Chapter 1

Introduction

In this chapter, we present the foundational theories, introductory information, symbols, and theory utilized in this thesis. We also explain the motivation behind the present study and give a detailed overview of all the chapters covered in this thesis.

1.1 Brief Literature Survey

Approximation theory is utilized in both theoretical and Applied mathematics, encompassing a broad range of topics from theoretical problems in real, complex, and functional analysis to practical applications in engineering. The primary goal of approximation theory, particularly focusing on the positive linear operators, is to estimate difficult-looking functions by using simpler, smoother functions.

The well-known theorem on the uniform convergence of the operator, originally established by Weierstrass [112], sets the required level of precision within a limited interval. The theorem asserts that any continuous function Ψ that takes real values on a closed interval $[a, b]$ can be uniformly approximated by polynomials. Mathematically, for $\epsilon > 0$, there exists a polynomial $\mathcal{P}(x)$ satisfying

$$|\Psi(x) - \mathcal{P}(x)| < \epsilon, \quad \forall x \in [a, b].$$

This theorem has been further proven by several prominent mathematicians such as Runge, Lebesgue, Landau, Fejér, and Jackson.

In 1912, in order to give a constructive proof of the above theorem, S. Bernstein [31] constructed an operator that generates a sequence of polynomials using the desired function, converging uniformly to that function. For $\Psi \in \mathfrak{C}[0,1]$, the Bernstein polynomials are defined by

$$\mathfrak{B}_m(\Psi;x) = \sum_{j=0}^{m} \binom{m}{j} x^j (1-x)^{m-j} \Psi\left(\frac{j}{m}\right). \tag{1.1}$$

We note that if S is the space of linear functionals, then a mapping $T : S \to S$ is said to be a positive linear operator if it satisfies the following conditions:

- $T(a\Psi + b\Phi) = aT(\Psi) + bT(\Phi)$
- $T(\Psi) \geq 0, \quad \forall \Psi \geq 0,$

where $\Psi, \Phi \in S$ and a, b are the elements of corresponding field. One can easily observe that the operator (1.1) is a positive linear operator.

The Bernstein operators (1.1) have been extensively used in multiple fields of mathematics and computer science by several researchers. The valuable structure and significance of Bernstein operators have led to the exploration of numerous generalizations and modifications.

The summation-type operators are not effective for approximating a function $\Psi \in L_p[0,1]$. This issue is then addressed using integral-type operators such as Kantorovich and Durrmeyer operators. For $\Psi \in L_p[0,1]$, the Kantorovich operators [70] are defined by:

$$\mathfrak{K}_m(\Psi;x) = (m+1)\sum_{j=0}^{m}\binom{m}{j}x^j(1-x)^{m-j}\int\limits_{\frac{j}{m+1}}^{\frac{j+1}{m+1}}\Psi(y)dy \tag{1.2}$$

and the Bernstein-Durrmeyer operator [55] is defined by:

$$\mathfrak{D}_m(\Psi;x) = (m+1)\sum_{j=0}^{m}\binom{m}{j}x^j(1-x)^{m-j}\int\limits_{0}^{1}\binom{m}{j}y^j(1-y)^{m-j}\Psi(y)dy. \tag{1.3}$$

An extension of the Bernstein operators to unbounded intervals, mainly to the interval $[0,\infty)$ was introduced by S. Mirakyan [85] (1941) and O. Szász [108] (1950). For $\Psi \in \mathfrak{C}[0,\infty)$, the Szász-Mirakyan operators are defined by:

$$S_m(\Psi;x) = \sum_{j=0}^{\infty} e^{-mx}\frac{(mx)^j}{j!}\Psi\left(\frac{j}{m}\right), \tag{1.4}$$

with the assumption that the right-hand side converges. The proof for these sequence of operators converging to the desired function became easy after the "Bohman-Korovkin theorem", which was introduced independently by Bohman and Korovkin. The theorem states that if a sequence of positive linear operators in $[a,b]$ preserves constant, linear, and quadratic polynomials, at least in limiting sense, then the sequence will converge to any continuous function in $[a,b]$. Later, this Korovkin-type result was extended for continuous functions defined on infinite intervals by Boyanov and Veselinov [36]. In 1976, A.D. Gadjiev [57] proved the Korovkin-type theorem for any interval using a weighted norm.

With the motivation of having better approximation, a different sequence of positive linear operators called Lototsky-Bernstein operators, which preserves the constant and quadratic polynomials was introduced by J.P. King in [73]. Various researchers have made contributions in this area by following the King's method. Using Bernstein polynomials, Cheng [45] studied the convergence results for functions with bounded variation. Bojanic and Vuilleumier [34, 35] investigated the approximation properties of Bernstein operators for functions with bounded variation using a probabilistic approach.

In 1972, Jain [67] introduced a new sequence of positive linear operators, which generalizes the Szász-Mirakyan operators. These operators were very popular and have been studied and generalized extensively (see [1, 16, 93, 95]). Later, in 2018, Başcanbaz-Tunca et al. [29] introduced a different generalization of the Jain operators using the Pochhammer symbol.

Recently, Bézier curves have been significantly utilized in various fields, such as computer-aided graphics and applied mathematics. In [114], Zeng and Piriou presented a Bézier variant of the Bernstein operators. Later on, a number of researchers identified different versions of Bézier operators and examined their convergence characteristics (see [18, 61, 62, 105, 106]). In 2010, Ye et al. [113] defined new Bézier bases with a shape parameter λ. Using these bases, Cai et al. [40], introduced the λ-Bernstein operators, from which the traditional Bernstein operators can be obtained by putting $\lambda = 0$. They studied different approximation properties and showed the modeling flexibility obtained due to the shape parameter λ using some examples. In 2017, Chen et al. [44] proposed another generalization of the Bernstein polynomials using a new shape parameter α. Many studies have been done recently regarding the modifications of these operators.

In 1932, Elena Voronovskaja, a doctoral student of S. Bernstein, established an asymptotic error term of the Bernstein operators. He proved that if $\Psi \in \mathfrak{C}[0,1]$ with Ψ being differentiable in some neighborhood of x and has second derivative Ψ'' for some $x \in [0,1]$, then

$$\lim_{m\to\infty} m[\mathfrak{B}_m(\Psi;x) - \Psi(x)] = \frac{x(1-x)}{2}\Psi''(x),$$

i.e. the rate of convergence of the Bernstein operators is the order of $\frac{1}{m}$. In [99], D. Popa proposed an intermediate Voronovskaja-type theorem for general positive linear operators. Also, in [100], he introduced the Lototsky-Kantorovich operators and studied a generalized Korovkin-type theorem as well as Voronovskaja-type results for those operators in general probability measure spaces.

In recent years, there has been significant research interest in approximation theory focusing on linear positive operators operating on a complex domain. This research area was pioneered by Bernstein in [32], where he introduced the study of complex Bernstein polynomials $\mathfrak{B}_m(\Psi;z)$ defined by

$$\mathfrak{B}_m(\Psi;z) = \sum_{j=0}^{m} \binom{m}{j} z^j (1-z)^{m-j} \Psi\left(\frac{j}{m}\right).$$

He also showed that, if $\Psi : U \to \mathbb{C}$ is an analytic function where U being an open set containing the set $\{z \in \mathbb{C} : |z| \leq 1\}$, then the complex Bernstein polynomials converge to Ψ in $\{z \in \mathbb{C} : |z| \leq 1\}$. However, the findings presented in that work lack quantitative estimates. Later, S. Gal [58] has contributed to this discourse by unveiling certain approximation properties with quantitative estimates of complex Bernstein and convolution-type operators within the complex domain. Further, Anastassiou and Gal [22] discussed approximation properties of complex perturbed Bernstein-type operators. In 2019, Çetin [42] discussed the geometric properties of the complex variant of the α-Bernstein operators. In recent years, the approximation and overconvergence properties of many complex operators have been studied intensively by several researchers, which can be seen in [14, 60, 63, 81, 82].

Another important concern about the study of approximation theory is its application.— Among all applications, we mainly focus on solving Integral equations using some Bernstein-type operators. Integral equations are significant both in theory and in practical applications across various fields, underscoring their importance as a focal point in the study of applied mathematics and theoretical physics. Hence, there is a growing interest nowadays in these equations. Various methods are available to solve these equations both directly as well as numerically. The Collocation method, Quadrature method, Finite element method, etc. are known methods to solve these kinds of equations numerically. We aim to solve the two important types of linear integral equations, namely, Volterra and Fredholm integral equations using the Bernstein approximation technique with the help of the λ-Bernstein operators.

1.2 Important Definitions and Notations

1.2.1 Modulus of continuity and Peetre's K-functionals

In 1910, H. Lebesgue introduced rth order modulus of continuity as follows

Definition 1.1. (rth order Modulus of Continuity)

Let $X \subset \mathbb{R}$ and $\Psi \in \mathfrak{U}(X)$. Then the modulus of continuity of rth order is defined by

$$\Omega_r(\Psi;\delta) = \sup_{0<|k|\leq\delta} \{|\Delta_k^r \Psi(x)| : x, x+rk \in X\}, \quad \delta > 0, \tag{1.5}$$

where $\Delta_k^r\Psi(x)$ is the rth forward difference with step length k.

For $r = 1$, we get the 1st order modulus of continuity:

$$\Omega_1(\Psi;\delta) = \sup_{0<|k|\leq\delta} \sup_{x,x+k\in X} |\Psi(x+k) - \Psi(x)|, \tag{1.6}$$

and for $r = 2$, we get the 2nd order modulus of continuity:

$$\Omega_2(\Psi;\delta) = \sup_{0<|k|\le\delta} \sup_{x,x+2k\in X} |\Psi(x+2k) - 2\Psi(x+k) + \Psi(x)|. \tag{1.7}$$

Proposition 1.2. *Let $X = [a,b]$ and $\Psi \in \mathfrak{C}(X)$. Then the 1st order modulus of continuity enjoys the following properties:*

- $\Omega_1(\Psi;.)$ *is non-negative, non-decreasing and uniformly continuous on* $(0,\infty)$.
- $\Omega_1(\Psi;\delta) \to 0$ *whenever* $\delta^+ \to 0$.
- *The relation* $|\Psi(x) - \Psi(t)| \le \Omega_1(\Psi;\delta)\left(\frac{|x-t|}{\delta} + 1\right)$ *holds for* $x, t \in [a,b]$.
- *For* $p \in [0,\infty)$, $\Omega_1(\Psi;p\delta) \le (1+p)\Omega_1(\Psi;\delta)$, *where* $\delta > 0$.

Definition 1.3. (Peetre's K-functional)

For $\Psi \in \mathfrak{U}(X)$ and $\delta > 0$, we define the Peetre's K-functional as

$$K_2(\Psi;\delta) = \inf_{\Phi\in\mathfrak{C}_B^2(X)} \left\{\|\Psi - \Phi\| + \delta\|\Phi''\|\right\}, \tag{1.8}$$

where $\mathfrak{C}_B^2(X) = \{\Phi \in \mathfrak{C}_B(X) : \Phi', \Phi'' \in \mathfrak{C}_B(X)\}$.

It is known from [50] that there is a connection between the 2nd order modulus of continuity and Peetre's K-functional, i,e. there exists a positive constant $\mathcal{M}$ such that the following relation holds:

$$K_2(\Psi;\delta) \le \mathcal{M}\Omega_2(\Psi;\sqrt{\delta}). \tag{1.9}$$

Remark 1.4. It is easy to observe that, for $X = [0,1]$, the spaces $\mathfrak{C}(X)$, $\mathfrak{C}_B(X)$ and $\mathfrak{U}(X)$ are the same.

1.2.2 Ditzian-Totik modulus of smoothness

The following definitions are given in [54].

Definition 1.5. (1st order modulus of smoothness)

Let X be a set and $\xi : X \to \mathbb{R}$ be an admissible weight function. Then for $\Psi \in \mathfrak{C}_B(X)$, the 1st order Ditzian-Totik modulus of smoothness is defined as

$$\Omega_1^{\xi}(\Psi;\delta) = \sup_{0<h\leq\delta} \sup_{x+h\xi(x)\in X} \left|\Psi\left(x+h\xi(x)\right) - \Psi\left(x\right)\right|. \tag{1.10}$$

The associated K-functional is given by

$$K_{1,\xi}(\Psi,\delta) = \inf_{\Phi\in\mathcal{W}_{\xi}^{1}(X)} \big\{\|\Psi-\Phi\| + \delta\|\xi\Phi'\| : \delta>0\big\}, \tag{1.11}$$

where $\mathcal{W}_{\xi}^{1}(X) = \{\Phi : \Phi \in \mathcal{AC}_l(X), \|\xi\Phi'\| < \infty\}$.

Definition 1.6. (2nd order modulus of smoothness)

The 2nd order Ditzian-Totik modulus of smoothness is defined as

$$\Omega_2^{\xi}(\Psi;\delta) = \sup_{0<h\leq\delta} \sup_{x\pm h\xi(x)\in X} \left|\Psi\left(x+h\xi(x)\right) - \Psi(x) + \Psi\left(x-h\xi(x)\right)\right| \tag{1.12}$$

The associated K-functional is given by

$$K_{2,\xi}(\Psi,\delta^2) = \inf_{\Phi\in\mathcal{W}_{\xi}^{2}(X)} \big\{\|\Psi-\Phi\| + \delta^2\|\xi\Phi''\| : \delta>0\big\}, \tag{1.13}$$

where $\mathcal{W}_{\xi}^{2}(X) = \{\Phi : \Phi' \in \mathcal{AC}_l(X), \|\xi\Phi''\| < \infty\}$.

It is shown in [54] that, the above moduli of smoothness are equivalent to their associated K-functionals, i,e. there exists positive constant $\mathcal{M}$ such that

$$\mathcal{M}^{-1}\Omega_1^{\xi}(\Psi;\delta) \leq K_{1,\xi}(\Psi,\delta) \leq \mathcal{M}\Omega_1^{\xi}(\Psi;\delta), \tag{1.14}$$

and

$$\mathcal{M}^{-1}\Omega_2^{\xi}(\Psi;\delta) \leq K_{2,\xi}(\Psi,\delta) \leq \mathcal{M}\Omega_2^{\xi}(\Psi;\delta). \tag{1.15}$$

1.2.3 Weighted spaces and a Weighed modulus of continuity

One can easily observe that the sup norm is not appropriate for assessing the accuracy of approximations for unbounded functions on the non-compact intervals. As a result, the concept of weighted space has been introduced to study how operators approximate functions on the noncompact intervals.

First of all, for the weight function $\sigma(x) = 1 + x^2$, we consider the space

$$B_\sigma(X) = \{\Psi : X \to \mathbb{R} \ : \ |\Psi(x)| \leq M_\Psi \sigma(x)\},$$

where the constant M_Ψ depends on Ψ only. The space is equipped with the weighed norm defined as $\|\Psi\|_\sigma = \sup\limits_{x\in X}\left|\frac{\psi(t)}{\sigma(t)}\right|$.

Also, we consider the following vector subspaces of the above space.

$$\begin{aligned}
\mathfrak{C}_\sigma(X) &= \{\Psi \in B_\sigma(X) : \Psi \text{ is continuous}\}\\
\mathfrak{U}_\sigma(X) &= \{\Psi \in \mathfrak{C}_\sigma(X) : \Psi \text{ is uniformly continuous}\}\\
\tilde{\mathfrak{C}}_\sigma(X) &= \{\Psi \in \mathfrak{C}_\sigma(X) : \lim_{x\to\infty}\frac{\Psi(x)}{\sigma(x)} = k_\Psi\}\\
\tilde{\mathfrak{U}}_\sigma(X) &= \{\Psi \in \mathfrak{U}_\sigma(X) : \lim_{x\to\infty}\frac{\Psi(x)}{\sigma(x)} = m_\Psi\},
\end{aligned}$$

where the constants k_Ψ and m_Ψ depend on Ψ.

The above spaces agree on the following inclusion:

$$\tilde{\mathfrak{U}}_\sigma(X) \subset \mathfrak{U}_\sigma(X) \subset \mathfrak{C}_\sigma(X) \subset \mathfrak{B}_\sigma(X). \tag{1.16}$$

In 2008, Holhoş [66] represented a paper that includes some quantitative estimates in weighted spaces by proposing a new modulus of continuity in a weighted space.

Let $X = [0,\infty)$ and $\zeta : X \to X$ be a unbounded strictly increasing continuous function such that $\exists \mathcal{M} > 0$ and $\alpha \in (0,1]$ with the property

$$|x-y| \leq \mathcal{M}|\zeta(x) - \zeta(y)|^\alpha.$$

Let us consider the function ζ to be the identity function on X. Clearly, $\sigma(x) = 1 + (\zeta(x))^2$.

Then, for $\Psi \in \mathfrak{C}_\sigma(X)$ and $x \in X$, a weighted modulus of continuity defined in [66] as

$$\omega_\zeta(\Psi,\delta) = \sup_{\substack{x,t\in[0,\infty) \\ |\zeta(t)-\zeta(x)|\leq\delta}} \frac{|\Psi(t)-\Psi(x)|}{\sigma(t)+\sigma(x)}, \quad \forall\delta \geq 0. \tag{1.17}$$

This weighed modulus of continuity also has the following properties:

- $\omega_\zeta(\Psi,\delta)$ is a non-negative and increasing function with respect to δ.
- $\lim\limits_{\delta\to 0} \omega_\zeta(\Psi,\delta) = 0$, for all $\Psi \in \mathfrak{C}_\sigma(X)$.

1.2.4 Definitions for functions of bounded variation

Let $BV'[a,b]$ be the collection of all continuous functions Ψ on $[a,b]$ such that its derivative Ψ' is equivalent to a function of bounded variation on $[a,b]$.

It is evident that any $\Psi \in BV'[a,b]$ can be expressed as

$$\Psi(x) = \int\limits_a^x \Phi(t)dt + \Psi(a), \tag{1.18}$$

where $\Phi \in BV[a,b]$.

The total variation $\mathfrak{T}_a^b(\Psi)$, of the function Ψ on the interval $[a,b]$ is defined by

$$\mathfrak{T}_a^b(\Psi) = \sup_{P \in \mathbb{P}} \left(\sum_{j=1}^{m} |\Psi(x_j) - \Psi(x_{j-1})| \right), \tag{1.19}$$

where $\mathbb{P}$ is the set of all partitions $P = \{a = x_0, x_1, ..., x_m = b\}$ of $[a,b]$. If $\mathfrak{T}_a^b(\Psi) < \infty$, then we say Ψ is a function of bounded variation, i,e. $\Psi \in BV[a,b]$. Also, for $c \in [a,b]$, it possesses the following characteristics:

$$\mathfrak{T}_a^b(\Psi) = \mathfrak{T}_a^c(\Psi) + \mathfrak{T}_c^b(\Psi).$$

1.2.5 Lipschitz class of functions

In [94, 96], the definitions of Lipschitz-type space is given as follows:

$$Lip_{M,\eta}^{*}(X) = \left\{ \Psi \in \mathfrak{C}_B(X) : |\Psi(t) - \Psi(x)| \leq M \frac{|t-x|^{\eta}}{(t+x)^{\eta/2}} : x,t \in X - \{0\} \right\}. \tag{1.20}$$

Also, for fixed $a, b > 0$, the Lipschitz-type space of two parameters is defined to be

$$Lip_{M,\eta}^{a,b}(X) = \left\{ \Psi \in \mathfrak{C}_B(X) : |\Psi(t) - \Psi(x)| \leq M \frac{|t-x|^{\eta}}{(t+ax^2+bx)^{\eta/2}} : x,t \in X - \{0\} \right\}. \tag{1.21}$$

Here $\eta \in (0,1]$, M is a positive real number, and the set X must be an interval containing non-negative real numbers.

1.2.6 Other important definitions

Definition 1.7. (Pochhammer symbol)

The Pochhammer symbol is denoted by $(a)_m, a \neq 0$ and is defined as

$$(a)_m = \begin{cases} a(a+1)...(a+m-1), & m \geq 1 \\ 1, & m = 0, \end{cases}$$

from which we also have

$$(a)_{-m} = \frac{1}{(a-1)(a-2)(a-3)\cdots(a-m)} = \frac{1}{(a-m)_m} = \frac{(-1)^m}{(1-a)_m}$$

when $m \in \mathbb{N}$ and $a \neq 1, 2, 3, \cdots, m$ (see, e.g., p.5 of [59]).

Definition 1.8. (Big-Θ notation)

For $m \in \mathbb{N}$, a sequence $\Psi(m)$ is said to be $\Theta(\Phi(m))$, if there exists constants $0 < K_1, K_2 < \infty$ and $m_0 \in \mathbb{N}$ such that $K_1\Phi(m) < \Psi(m) < K_2\Phi(m)$, $\forall m > m_0$.

Definition 1.9. (Big-O notation)

For $m \in \mathbb{N}$, a sequence $\Psi(m)$ is said to be $O(\Phi(m))$, if there exists a constant $0 < K < \infty$ and $m_0 \in \mathbb{N}$ such that $|\Psi(m)| \leq K\Phi(m)$, $\forall m > m_0$.

Definition 1.10. (Volterra Integral Equation)

A linear integral equation of the form

$$f(x)\Psi(x) = \Phi(x) + \mu \int_0^x \kappa(x,t)\Psi(t)dt$$

is called Volterra integral equation, where $f(x)$, $\Phi(x)$ and $\kappa(x,t)$ are known functions, μ is a non-zero real or complex parameter. Here, $x, t \in [0,1]$ and $\Psi(x)$ is the unknown function.

For $f(x) = 0$, the above equation is called Volterra integral equation of 1st kind and for $f(x) = 1$, the above equation is called Volterra integral equation of 2nd kind.

Definition 1.11. (Fredholm Integral Equation)

A linear integral equation of the form

$$f(x)\Psi(x) = \Phi(x) + \mu \int\limits_{0}^{1} \kappa(x,t)\Psi(t)dt$$

is called Fredholm integral equation, where $f(x)$, $\Phi(x)$ and $\kappa(x,t)$ are known functions, μ is a non-zero real or complex parameter. Here, $x, t \in [0,1]$ and $\Psi(x)$ is the unknown function.

Similar to the previous definition, for $f(x) = 0$, the above equation is called Fredholm integral equation of 1st kind and for $f(x) = 1$, the above equation is called Fredholm integral equation of 2nd kind.

1.3 Motivation and Objective

There has been a growing research interest in the study of positive linear operators as well as complex type operators.

Positive linear operators are significant because they maintain positivity, a valuable characteristic in various practical fields such as probability theory, statistical mechanics, and economics. These operators frequently appear in constructive approximation, where the aim is to create sequences of approximations that converge to a specific function. In contrast, complex-type operators extend real-valued operators into the complex plane, providing a wider range of analysis and applications, particularly in the areas like signal processing and quantum mechanics.

The objective of present research work in this domain is to enhance the theoretical understanding of these operators, investigate their properties, and create new methods for function approximation. By broadening the theory surrounding positive linear operators and complex-type operators, we can refine current approximation techniques, uncover new ones, and leverage these improvements to address real-world challenges.

Another objective of studying positive linear operators is to solve integral equations. Though in the existing literature the solution of Integral equations using the Bernstein technique is there, a natural question arises whether applying the λ-Bernstein operators provides some better approximation at least for some values of λ. In Chapter 6, we have solved the Volterra Integral Equations using the λ- Bernstein operators, which have better modeling flexibility than the usual Bernstein operators and shown that for some test examples, different values of λ provide less error of approximation than the usual Bernstein approximation technique.

1.4 Organisation of this Thesis

This thesis consists of seven chapters; the organization of these chapters is as follows.

Chapter 1 is reserved for the introduction and basic definitions of this thesis that are being used throughout this thesis. It consists of the literature survey and recent developments in the field of Approximation theory. It also mentions the contributions made in this thesis.

Chapter 2 presents some study of positive linear operators of Bernstein type i,e. for finite interval. It has two sub-sections; the first one is related to the study of the convergence properties of a modified Kantoravich operator, which uniformly approximates integrable functions in the interval $(0,1)$. Some examples are added to validate the convergence. The second one is dedicated to the study of a blending type Bernstein-Beta operator which generates several operators as a special case. Its convergence properties for some special classes of functions have also been explored.

Chapter 3 proposes the study of some positive linear operators in the unbounded interval $[0,\infty)$. It also has two subchapters: the first one is about the convergence properties of a Szasz-Kantorovich type operator associated with d-symmetric d-orthogonal polynomials of Brenke type. In the second subsection, we introduce a Stancu-type integral generalization of a modified Jain operator and examined its approximation properties. Some quantitative upper estimation using a weighted modulus of continuity is also provided along with some illustrative examples.

Chapter 4 consists of a generalized Kantorovich-type modification of the α-Bernstein operators. A generalized Korovkin-type theorem, as well as an intermediate Voronovskaja-type theorem, is studied in general probability measure space. In addition, an error estimation for the functions having derivative of bounded variation has been established.

Chapter 5 is devoted to the study of a complex variant of the α-Bernstein-Durrmeyer operators and their convergence to analytic functions. The quantitative upper estimation and Voronovskaja type result have also been established to validate the natural extension of the α-Bernstein-Durrmeyer operators from the real domain to compact discs in the complex argand plane. Along with the analytic functions, a note on the order of convergence of their higher-order derivatives is also highlighted.

Chapter 6 is dedicated to an important application of the positive linear operators. The Volterra and Fredholm integral equations of 1st and 2nd kind have been solved using the λ-Bernstein operators. The convergence is confirmed by some quantitative error estimation. the relevance of the shape parameter λ is shown by some numerical examples, where the approximation using the λ-Bernstein operators yields less error as compared to the use of usual Bernstein polynomials.

Chapter 7 wraps up the thesis and provides suggestions for potential future research directions.

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