langle x',\eta'\rangle}J_\beta(x_{n+1}\eta_{n+1})a(x,\eta)]\big|d\mu_\beta(x). \tag{5.3.5}$$

In veiw of (5.3.2), we have

$$\begin{aligned}
&(1+\|\xi\|^2)^p|a_\eta(\xi)| \\
&\quad\leq \sum_{\alpha=0}^{p}\binom{p}{\alpha}\int_{\mathbb{R}_+^{n+1}} E''_{\beta,\alpha}(1+\|\eta\|^2)^\alpha e^{rM(\eta)}\|x\|^\alpha e^{-lM(x)}\,d\mu_\beta(x) \\
&\quad\leq E'''_{\beta,p}(1+\|\eta\|^2)^p e^{rM(\eta)}\int_{\mathbb{R}_+^{n+1}}\|x\|^p e^{-lM(x)}\,d\mu_\beta(x).
\end{aligned}$$

Therefore,

$$|a_\eta(\xi)| \leq E'''_{\beta,p}(1+\|\eta\|^2)^p e^{rM(\eta)}(1+\|\xi\|^2)^{-p}.$$

Theorem 5.3.4. *Let $a(x,\xi)$ be any symbol in $S^{r,l}_{(M)}$, then the pseudo-differential operator $A(x,D)$ admits the representation:*

$$(A(x,D)u)(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi\rangle}J_\beta(x_{n+1}\xi_{n+1})\Big(\int_{\mathbb{R}_+^{n+1}} a_\eta(\xi)(\mathcal{F}_w u)(\eta)\,d\mu_\beta(\eta)\Big)d\mu_\beta(\xi) \tag{5.3.6}$$

$\forall\ u \in S(\mathbb{N}_k,\mathbb{M}_k;\mathbb{R}_+^{n+1})$.

Proof: From (5.3.3), we have

$$\begin{aligned}
a_\eta(\xi) &= \int_{\mathbb{R}_+^{n+1}} e^{-i\langle x',\xi'\rangle}J_\beta(x_{n+1}\xi_{n+1})[e^{i\langle x',\eta'\rangle}J_\beta(x_{n+1}\eta_{n+1})a(x,\eta)]d\mu_\beta(x) \\
&= \mathcal{F}_w[e^{i\langle x',\eta'\rangle}J_\beta(x_{n+1}\eta_{n+1})a(x,\eta)](\xi).
\end{aligned}$$

By the inversion property of Weinstein transform, we have

$$\mathcal{F}_w^{-1}(a_\eta(\xi))(x) = e^{i\langle x',\eta'\rangle}J_\beta(x_{n+1}\eta_{n+1})a(x,\eta).$$

Therefore,

$$\int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}J_\beta(x_{n+1}\xi_{n+1})a_\eta(\xi)d\mu_\beta(\xi) = e^{i\langle x',\eta'\rangle}J_\beta(x_{n+1}\eta_{n+1})a(x,\eta). \tag{5.3.7}$$

Next, we find

$$|a_\eta(\xi)(F_w u)(\eta)| \leq |a_\eta(\xi)| \cdot |(F_w u)(\eta)|.$$

In view of (5.2.13) and (5.3.4), we obtain

$$\begin{aligned}|a_\eta(\xi)(F_w u)(\eta)| &\leq E'''_{\beta,p}(1+\|\eta\|^2)^p e^{rM(\eta)}(1+\|\xi\|^2)^{-p} D e^{-M(\frac{\eta}{A})} \\ &= E'''_{\beta,p} D (1+\|\xi\|^2)^{-p}(1+\|\eta\|^2)^p e^{rM(\eta)} e^{-M(\frac{\eta}{A})} \\ &= E''''_{\beta,p}(1+\|\xi\|^2)^{-p}(1+\|\eta\|^2)^p e^{rM(A.\frac{\eta}{A})} e^{-M(\frac{\eta}{A})}.\end{aligned}$$

For $A \geq 1$, from (5.2.3), we get

$$\begin{aligned}&|a_\eta(\xi)(F_w u)(\eta)| \\ &\quad \leq E''''_{\beta,p}(1+\|\xi\|^2)^{-p}(1+\|\eta\|^2)^p e^{\frac{3}{2}ArM(\frac{\eta}{A})} e^{-M(\frac{\eta}{A})} \\ &\quad = E''''_{\beta,p}(1+\|\xi\|^2)^{-p}(1+\|\eta\|^2)^p e^{-(1-\frac{3}{2}Ar)M(\frac{\eta}{A})} \in L^1(\mathbb{R}^{n+1}_+) \ for \ r < \frac{2}{3A}.\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^{n+1}_+} |a_\eta(\xi)(F_w u)(\eta)| d\mu_\beta(\eta) \leq E''''_{\beta,p}(1+\|\xi\|^2)^{-p} \int_{\mathbb{R}^{n+1}_+} (1+\|\eta\|^2)^p e^{-(1-\frac{3}{2}Ar)M(\frac{\eta}{A})} d\mu_\beta(\eta).$$

From above, the right hand side of the integral exists on $\mathbb{R}^{n+1}_+$. Therefore

$$f(\xi) = \int_{\mathbb{R}^{n+1}_+} a_\eta(\xi)(F_w u)(\eta)\, d\mu_\beta(\eta) \ \in L^1(\mathbb{R}^{n+1}_+).$$

Taking the inverse Weinstein transform we have

$$\begin{aligned}
&\int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})\big[\int_{\mathbb{R}_+^{n+1}} a_\eta(\xi)(F_w u)(\eta) d\mu_\beta(\eta)\big] d\mu_\beta(\xi)\\
&\quad = \int_{\mathbb{R}_+^{n+1}} (F_w u)(\eta)\big[\int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1}) a_\eta(\xi) d\mu_\beta(\xi)\big] d\mu_\beta(\eta)\\
&\quad = \int_{\mathbb{R}_+^{n+1}} (F_w u)(\eta)\big[e^{i\langle x',\eta'\rangle} J_\beta(x_{n+1}\eta_{n+1}) a(x,\eta)\big] d\mu_\beta(\eta)\\
&\quad = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\eta'\rangle} J_\beta(x_{n+1}\eta_{n+1}) a(x,\eta)(F_w u)(\eta) d\mu_\beta(\eta)\\
&\quad = (A(x,D)u)(x).
\end{aligned}$$

From above the pseudo-differential operator $A(x, D)$ admits the integral representation. Then we have

$$(A(x,D)u)(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi\rangle} J_\beta(x_{n+1}\xi_{n+1})\big(\int_{\mathbb{R}_+^{n+1}} a_\eta(\xi)(\mathcal{F}_w u)(\eta)\, d\mu_\beta(\eta)\big) d\mu_\beta(\xi).$$

Definition 5.3.5. For $s \in \mathbb{R}$, and $u \in S(N_k, M_k; \mathbb{R}_+^{n+1})$, then norm of u is defined by

$$\|u\|_M^{s,p} = \left(\int_{\mathbb{R}_+^{n+1}} |e^{sM(\xi)}(\mathcal{F}_w u)(\xi)|^p \, d\mu_\beta(\xi)\right)^{\frac{1}{p}}, \quad 1 \le p < \infty. \tag{5.3.8}$$

Theorem 5.3.6. *Let $a(x,\xi) \in S^{r,l}_{(M)}$ and let $A(x, D)$ be the associated pseudo-differential operator. Then for all $u \in S(N_k, M_k; \mathbb{R}_+^{n+1})$ we have*

$$\|A(x,D)u\|_M^{s,2} \le E_{\beta,p,s}\|u\|_M^{1,2}. \tag{5.3.9}$$

Proof: From (5.3.6), we have

$$(A(x,D)u)(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi\rangle} J_\beta(x_{n+1}\xi_{n+1})\left(\int_{\mathbb{R}_+^{n+1}} a_\eta(\xi)(\mathcal{F}_w u)(\eta)\, d\mu_\beta(\eta)\right) d\mu_\beta(\xi)$$

$$k = \mathcal{F}^{-1}\left(\int_{\mathbb{R}_+^{n+1}} (a_\eta(\xi)(\mathcal{F}_w u)(\eta)\, d\mu_\beta(\eta)\right)(x).$$

Taking the inverse Weinstein transform we get

$$\mathcal{F}\big(A(x,D)u\big)(\xi) = \int_{\mathbb{R}_+^{n+1}} (a_\eta(\xi)(\mathcal{F}_w u)(\eta)\, d\mu_\beta(\eta).$$

Then,

$$\begin{aligned}\big|\mathcal{F}\big(A(x,D)u\big)(\xi)\big| &= \Big| \int_{\mathbb{R}_+^{n+1}} (a_\eta(\xi)(\mathcal{F}_w u)(\eta)\, d\mu_\beta(\eta)\Big| \\ &\leq \int_{\mathbb{R}_+^{n+1}} |(a_\eta(\xi)| \cdot |(\mathcal{F}_w u)(\eta)|\, d\mu_\beta(\eta).\end{aligned}$$

From (5.3.4), we have

$$\begin{aligned}\big|\mathcal{F}\big(A(x,D)u\big)(\xi)\big| \leq & \int_{\mathbb{R}_+^{n+1}} E'''_{\beta,p}(1+\|\eta\|^2)^p e^{rM(\eta)}(1+\|\xi\|^2)^{-p}|(\mathcal{F}_w u)(\eta)|\, d\mu_\beta(\eta) \\ =& E'''_{\beta,p}(1+\|\xi\|^2)^{-p}\int_{\mathbb{R}_+^{n+1}} (1+\|\eta\|^2)^p e^{rM(\eta)}|(\mathcal{F}_w u)(\eta)|\, d\mu_\beta(\eta) \\ =& E'''_{\beta,p}(1+\|\xi\|^2)^{-p}\int_{\mathbb{R}_+^{n+1}} (1+\|\eta\|^2)^p e^{rM(\eta)} e^{-M(\frac{\eta}{A})} \\ & \times e^{M(\frac{\eta}{A})}|(\mathcal{F}_w u)(\eta)|\, d\mu_\beta(\eta).\end{aligned}$$

Using (5.2.3), we get

$$\begin{aligned}\big|\mathcal{F}\big(A(x,D)u\big)(\xi)\big| \leq & E'''_{\beta,p}(1+\|\xi\|^2)^{-p}\int_{\mathbb{R}_+^{n+1}} (1+\|\eta\|^2)^p e^{-(1-\frac{3}{2}Ar)M(\frac{\eta}{A})} \\ & \times |e^{M(\frac{\eta}{A})}(\mathcal{F}_w u)(\eta)|\, d\mu_\beta(\eta).\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|e^{sM(\xi)}\mathcal{F}\big(A(x,D)u\big)(\xi)\right| \\
&\quad \leq E'''_{\beta,p}e^{sM(\xi)}(1+\|\xi\|^2)^{-p}\int_{\mathbb{R}^{n+1}_+}(1+\|\eta\|^2)^p e^{-(1-\frac{3}{2}Ar)M(\frac{\eta}{A})} \\
&\quad \times |e^{M(\frac{\eta}{A})}(\mathcal{F}_w u)(\eta)|\,d\mu_\beta(\eta) \\
&\quad \leq E'''_{\beta,p}e^{sM(\xi)}(1+\|\xi\|^2)^{-p}\bigg(\int_{\mathbb{R}^{n+1}_+}\left|(1+\|\eta\|^2)^p e^{-(1-\frac{3}{2}Ar)M(\frac{\eta}{A})}\right|^2 d\mu_\beta\bigg)^{\frac{1}{2}} \\
&\quad \times \bigg(\int_{\mathbb{R}^{n+1}_+}|e^{M(\frac{\eta}{A})}(\mathcal{F}_w u)(\eta)|^2\,d\mu_\beta(\eta)\bigg)^{\frac{1}{2}} \\
&\quad \leq E'''_{\beta,p}e^{sM(\xi)}(1+\|\xi\|^2)^{-p}\left\|(1+\|\eta\|^2)^p e^{-(1-\frac{3}{2}Ar)M(\frac{\eta}{A})}\right\|_{L^2(\mathbb{R}^{n+1}_+)} \\
&\quad \times \left\|e^{M(\frac{\eta}{A})}(\mathcal{F}_w u)(\eta)\right\|_{L^2(\mathbb{R}^{n+1}_+)}.
\end{aligned}$$

For $r < \frac{2}{3A}$, we get

$$|e^{sM(\xi)}\mathcal{F}\big(A(x,D)u\big)(\xi)| \leq E'''_{\beta,p}D_p e^{sM(\xi)}(1+\|\xi\|^2)^p\left\|e^{M(\frac{\eta}{A})}(\mathcal{F}_w u)(\eta)\right\|_{L^2(\mathbb{R}^{n+1}_+)}.$$

Now, from (5.3.8), we obtain

$$\begin{aligned}
&\left\|A(x,D)u\right\|_M^{s,2} \\
&\quad = \bigg(\int_{\mathbb{R}^{n+1}_+}|e^{sM(\xi)}\mathcal{F}\big(A(x,D)u\big)(\xi)|^2\,d\mu_\beta(\xi)\bigg)^{\frac{1}{2}} \\
&\quad \leq E'''_{\beta,p}\cdot D_p\bigg(\int_{\mathbb{R}^{n+1}_+}|e^{sM(\xi)}(1+\|\xi\|^2)^p|^2\,d\mu_\beta(\xi)\bigg)^{\frac{1}{2}}\left\|u\right\|_M^{1,2}.
\end{aligned}$$

For $s < 0$, we have

$$\left\|A(x,D)u\right\|_M^{s,2} \leq E'''_{\beta,p}\times D_p\times E_s\left\|u\right\|_M^{1,2}.$$

Which shows that

$$\left\|A(x,D)u\right\|_M^{s,2} \leq E_{\beta,p,s}\left\|u\right\|_M^{1,2}.$$

Next, we discuss symbol and properties with growth governed by the indicatrix ω.

Definition 5.3.7. The symbol $a(x,\xi)$ belongs to the class $S^{r,l}_{(\omega)}$ if and only if $a(x,\xi) \in C^\infty(\mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+)$ and for $r \in \mathbb{R},\ l > 0$ and $\forall \alpha, \beta \in \mathbb{N}^{n+1}$, there are positive constants L and $C_{\alpha,l,r}$ such that

$$|(\frac{\partial}{\partial \xi})^\alpha (\frac{\partial}{\partial x})^\beta a(x,\xi)| \leq C_{\alpha,l,r} L^{|\beta|}(|\beta|)!\ e^{[r\omega(\xi)-l\omega(x)]}. \tag{5.3.10}$$

Lemma 5.3.8. *Let the symbol* $a(x,\xi) \in S^{r,l}_{(\omega)}$. *Then for* $\alpha \in \mathbb{N}^{n+1}$ *and* $\omega \in M$ *we find the following estimate*

$$\left|(\Delta_{n,\beta})^\alpha_x [e^{i\langle x', \eta' \rangle} J_\beta(x_{n+1}\eta_{n+1}) a(x,\eta)]\right| \leq E''_{\beta,\alpha}(1+\|\eta\|^2)^\alpha e^{r\omega(\eta)} \|x\|^\alpha e^{-l\omega(x)}, \tag{5.3.11}$$

where $E''_{\beta,\alpha}$ *is a positive constant.*

Proof: From (1.4.15), we find that

$$\begin{aligned}
&(\Delta_{n,\beta})^\alpha_x [e^{i\langle x', \eta' \rangle} J_\beta(x_{n+1}\eta_{n+1}) a(x,\eta)] \\
&= \sum_{j=0}^{\alpha} \sum_{m=1}^{2j} \sum_{q=0}^{m} \sum_{|\rho'| \leq 2(\alpha-j)} \sum_{\delta_1,\delta_2,\ldots,\delta_n \geq 0} \binom{\alpha}{j} \binom{m}{q} \binom{\alpha-j}{\delta_1,\delta_2,....,\delta_n} \frac{1}{\rho'!} E'_{\beta,m} \\
&\times x_{n+1}^{m-\alpha} \left(D_x^{\rho'+q} e^{i\langle x', \eta' \rangle} J_\beta(x_{n+1}\eta_{n+1}) \right) \left(D_x^{\rho'+2\delta'+m-q} a(x,\eta) \right),
\end{aligned}$$

for $E'_{\beta,m}$ is a constant for $m \in \{0,1,......,\alpha\}$ which depends on β.

Therefore,

$$\begin{aligned}
&\left|(\Delta_{n,\beta})^\alpha_x [e^{i\langle x', \eta' \rangle} J_\beta(x_{n+1}\eta_{n+1}) a(x,\eta)]\right| \\
&= \Bigg| \sum_{j=0}^{\alpha} \sum_{m=1}^{2j} \sum_{q=0}^{m} \sum_{|\rho'| \leq 2(\alpha-j)} \sum_{\delta_1,\delta_2,\ldots,\delta_n \geq 0} \binom{\alpha}{j} \binom{m}{q} \binom{\alpha-j}{\delta_1,\delta_2,....,\delta_n} \frac{1}{\rho'!} E'_{\beta,m} \\
&\times x_{n+1}^{m-\alpha} \left(D_x^{\rho'+q} e^{i\langle x', \eta' \rangle} J_\beta(x_{n+1}\eta_{n+1}) \right) \left(D_x^{\rho'+2\delta'+m-q} a(x,\eta) \right) \Bigg|.
\end{aligned}$$

Using (1.3.30) and (5.3.10), above yeilds

$$
\begin{aligned}
&\left|(\Delta_{n,\beta})_x^{\alpha}[e^{i\langle x',\eta'\rangle}J_\beta(x_{n+1}\eta_{n+1})a(x,\eta)]\right|\\
&\leq \sum_{j=0}^{\alpha}\sum_{m=1}^{2j}\sum_{q=0}^{m}\sum_{|\rho'|\leq 2(\alpha-j)}\sum_{\delta_1,\delta_2,\ldots,\delta_n\geq 0}\binom{\alpha}{j}\binom{m}{q}\binom{\alpha-j}{\delta_1,\delta_2,....,\delta_n}\frac{1}{\rho'!}E'_{\beta,m}\\
&\times\left|x_{n+1}\right|^{m-\alpha}\|\eta\|^{|\rho'+q|}\left|C_{0,l,m}L^{|\rho'+2\delta'+m-q|}(|\rho'+2\delta'+m-q|)!\right|e^{r\omega(\eta)-l\omega(x)}\\
&\leq \sum_{j=0}^{\alpha}\sum_{m=1}^{2j}\sum_{q=0}^{m}\sum_{|\rho'|\leq 2(\alpha-j)}\binom{\alpha}{j}\binom{m}{q}\frac{1}{\rho'!}E'_{\beta,m}\left|C_{0,l,m}L^{|\rho'+m-q|}(|\rho'+m-q|)!\right|\\
&\times\left|x_{n+1}\right|^{m-\alpha}\|\eta\|^{|\rho'+q|}e^{r\omega(\eta)-l\omega(x)}\\
&\leq \sum_{j=0}^{\alpha}\sum_{m=1}^{2j}\sum_{q=0}^{m}\binom{\alpha}{j}\binom{m}{q}\frac{1}{2(\alpha-j)!}E'_{\beta,m}\left|C_{0,l,m}L^{|2(\alpha-j)+m-q|}\right.\\
&\times\left.(|2(\alpha-j)+m-q|)!\right|\left|x_{n+1}\right|^{m-\alpha}\|\eta\|^{|2(\alpha-j)+q|}e^{r\omega(\eta)-l\omega(x)}\\
&\leq \sum_{j=0}^{\alpha}\sum_{m=1}^{2j}\binom{\alpha}{j}\frac{1}{2(\alpha-j)!}E'_{\beta,m}\left|C_{0,l,m}L^{|2(\alpha-j)|}(|2(\alpha-j)|)!\right|\left|x_{n+1}\right|^{m-\alpha}\\
&\times\|\eta\|^{|2(\alpha-j)+m|}e^{r\omega(\eta)-l\omega(x)}\\
&\leq \sum_{j=0}^{\alpha}\binom{\alpha}{j}\frac{1}{2(\alpha-j)!}E'_{\beta,j}\left|C_{0,l,m}L^{|2(\alpha-j)|}(|2(\alpha-j)|)!\right|\left|x_{n+1}\right|^{2j-\alpha}\\
&\times\|\eta\|^{|2\alpha|}e^{r\omega(\eta)-l\omega(x)}\\
&\leq E'_{\beta,\alpha}\left|C_{0,l,\alpha}L^0\right|\left|x_{n+1}\right|^{\alpha}\|\eta\|^{|2\alpha}le^{r\omega(\eta)-l\omega(x)}\\
&\leq E''_{\beta,\alpha}\|x\|^{\alpha}(1+\|\eta\|^2)^{\alpha}e^{r\omega(\eta)-l\omega(x)},
\end{aligned}
$$

where $E''_{\beta,\alpha}=E'_{\beta,\alpha}\left|C_{0,l,r}L^0\right|$, which is a positive constant.

Hence, above expression shows that

$$\left|(\Delta_{n,\beta})_x^{\alpha}[e^{i\langle x',\eta'\rangle}J_\beta(x_{n+1}\eta_{n+1})a(x,\eta)]\right|\leq E''_{\beta,\alpha}(1+\|\eta\|^2)^{\alpha}e^{r\omega(\eta)}\|x\|^{\alpha}e^{-l\omega(x)}.$$

Let $a(x,\xi) \in S^{r,l}_{(\omega)}$. Then we define the function $a_\eta(\xi)$ associated with the symbol $a(x,\xi)$ is

$$a_\eta(\xi) = \int_{\mathbb{R}^{n+1}_+} e^{-i\langle x',\xi'\rangle} J_\beta(x_{n+1}\xi_{n+1})[e^{i\langle x',\eta'\rangle} J_\beta(x_{n+1}\eta_{n+1}) a(x,\eta)] d\mu_\beta(x). \quad (5.3.12)$$

Theorem 5.3.9. *Let the symbol $a(x,\xi) \in S^{r,l}_{(\omega)}$. Then for a positive constant C' and multi-index α, the function $a_\eta(\xi)$ is defined by (5.3.12), satisfies the following estimate*

$$|a_\eta(\xi)| \leq E'''_{\beta,p}(1+\|\eta\|^2)^p e^{r\omega(\eta)}(1+\|\xi\|^2)^{-p}. \quad (5.3.13)$$

Proof: For, the proof of above theorem, we can refer the theorem 5.3.3.

Theorem 5.3.10. *Let $a(x,\xi) \in S^{r,\,l}_{(\omega)}$, then the pseudo-differential operator $A(x,D)$ satisfies*

$$\|A(x,D)u\|^{s,2}_\omega \leq E_{\beta,p,s}\|u\|^{1,2}_\omega, \quad \forall\ u \in S_\omega(\mathbb{R}^{n+1}_+). \quad (5.3.14)$$

Proof: Using Theorem 5.3.6, we can proof the Theorem 5.3.10.

***