Approximation Methods for Fractional Variational and Sturm-Liouville Problems

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DOCTOR OF PHILOSOPHY

by

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It is certified that the work contained in this thesis titled "***Approximation Methods for Fractional Variational and Sturm-Liouville Problems***" by ***Divyansh Pandey*** has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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I, ***Divyansh Pandey***, certify that the work embodied in this thesis is my own bonafide work and carried out by me under the supervision of ***Dr. Rajesh K. Pandey*** from ***January, 2020*** to ***April, 2024*** at the ***Department of Mathematical Sciences, Indian Institute of Technology (Banaras Hindu University), Varanasi***. The matter embodied in this thesis has not been submitted for the award of any other degree/diploma. I declare that I have faithfully acknowledged and given credits to the research workers wherever their works have been cited in my work in this thesis. I further declare that I have not willfully copied any other's work, paragraphs, text, data, results, *etc.*, reported in journals, books, magazines, reports dissertations, theses, *etc.*, or available at websites and have not included them in this thesis and have not cited as my own work.

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It is certified that the above statement made by the student is correct to the best of my/our knowledge.

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Divyansh Pandey

Dedicated

to

My Nani Maa

Smt. Indravati Upadhyay

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Abbreviations

FVP(s)	**F**ractional **V**ariational **P**roblem(s)
ICFVP(s)	**I**soperimetric **C**onstraint **F**ractional **V**ariational **P**roblem(s)
GICFVP(s)	**G**eneralized **I**soperimetric **C**onstraint **F**ractional **V**ariational **P**roblem(s)
SLP(s)	**S**turm-**L**iouville **P**roblem(s)
FSLP(s)	**F**ractional **S**turm-**L**iouville **P**roblem(s)
FC	**F**ractional **C**alculus
FD(s)	**F**ractional **D**erivative(s)
FDE(s)	**F**ractional **D**ifferential **E**quation(s)
CV(s)	**C**alculus of **V**ariation(s)
FCV(s)	**F**ractional **C**alculus of **V**ariation(s)
CFD(s)	**C**aputo **F**ractional **D**erivative(s)
GFD(s)	**G**eneralized **F**ractional **D**erivative(s)
RLFD(s)	**R**iemann-**L**iouville **F**ractional **D**erivative(s)
R-L	**R**iemann-**L**iouville
BC(s)	**B**oundary **C**ondition(s)
EV(s)	**E**igenvalue(s)
EF(s)	**E**igenfunction(s)

Symbols

$\mathbb{R}$	Set of real numbers
$\mathbb{R}^+$	Set of positive real numbers
$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}^+$	Set of positive integers
α	Fractional order
Γ	Euler gamma function
$AC[c,d]$	Absolutely continuous on $[c,d]$
$C[c,d]$	Continuous on $[c,d]$
$C^n[c,d]$	n^{th} derivative continuous on $[c,d]$
$L^p(p \in [1,\infty])$	L^p space
$C_H^{\beta}[c,d]$	Class of Hölder continuous function with exponent β

PREFACE

The concept of calculus of variations (CVs) originated during 17th century while optimizing a functional which was later known as brachistochrone problem. These functionals are generally expressed as integrals involving one or more functions and their derivatives. This mathematical domain has gained significance due to its diverse applications in practical scenarios, arising from its ability to address problems of extremization, be it in the form of minimization or maximization.

Fractional variational calculus is a contemporary mathematical field focused on optimizing functionals that involve fractional operators, encompassing both integrals and derivatives. This area was pioneered by Riewe in 1996, where the traditional calculus of variations was extended by incorporating fractional derivatives [1, 2]. This extension enables the derivation of the conservation laws in presence of nonconservative forces, such as friction. Subsequent research has delved into various aspects of fractional calculus of variations exploring different fractional operators such as Riemann–Liouville, Caputo, Grünwald–Letnikov, Weyl, Marchaud, or Hadamard fractional derivatives [3, 4, 5, 6, 7].

Sturm-Liouville problems (SLPs) were developed in the 19th century by French mathematician Jacques Charles François Sturm and Joseph Liouville. SLPs are a certain type of second-order linear ordinary differential equation with specific boundary conditions (BCs). It has wide applications in mathematics, science, and engineering. Notably, SLPs have been prominently utilized in classical and quantum physics. In the past decade, there has been a notable extension of SLPs by incorporating fractional derivatives. This has sparked significant interest among researchers, leading to the generalization of theories of integer order SLPs to the fractional-order SLPs.

In this thesis, we develop approximation methods for solving the fractional variational problems (FVPs). Furthermore, using the variational approach, we study the main properties of FSLPs. Various forms of fractional integrals and derivatives have been established in the literature. However, this thesis explicitly investigates FVPs and FSLPs utilizing Caputo fractional derivatives (CFDs), Riemann-Liouville fractional derivatives, and the generalized fractional derivatives [8].

The initial focus of this thesis involves solving FVPs with several dependent variables expressed in the context of CFDs. Chapter 2 employs the Jacobi poly-fractonomial and adopts the Rayleigh-Ritz method to address this problem. Further, three numerical schemes are discussed in chapter 3 for solving the isoperimetric constraint fractional variational problem (ICFVP) defined in terms of generalized fractional derivative (GFD).

Chapters 4 and 5 are based on solving the FSLPs by using variational theory. The higher-order FSLP is considered in chapter 4. Variational and numerical approximation methods are used to study the properties of FSLP. Utilizing the variational approach, we demonstrate that FSLP exhibits an infinite increasing sequence of eigenvalues. Additionally, for each eigenvalue, there exists a distinct eigenfunction, and the collection of these eigenfunctions constitutes an orthogonal set. The numerical results corresponding to the problem are also presented, aligning well with the theoretical findings. Chapter 5 deals with the higher dimensional FSLPs. Here, we also prove the existence of infinite eigenvalues for N-dimensional FSLP as presented in chapter 4.