Chapter 5

A Note on Stability and Fractal Dimension of Bivariate α-Fractal Functions

The aim of this chapter is to investigate the stability of bivariate α-fractal function with respect to the parameters, which is inspired by the results available in the thesis [90]. Furthermore, we also try to estimate the bounds for the dimension of its graph under certain conditions on the parameters.

This chapter is organized as follows. In the next section, we recall the framework of α-fractal functions given in [43]. Section 5.1 is devoted to present the continuous dependence of α-fractal function f^α on the parameters α, s and $\triangle$. In Section 5.2, we estimate the dimension of α-fractal function. In the last part, we focus on some properties of the associated fractal operator.

5.1 Continuous Dependence of IFS Parameters

Theorem 5.1.1. *For fixed* $f \in \mathcal{C}(\square)$*, net* $\triangle$ *and* $s \in \mathcal{C}(\square)$ *satisfying* $s(\boldsymbol{x}) = f(\boldsymbol{x})$, $\forall\ \boldsymbol{x} \in \partial\Delta$*, the map* $\mathcal{W} : S \to \mathcal{C}(\square)$ *defined by*

$$\mathcal{W}(\alpha) = f^{\alpha}_{\triangle,s},$$

is continuous, where $S = \{\alpha \in \mathcal{C}(\square) : \|\alpha\|_\infty \le q < 1$ *and* q *is a fixed number*$\}$.

Proof. For a fixed net $\triangle$, scale function α, and base function s, the map $f^{\alpha}_{\triangle,s}$ is unique, and being the fixed point of the Read-Bajraktarević operator (see Equation (1.11.1)) the α-fractal function $f^{\alpha}_{\triangle,s}$ satisfies the functional equation: for all $\boldsymbol{x} \in \square_{ij}$, where $(i,j) \in \Sigma$, we have

$$f^{\alpha}_{\Delta,s}(\boldsymbol{x}) = f(\boldsymbol{x}) + \alpha(\boldsymbol{x}) f^{\alpha}_{\Delta,s}(Q_{ij}(\boldsymbol{x})) - \alpha(\boldsymbol{x}) s(Q_{ij}(\boldsymbol{x})). \tag{5.1.1}$$

It is obvious that S is well defined. Let $\alpha, \beta \in S$ then from the above functional equation, we have

$$\mathcal{W}(\alpha)(\boldsymbol{x}) = f(\boldsymbol{x}) + \alpha(\boldsymbol{x}) f^{\alpha}_{\Delta,s}(Q_{ij}(\boldsymbol{x})) - \alpha(\boldsymbol{x}) s(Q_{ij}(\boldsymbol{x})),$$

and

$$\mathcal{W}(\beta)(\boldsymbol{x}) = f(\boldsymbol{x}) + \beta(\boldsymbol{x}) f^{\beta}_{\Delta,s}(Q_{ij}(\boldsymbol{x})) - \beta(\boldsymbol{x}) s(Q_{ij}(\boldsymbol{x})).$$

We shall show that $\mathcal{W}$ is continuous at α. For this, subtract one from other of the above two equations, for $\boldsymbol{x} \in \square_{ij}$, where $(i,j) \in \Sigma$, we have

$$\begin{aligned}
&\mathcal{W}(\alpha)(\boldsymbol{x}) - \mathcal{W}(\beta)(\boldsymbol{x})\\
&\quad= \Big(\alpha(\boldsymbol{x}) f^{\alpha}_{\triangle,s}(Q_{ij}(\boldsymbol{x})) - \alpha(\boldsymbol{x}) s(Q_{ij}(\boldsymbol{x}))\Big)\\
&\qquad - \Big(\beta(\boldsymbol{x}) f^{\beta}_{\triangle,s}(Q_{ij}(\boldsymbol{x})) - \beta(\boldsymbol{x}) s(Q_{ij}(\boldsymbol{x}))\Big)\\
&\quad= \alpha(\boldsymbol{x}) f^{\alpha}_{\triangle,s}(Q_{ij}(\boldsymbol{x})) - \beta(\boldsymbol{x}) f^{\beta}_{\triangle,s}(Q_{ij}(\boldsymbol{x}))\\
&\qquad + (\beta-\alpha)(\boldsymbol{x}) s(Q_{ij}(\boldsymbol{x}))\\
&\quad= \alpha(\boldsymbol{x}) f^{\alpha}_{\triangle,s}(Q_{ij}(\boldsymbol{x})) - \beta(\boldsymbol{x}) f^{\alpha}_{\triangle,s}(Q_{ij}(\boldsymbol{x}))\\
&\qquad + \beta(\boldsymbol{x}) f^{\alpha}_{\triangle,s}(Q_{ij}(\boldsymbol{x})) - \beta(\boldsymbol{x}) f^{\beta}_{\triangle,s}(Q_{ij}(\boldsymbol{x}))\\
&\qquad + (\beta-\alpha)(\boldsymbol{x}) s(Q_{ij}(\boldsymbol{x})).
\end{aligned}$$

Now using triangle inequality and definition of uniform norm, we have

$$\begin{aligned}
\Big|\mathcal{W}(\alpha)(\boldsymbol{x}) - \mathcal{W}(\beta)(\boldsymbol{x})\Big| &\le \Big|(\alpha-\beta)(\boldsymbol{x}) f^{\alpha}_{\triangle,s}(Q_{ij}(\boldsymbol{x}))\\
&\qquad + \beta(\boldsymbol{x})(f^{\alpha}_{\triangle,s} - f^{\beta}_{\triangle,s})(Q_{ij}(\boldsymbol{x}))\Big|\\
&\qquad + \Big|(\beta-\alpha)(\boldsymbol{x}) s(Q_{ij}(\boldsymbol{x}))\Big|\\
&\le \|\alpha-\beta\|_\infty \|f^{\alpha}_{\triangle,s}\|_\infty + \|\beta\|_\infty \|f^{\alpha}_{\triangle,s} - f^{\beta}_{\triangle,s}\|_\infty\\
&\qquad + \|\beta-\alpha\|_\infty \|s\|_\infty\\
&= \|\alpha-\beta\|_\infty \big(\|f^{\alpha}_{\triangle,s}\|_\infty + \|s\|_\infty\big) + \|\beta\|_\infty \|f^{\alpha}_{\triangle,s} - f^{\beta}_{\triangle,s}\|_\infty.
\end{aligned}$$

It follows that for all $\boldsymbol{x} \in \square$, we get

$$|\mathcal{W}(\alpha)(\boldsymbol{x}) - \mathcal{W}(\beta)(\boldsymbol{x})| \le \|\alpha-\beta\|_\infty \big(\|f^{\alpha}_{\triangle,s}\|_\infty + \|s\|_\infty\big) + \|\beta\|_\infty \|f^{\alpha}_{\triangle,s} - f^{\beta}_{\triangle,s}\|_\infty.$$

The above implies that

$$\|\mathcal{W}(\alpha) - \mathcal{W}(\beta)\|_\infty \le \|\alpha-\beta\|_\infty \big(\|f^{\alpha}_{\triangle,s}\|_\infty + \|s\|_\infty\big) + \|\beta\|_\infty \|\mathcal{W}(\alpha) - \mathcal{W}(\beta)\|_\infty.$$

Using $1 - \|\beta\|_\infty \ge 1 - q$, we have

$$\|\mathcal{W}(\alpha)-\mathcal{W}(\beta)\|_\infty \leq \frac{\|\alpha-\beta\|_\infty}{1-q}\big(\|f^\alpha_{\triangle,s}\|_\infty+\|s\|_\infty\big).$$

Since α is fixed and $\|f^\alpha_{\triangle,s}\|_\infty$ is bounded, we have $\mathcal{W}$ is continuous at α. Since α was taken arbitrarily, $\mathcal{W}$ is continuous on S. $\square$

Theorem 5.1.2. *Let* $f \in \mathcal{C}(\square)$, *a partition* $\triangle$, *a scale function* $\alpha \in \mathcal{C}(\square)$ *with* $\|\alpha\|_\infty < 1$ *and* $X = \{s \in \mathcal{C}(\square) : s(\boldsymbol{x}) = f(\boldsymbol{x}),\ \forall\ \boldsymbol{x} \in \partial\Delta\}$. *Then the map* $\mathcal{V} : X \to \mathcal{C}(\square)$ *defined by* $\mathcal{V}(s) = f^\alpha_{\triangle,s}$ *is Lipschitz continuous.*

Proof. We know that for a fixed partition $\triangle$, a scale function α, and a suitable function $s \in \mathcal{C}(\square)$, the map $f^\alpha_{\triangle,s}$ is unique. Further, being fixed point of Read-Bajraktarević operator, $f^\alpha_{\triangle,s}$ satisfies the functional equation: for all $\boldsymbol{x} \in \square_{ij}$, where $(i,j) \in \Sigma$, we have

$$f^\alpha_{\triangle,s}(\boldsymbol{x}) = f(\boldsymbol{x}) + \alpha(\boldsymbol{x}) f^\alpha_{\triangle,s}(Q_{ij}(\boldsymbol{x})) - \alpha(\boldsymbol{x}) s(Q_{ij}(\boldsymbol{x})). \tag{5.1.2}$$

It is obvious that $\mathcal{V}$ is well defined. Let $s, r \in X$. For reader's convenience, we write

$$\mathcal{V}(s)(\boldsymbol{x}) = f(\boldsymbol{x}) + \alpha(\boldsymbol{x}) f^\alpha_{\triangle,s}(Q_{ij}(\boldsymbol{x})) - \alpha(\boldsymbol{x}) s(Q_{ij}(\boldsymbol{x}))$$

and

$$\mathcal{V}(r)(\boldsymbol{x}) = f(\boldsymbol{x}) + \alpha(\boldsymbol{x}) f^\alpha_{\triangle,r}(Q_{ij}(\boldsymbol{x})) - \alpha(\boldsymbol{x}) r(Q_{ij}(\boldsymbol{x})).$$

On subtracting one from other of the above two equations, we get for $\boldsymbol{x} \in \square_{ij}$,

$$\begin{aligned}\mathcal{V}(s)(\boldsymbol{x}) - \mathcal{V}(r)(\boldsymbol{x}) &= \alpha(\boldsymbol{x}) f^\alpha_{\triangle,s}(Q_{ij}(\boldsymbol{x})) - \alpha(\boldsymbol{x}) s(Q_{ij}(\boldsymbol{x})) \\ &\quad - \alpha(\boldsymbol{x}) f^\alpha_{\triangle,r}(Q_{ij}(\boldsymbol{x})) + \alpha(\boldsymbol{x}) r(Q_{ij}(\boldsymbol{x})) \\ &= \alpha(\boldsymbol{x})\ (f^\alpha_{\triangle,s} - f^\alpha_{\triangle,r}) \circ (Q_{ij}(\boldsymbol{x})) \\ &\quad + \alpha(\boldsymbol{x})(r-s) \circ (Q_{ij}(\boldsymbol{x})).\end{aligned}$$

Now using triangle inequality and definition of uniform norm, we have

$$\big|\mathcal{V}(s)(\boldsymbol{x}) - \mathcal{V}(r)(\boldsymbol{x})\big| \leq \|\alpha\|_\infty \|f^\alpha_{\triangle,s} - f^\alpha_{\triangle,r}\|_\infty + \|\alpha\|_\infty \|q - s\|_\infty.$$

The above inequality holds for all $\boldsymbol{x} \in \square$. Therefore,

$$\|\mathcal{V}(s) - \mathcal{V}(r)\|_\infty \leq \|\alpha\|_\infty \|f^\alpha_{\triangle,s} - f^\alpha_{\triangle,r}\|_\infty + \|\alpha\|_\infty \|s - r\|_\infty.$$

This can be recasted as $\|\mathcal{V}(s) - \mathcal{V}(r)\|_\infty \leq \frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty} \|s - r\|_\infty$. It follows that

$$\|\mathcal{V}(s) - \mathcal{V}(r)\|_\infty \leq \frac{\|\alpha\|_\infty}{1 - \|\alpha\|_\infty} \|s - r\|_\infty.$$

This shows that $\mathcal{V}$ is a Lipschitz continuous map with Lipschitz constant $\frac{\|\alpha\|_\infty}{1-\|\alpha\|_\infty}$. $\square$

Our next target is to show that $f^\alpha_{\Delta,s}$ continuously depends on the partition Δ. We shall show this with an additional assumption that the germ function f and function s are Lipschitz functions. First let us remind the following well-known theorem by Barnsley [8].

Theorem 5.1.3. *Let (X, d) be a complete metric space. Let $\{X; w_1, w_2, \dots w_N\}$ be a hyperbolic IFS. For $n = 1, 2, \dots N$, let w_n depend on the parameter $p \in (P, d_p)$ subject to the condition $d\big(w_{n_p}(x), w_{n_q}(x)\big) \leq K\ d(p, q)$ for all $x \in X$ with K independent of n, p, or x, where (P, d_p) is a metric space. Then the attractor $A(p) \in H(X)$ depends continuously on the parameter $p \in P$, with respect to the Hausdorff metric h induced by d.*

As a prelude, we write the following.

Note 5.1.1. Let us endow $\mathcal{C}(\square)$ with the following metrics:

$$d_1(f,g) := \sup\big\{|f(\boldsymbol{x}) - g(\boldsymbol{x})| : \boldsymbol{x} \in \square\big\} \text{ and } d_2(f,g) = h_\rho\Big(Gr(f), Gr(g)\Big),$$

where h_ρ is the Hausdorff metric induced by the box metric ρ (or any equivalent metric):

$$\rho\Big((x,y,z),(\bar{x},\bar{y},\bar{z})\Big) = \max\big\{|x-\bar{x}|, |y-\bar{y}|, |z-\bar{z}|\big\}.$$

It is known [91] that d_1 and d_2 are equivalent metrics on $\mathcal{C}(\square)$.

Theorem 5.1.4. *Let $f, s \in \mathcal{C}(\square)$ be Lipschitz continuous with Lipschitz constant k_f, k_s respectively, satisfying $s(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \forall\ \boldsymbol{x} \in \partial\Delta$. Assume that scale function $\alpha \in \mathcal{C}(\square)$ is Lipschitz function with Lipschitz constant k_α such that $\|\alpha\|_\infty < 1$. Then a mapping $\mathcal{T} : Y \to \mathcal{C}(\square)$ defined by $\mathcal{T}(\triangle) = f^{\alpha}_{\triangle,s}$ is continuous, where $Y = \{\triangle \in \mathbb{R}^{N+M+2} : \triangle$ is a partition of $\square\}$.*

Proof. In view of Remark 1.11.2, let $X = \square \times [-M, M]$ and $W_{ij}(x,y,z) = \big(P_{ij}(\boldsymbol{x}), F_{ij}(x,y,z)\big)$, where $u_i(x) = a_i x + b_i, v_j(y) = c_j y + d_j$ and $F_{ij}(x,y,z) = \alpha\big(P_{ij}(\boldsymbol{x})\big)z + f\big(P_{ij}(\boldsymbol{x})\big) - \alpha\big(P_{ij}(\boldsymbol{x})\big)s(\boldsymbol{x})$. Let us work with the following metric on $\mathbb{R}^3$ (denoted by ρ again), which is equivalent to ρ given before,

$$\rho\big((\boldsymbol{x},z),(\boldsymbol{y},z')\big) = \|\boldsymbol{x}-\boldsymbol{y}\|_2 + \theta|z-z'|,$$

where θ is a suitable constant to be determined later. Further, with $A := \max\{|a_i|,\ |c_j| : (i,j) \in \Sigma_N \times \Sigma_M\}$, we get

$$|u_i(x) - u_i(x')| = |a_i|\ |x - x'| \le A|x - x'|$$

and

$$|v_j(y) - v_j(y')| = |c_j|\ |y - y'| \le A|y - y'|.$$

Similarly,

$$\begin{aligned}
|F_{ij}(\boldsymbol{x}, z) - F_{ij}(\boldsymbol{y}, z')| &= \Big|\alpha\big(P_{ij}(\boldsymbol{x})\big)z + f\big(P_{ij}(\boldsymbol{x})\big) \\
&\quad - \alpha\big(P_{ij}(\boldsymbol{x})\big)s(\boldsymbol{x}) - [\alpha\big(P_{ij}(\boldsymbol{y})\big)z' \\
&\quad + f\big(P_{ij}(\boldsymbol{y})\big) - \alpha\big(P_{ij}(\boldsymbol{y})\big)s(x', y')]\Big| \\
&\le |\alpha(P_{ij}(\boldsymbol{x}))z - \alpha(P_{ij}(\boldsymbol{x}))z'| \\
&\quad + |\alpha(P_{ij}(\boldsymbol{x}))z' - \alpha(P_{ij}(\boldsymbol{y}))z'| \\
&\quad + \big|f\big(P_{ij}(\boldsymbol{x})\big) - f\big(P_{ij}(\boldsymbol{y})\big)\big| \\
&\quad + |\alpha\big(P_{ij}(\boldsymbol{x})\big)s(\boldsymbol{x}) - \alpha\big(P_{ij}(\boldsymbol{x})\big)s(\boldsymbol{y})| \\
&\quad + |\alpha\big(P_{ij}(\boldsymbol{x})\big)s(\boldsymbol{y}) - \alpha\big(P_{ij}(\boldsymbol{y})\big)s(\boldsymbol{y})| \\
&\le \|\alpha\|_\infty|z - z'| + Ak_\alpha\|\boldsymbol{x} - \boldsymbol{y}\|_2|z'| \\
&\quad + Ak_f\|\boldsymbol{x} - \boldsymbol{y}\|_2 + \|\alpha\|_\infty k_s\|\boldsymbol{x} - \boldsymbol{y}\|_2 \\
&\quad + \|s\|_\infty k_\alpha\|\boldsymbol{x} - \boldsymbol{y}\|_2 \\
&\le \|\alpha\|_\infty|z - z'| + AMk_\alpha\|\boldsymbol{x} - \boldsymbol{y}\|_2 \\
&\quad + Ak_f\|\boldsymbol{x} - \boldsymbol{y}\|_2 + \|\alpha\|_\infty k_s\|\boldsymbol{x} - \boldsymbol{y}\|_2 \\
&\quad + A\|s\|_\infty k_\alpha\|\boldsymbol{x} - \boldsymbol{y}\|_2 \\
&\le \|\alpha\|_\infty|z - z'| + \Big(AMk_\alpha + Ak_f + \|\alpha\|_\infty k_s \\
&\quad + A\|s\|_\infty k_\alpha\Big)\|\boldsymbol{x} - \boldsymbol{y}\|_2.
\end{aligned}$$

Taking $K = \max\big\{\|\alpha\|_\infty, A + \theta[AMk_\alpha + Ak_f + \|\alpha\|_\infty k_s + A\|s\|_\infty k_\alpha]\big\}$, we infer that

$$\rho\big(W_{ij}(\boldsymbol{x}, z), W_{ij}(\boldsymbol{y}, z')\big) \le K\rho\big((\boldsymbol{x}, z), (\boldsymbol{y}, z')\big).$$

Note that with $\theta < \dfrac{1-A}{AMk_\alpha + Ak_f + \|\alpha\|_\infty k_s + A\|s\|_\infty k_\alpha}$, the mapping W_{ij} is a contraction and $\{X; W_{ij}\}$ is a hyperbolic IFS. Let $\Delta := \{x_i : i = 0,1,2,\dots,N; j = 0,1,\dots,M, x_0 < x_2 < \cdots < x_N : y_0 < y_2 < \cdots < y_M\}$ and $\tilde{\Delta} := \{\tilde{x}_i : i = 0,1,2,\dots,N; j = 0,1,\dots,M, x_0 = \tilde{x}_0 < \tilde{x}_1 < \cdots < \tilde{x}_N = x_N; x_0 = \tilde{y}_0 < \tilde{y}_1 < \cdots < \tilde{y}_M = y_M\}$ be two nets of $\square$. Note that the maps u_i, v_j, F_{ij} and W_{ij} depend on the net chosen, hence we denote u_i, v_j, F_{ij} and W_{ij} corresponding to the net Δ by $u_{i,\Delta}, v_{j,\Delta}, F_{ij,\Delta}$ and $W_{ij,\Delta}$ respectively. Further

$$\begin{aligned}
&\rho\big(W_{ij,\Delta}(\boldsymbol{x},z), W_{ij,\tilde{\Delta}}(\boldsymbol{x},z)\big)\\
&\quad= \rho\Big(\big(u_{i,\Delta}(x), v_{j,\Delta}(y), F_{ij,\Delta}(\boldsymbol{x},z)\big), \big(u_{i,\tilde{\Delta}}(x), v_{j,\tilde{\Delta}}(y), F_{ij,\tilde{\Delta}}(\boldsymbol{x},z)\big)\Big).
\end{aligned}$$

If $i \in \Sigma_N$ is an odd number then we have

$$u_{i,\Delta}(x) = \frac{x_i - x_{i-1}}{x_N - x_0}x + \frac{x_N x_{i-1} - x_0 x_i}{x_N - x_0}, \quad u_{i,\tilde{\Delta}}(x) = \frac{\tilde{x}_i - \tilde{x}_{i-1}}{x_N - x_0}x + \frac{x_N \tilde{x}_{i-1} - x_0 \tilde{x}_i}{x_N - x_0}.$$

Therefore,

$$\begin{aligned}
\big|u_{i,\Delta}(x) - u_{i,\tilde{\Delta}}(x)\big| &= \frac{1}{x_N - x_0}\big|(x_i - \tilde{x}_i)(x - x_0) + (\tilde{x}_{i-1} - x_{i-1})(x - x_N)\big|\\
&\le \frac{1}{x_N - x_0}\Big(|x_i - \tilde{x}_i||x - x_0| + |\tilde{x}_{i-1} - x_{i-1}||x - x_N|\Big)\\
&\le |x_i - \tilde{x}_i| + |\tilde{x}_{i-1} - x_{i-1}|\\
&\le 2\|\Delta - \tilde{\Delta}\|_2.
\end{aligned}$$

We can deduce the similar expression when i is an even number. Furthermore, one gets similar expression for $\big|v_{j,\Delta}(y) - v_{j,\tilde{\Delta}}(y)\big|$. Now

$$\begin{aligned}\big|F_{ij,\Delta}(\boldsymbol{x},z) - F_{ij,\tilde{\Delta}}(\boldsymbol{x},z)\big| &= \Big|\alpha\big(P_{ij,\Delta}(\boldsymbol{x})\big)z + f\big(P_{ij,\Delta}(\boldsymbol{x})\big) \\ &\quad - \alpha\big(P_{ij,\Delta}(\boldsymbol{x}))\big)s(\boldsymbol{x}) \\ &\quad - \Big(\alpha\big(P_{ij,\tilde{\Delta}}(\boldsymbol{x})\big)z + f\big(P_{ij,\tilde{\Delta}}(\boldsymbol{x})\big) \\ &\quad - \alpha\big(P_{ij,\tilde{\Delta}}(\boldsymbol{x})\big)s(\boldsymbol{x})\Big)\Big| \\ &\leq k_\alpha(M + \|s\|_\infty)\Big\|P_{ij,\Delta}(\boldsymbol{x}) - P_{ij,\tilde{\Delta}}(\boldsymbol{x})\Big\|_2 \\ &\quad + k_f\Big\|P_{ij,\Delta}(\boldsymbol{x}) - P_{ij,\tilde{\Delta}}(\boldsymbol{x})\Big\|_2 \\ &\leq 2\sqrt{2}\big(k_f + k_\alpha(M + \|s\|_\infty)\big)\|\Delta - \tilde{\Delta}\|_2,\end{aligned}$$

where $P_{ij,\Delta}(\boldsymbol{x}) = \big(u_{i,\tilde{\Delta}}(x), v_{j,\Delta}(y)\big)$ and $P_{ij,\tilde{\Delta}}(\boldsymbol{x}) = \big(u_{i,\tilde{\Delta}}(x), v_{j,\tilde{\Delta}}(y)\big)$. Consequently,

$$\rho\big(W_{ij,\Delta}(\boldsymbol{x},z), W_{ij,\tilde{\Delta}}(\boldsymbol{x},z)\big) \leq 2\sqrt{2}(1 + \theta k_f)\|\Delta - \tilde{\Delta}\|_2.$$

The previous inequality reveals that the IFS maps W_{ij} depend continuously on the net $\Delta \in Y$. Consequently, Theorem 5.1.3 yields that the attractor $G(\triangle) \in H(X)$ depends continuously on $\Delta \in P$, with respect to the Hausdorff metric h_ρ. Let $g, h \in \mathcal{C}(\square)$ and $Gr(g), Gr(h)$ be the graphs of g and h respectively. By Note 5.1.1, it follows that

$$\|g_n - g\|_\infty \to 0 \ \text{ if and only if } \ h_\rho\big(Gr(g_n), Gr(g)\big) \to 0.$$

Therefore, for a given $\epsilon > 0$ and $\Delta \in Y$, we have

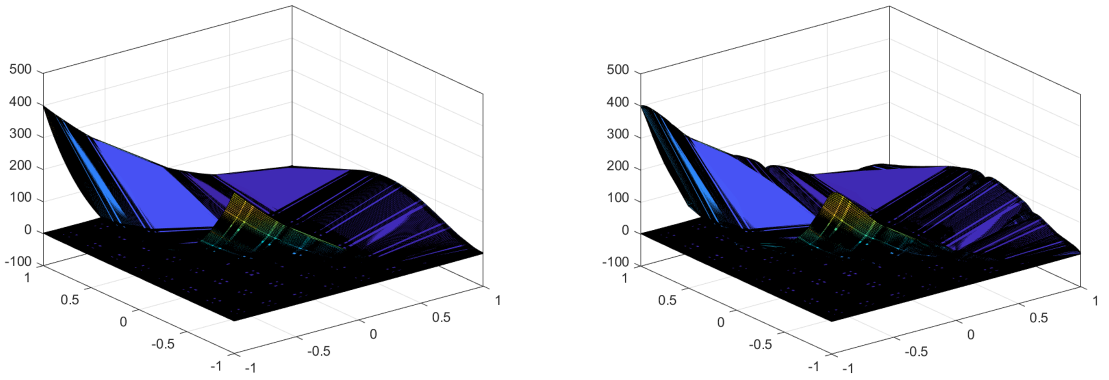
$$\|\mathcal{T}(\tilde{\Delta}) - \mathcal{T}(\Delta)\|_\infty = \|f^\alpha_{\tilde{\Delta},s} - f^\alpha_{\Delta,s}\|_\infty < \epsilon \ \text{ whenever } \ \|\tilde{\Delta} - \Delta\|_2 < \delta.$$

That is, $\mathcal{T}$ is continuous at Δ. Since $\Delta \in Y$ is arbitrary, $\mathcal{T}$ is continuous on Y.

□

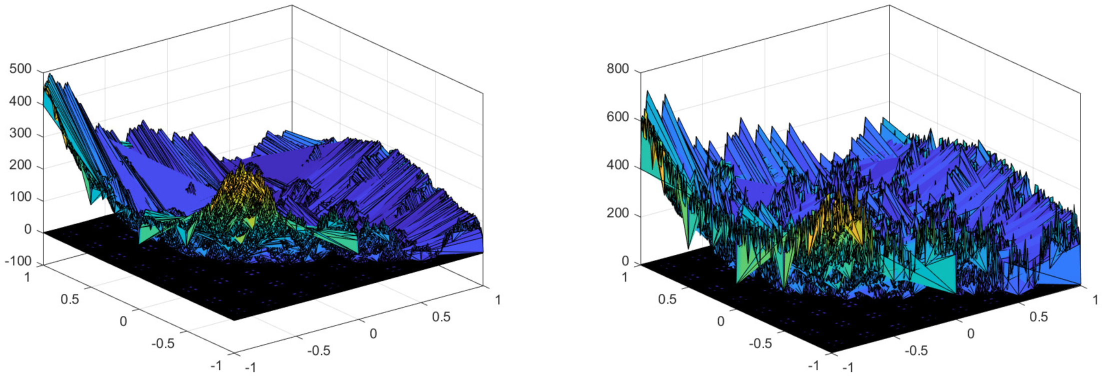
We plot the α- fractal function of the original function on the domain $[-1,1] \times [-1,1]$.

The Δ is taken by the partition $\{-1, -0.5, 0, 0.5, 1\}$ of $[-1, 1]$. We choose the different scaling function $\alpha(x, y)$ corresponding to the original bivariate function $f(x, y) = 100(y - x^2)^2 + (1 - x)^2$ and base function $s(x, y) = x^2y^2f(x, y)$. The following graphs depict that α-fractal function f^α has continuous dependence on the scaling function $\alpha(x, y)$, net Δ of rectangular grid, and the base function $s(x, y)$.



Original function f(x,y)

f^α at $\alpha = 0.1$



f^α at $\alpha = 0.5$

f^α at $\alpha = 0.9$

FIGURE 5.1.1: $f(x, y) = 100(y - x^2)^2 + (1 - x)^2$, $s(x, y) = x^2y^2f(x, y)$.

5.2 Fractal Dimension of α-Fractal Function

We write the following note which is a modification of [92, Proposition 1].

Remark 5.2.1. We equip the set $\square \times \mathbb{R}$ with a metric d defined as follows:

$$d((\boldsymbol{x}, t), (\boldsymbol{y}, u)) = r_1 \|\boldsymbol{x} - \boldsymbol{y}\|_2 + r_2 |(t - f^\alpha(\boldsymbol{x})) - (u - f^\alpha(\boldsymbol{y}))|,$$

for every $(\boldsymbol{x}, t), (\boldsymbol{y}, u) \in \square \times \mathbb{R}$, and for fixed but arbitrary $r_1, r_2 > 0$. Then the metric d makes the space $\square \times \mathbb{R}$ complete.

We should mention that the techniques involved in the proof of the next theorem is same as in that of [92, Theorem 4].

Theorem 5.2.1. *The map $W_{ij} : \square \times [-M, M] \to \square \times [-M, M]$ is a contraction map with respect to the above metric provided the scale function $\alpha : \square \to \mathbb{R}$ satisfies $|\alpha(\boldsymbol{x}) - \alpha(\boldsymbol{y})| \le k_\alpha \|\boldsymbol{x} - \boldsymbol{y}\|_2$, and*

$$\max\left\{ \max\{|a_i|, |c_j|\} + \frac{2 r_2 M k_\alpha \max\{|a_i|, |c_j|\}}{r_1}, \|\alpha\|_\infty \right\} < 1.$$

Proof. Let $(\boldsymbol{x}, t), (\boldsymbol{y}, u) \in \square \times [-M, M]$. Then we have

$$\begin{aligned}
& d\big(W_{ij}(\boldsymbol{x}, t), W_{ij}(\boldsymbol{y}, u)\big) - r_1 \big\| \big(P_{ij}(\boldsymbol{x})\big) - \big(P_{ij}(\boldsymbol{y})\big) \big\|_2 \\
&= r_2 \Big| \Big(\alpha\big(P_{ij}(\boldsymbol{x})\big) t + f\big(P_{ij}(\boldsymbol{x})\big) - \alpha\big(P_{ij}(\boldsymbol{x})\big) s(\boldsymbol{x}) \\
&\qquad - f^\alpha\big(P_{ij}(\boldsymbol{x})\big) \Big) - \Big(\alpha\big(P_{ij}(\boldsymbol{y})\big) u + f\big(P_{ij}(\boldsymbol{y})\big) \\
&\qquad - \alpha\big(P_{ij}(\boldsymbol{y})\big) s(\boldsymbol{y}) - f^\alpha\big(P_{ij}(\boldsymbol{y})\big) \Big) \Big| \\
&= r_2 \Big| \alpha\big(P_{ij}(\boldsymbol{x})\big) \big(t - f^\alpha(\boldsymbol{x})\big) - \alpha\big(P_{ij}(\boldsymbol{y})\big) \big(u - f^\alpha(\boldsymbol{y})\big) \Big| \\
&= r_2 \Big| \alpha\big(P_{ij}(\boldsymbol{x})\big) \big(t - f^\alpha(\boldsymbol{x})\big) - \alpha\big(P_{ij}(\boldsymbol{x})\big) \big(u - f^\alpha(\boldsymbol{y})\big) \\
&\qquad + \alpha\big(P_{ij}(\boldsymbol{x})\big) \big(u - f^\alpha(\boldsymbol{y})\big) - \alpha\big(P_{ij}(\boldsymbol{y})\big) \big(u - f^\alpha(\boldsymbol{y})\big) \Big| \\
&\le r_2 \big|\alpha\big(P_{ij}(\boldsymbol{x})\big)\big| \, \big| \big(t - f^\alpha(\boldsymbol{x})\big) - \big(u - f^\alpha(\boldsymbol{y})\big) \big| \\
&\qquad + r_2 \big|\alpha\big(P_{ij}(\boldsymbol{x})\big) - \alpha\big(P_{ij}(\boldsymbol{y})\big)\big| \, |u - f^\alpha(\boldsymbol{y})|.
\end{aligned}$$

Hence,

$$\begin{aligned}
&d\big(W_{ij}(\boldsymbol{x},t),W_{ij}(\boldsymbol{y},u)\big)-r_1\|\big(P_{ij}(\boldsymbol{x})\big)-\big(P_{ij}(\boldsymbol{y})\big)\|_2\\
&\le r_2\|\alpha\|_\infty\big|\big(t-f^\alpha(\boldsymbol{x})\big)-\big(u-f^\alpha(\boldsymbol{y})\big)\big|\\
&\quad+2r_2Mk_\alpha\max\{|a_i|,|c_j|\}\|\boldsymbol{x}-\boldsymbol{y}\|_2.
\end{aligned}$$

Further, we get

$$\begin{aligned}
&d\big(W_{ij}(\boldsymbol{x},t),W_{ij}(\boldsymbol{y},u)\big)\\
&\le r_1\|\big(P_{ij}(\boldsymbol{x})\big)-\big(P_{ij}(\boldsymbol{y})\big)\|_2+r_2\|\alpha\|_\infty\big|\big(t-f^\alpha(\boldsymbol{x})\big)-\big(u-f^\alpha(\boldsymbol{y})\big)\big|\\
&\quad+2r_2Mk_\alpha\max\{|a_i|,|c_j|\}\|\boldsymbol{x}-\boldsymbol{y}\|_2\\
&=r_1\max\{|a_i|,|c_j|\}\|\boldsymbol{x}-\boldsymbol{y}\|_2+r_2\|\alpha\|_\infty\big|\big(t-f^\alpha(\boldsymbol{x})\big)-\big(u-f^\alpha(\boldsymbol{y})\big)\big|\\
&\quad+2r_2Mk_\alpha\max\{|a_i|,|c_j|\}\|\boldsymbol{x}-\boldsymbol{y}\|_2\\
&\le\left(\max\{|a_i|,|c_j|\}+\frac{2r_2Mk_\alpha\max\{|a_i|,|c_j|\}}{r_1}\right)r_1\|\boldsymbol{x}-\boldsymbol{y}\|_2\\
&\quad+r_2\|\alpha\|_\infty\big|\big(t-f^\alpha(\boldsymbol{x})\big)-\big(u-f^\alpha(\boldsymbol{y})\big)\big|.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&d\big(W_{ij}(\boldsymbol{x},t),W_{ij}(\boldsymbol{y},u)\big)\\
&\le\ \max\left\{\max\{|a_i|,|c_j|\}+\frac{2r_2Mk_\alpha\max\{|a_i|,|c_j|\}}{r_1},\|\alpha\|_\infty\right\}\Big(r_1\|\boldsymbol{x}-\boldsymbol{y}\|_2\\
&\quad+r_2\big(t-f^\alpha(\boldsymbol{x})\big)-\big(u-f^\alpha(\boldsymbol{y})\big)\big|\Big)\\
&=\max\left\{\max\{|a_i|,|c_j|\}+\frac{2r_2Mk_\alpha\max\{|a_i|,|c_j|\}}{r_1},\|\alpha\|_\infty\right\}d\big((\boldsymbol{x},t),(\boldsymbol{y},u)\big).
\end{aligned}$$

This completes the proof. □

Remark 5.2.1. *The above theorem conveys us that when the scale function α satisfies the Lipschitz condition, we can define a metric d with respect to all maps $W_{ij}:\square\times\mathbb{R}\to\square\times\mathbb{R}$ are contraction map. By the definition of d, we could easily* *say that metric d and Euclidean metric are equivalent. If one would like to show the stability of fractal perturbation with respect to parameters α and s, then one will not be able to describe the stability by above theorem because here metric also depends on parameters.*

Now let us introduce the notion of separation properties for an IFS in terms of W_{ij}. We say that the W_{ij} satisfy the Open Set Condition (OSC) if there exists a non-empty bounded open set U with

$$\cup_{(i,j)\in\Sigma} W_{ij}(U) \subset U,$$

and terms present in the above union are disjoint. Further, if the above U satisfies $U \cap Gr(f^\alpha) \neq \emptyset$, then we call that W_{ij} satisfy the Strong Open Set Condition (SOSC).

Theorem 5.2.2. *If the IFS $\{\square \times \mathbb{R} : W_{ij}, (i,j) \in \Sigma\}$ as defined earlier satisfies the following condition:*

$$c_{ij} \|(\boldsymbol{x}, t) - (\boldsymbol{y}, u)\|_2 \leq \|W_{ij}(\boldsymbol{x}, t) - W_{ij}(\boldsymbol{y}, u)\|_2 \leq C_{ij} \|(\boldsymbol{x}, t) - (\boldsymbol{y}, u)\|_2,$$

for every $(\boldsymbol{x}, t), (\boldsymbol{y}, u) \in \square \times \mathbb{R}$, where $0 < c_{ij} \leq C_{ij} < 1 \ \forall \ (i,j) \in \Sigma$. Then $s_* \leq \dim_H(Gr(f^\alpha)) \leq \overline{\dim_B}(Gr(f^\alpha)) \leq s^*$, where s_* and s^* are such that $\sum_{(i,j)\in\Sigma} c_{ij}^{s_*} = 1$ and $\sum_{(i,j)\in\Sigma} C_{ij}^{s^*} = 1$, and f^α is the FIF corresponding to the IFS $\{\square \times \mathbb{R} : W_{ij}, (i,j) \in \Sigma\}$.

Proof. From [5, Proposition 9.6], we deduce that $\overline{\dim_B}(Gr(f^\alpha)) \leq s^*$. To get the required lower bound for $\dim_H(Gr(f^\alpha))$, one may begin as follows.

We choose an open set $U = \text{int}\big(\square\big) \times \mathbb{R}$, where $\text{int}(\square)$ denotes interior of the rectangular domain $\square$. It is obvious to see that

$$u_i(\text{int}(I)) \cap u_k(\text{int}(I)) = \emptyset, \quad i \neq k$$

and

$$v_j(\text{int}(J)) \cap v_l(\text{int}(J)) = \emptyset, \quad j \neq l.$$

This in turn yields that $W_{ij}(U) \cap W_{kl}(U) = \emptyset$, for each $(i,j) \neq (k,l)$. Note also that $U \cap Gr(f^\alpha) \neq \emptyset$, that is, the IFS satisfies the SOSC. Let us use $\Sigma = \Sigma$ and for $\xi = (i,j) \in \Sigma$, $W_\xi = W_{ij}, \quad c_\xi = c_{ij}$. Since $U \cap Gr(f^\alpha) \neq \emptyset$, we can choose a word $w \in \Sigma^*$ such that $Gr(f^\alpha)_w \subset U$, where $\Sigma^* := \cup_{m\in\mathbb{N}}\Sigma^m$ and $Gr(f^\alpha)_w := W_w(Gr(f^\alpha)) := W_{w_1} \circ W_{w_2} \circ \cdots \circ W_{w_m}(Gr(f^\alpha))$ for $w \in \Sigma^m$ and $m \in \mathbb{N}$. Further, for n and $\xi \in \Sigma^n$, the sets $Gr(f^\alpha)_{\xi w}$ are disjoint. Therefore, the IFS $\{W_{\xi w} : \xi \in \Sigma^n\}$ satisfies the hypotheses of [5, Proposition 9.7]. Hence, by defining $c_\xi := c_{\xi_1}c_{\xi_2}\dots c_{\xi_n}$, one has $s_n \leq \dim_H(B_n)$, where $B_n = \cup_{\xi\in\Sigma^n}W_{\xi w}(B_n)$ and $\sum_{\xi\in\Sigma^n} c_{\xi w}^{s_n} = 1$. Since $B_n \subset Gr(f^\alpha)$, we have $s_n \leq \dim_H(B_n) \leq \dim_H(Gr(f^\alpha))$. If possible assume that $\dim_H(Gr(f^\alpha)) < s_*$, where $\sum_{\xi\in\Sigma} c_\xi^{s_*} = 1$. This provides $s_n < s_*$. Consequently, one gets

$$c_w^{-s_n} = \sum_{\xi\in\Sigma^n} c_\xi^{s_n} \geq \sum_{\xi\in\Sigma^n} c_\xi^{\dim_H(Gr(f^\alpha))} \geq \sum_{j\in\Sigma^n} c_\xi^{s_*} c_{max}^{n(\dim_H(Gr(f^\alpha))-s_*)} = c_{max}^{n(\dim_H(Gr(f^\alpha))-s_*)},$$

where $c_{max} = \max_{\xi\in\Sigma}\{c_\xi\}$. Since $c_{max} < 1$, we arrive at a contradiction as n tends to infinity. This further produces $\dim_H(Gr(f^\alpha)) \geq s_*$, completing the proof. $\square$

Remark 5.2.2. *Under the assumptions of Theorem 5.2.1, we may find a upper bound for the Hausdorff dimension of graph of α-fractal function f^α using the above theorem. For example, if the scaling function α is a constant function then $k_\alpha = 0$.*

Define

$$C_{ij} = \max\left\{ \max\{|a_i|, |c_j|\} + \frac{2r_2 M k_\alpha \max\{|a_i|, |c_j|\}}{r_1}, \|\alpha\|_\infty \right\}$$
$$= \max\left\{ \max\{|a_i|, |c_j|\}, \|\alpha\|_\infty \right\}.$$

Now from the above theorem, we have $\dim_H(Gr(f^\alpha)) \leq \overline{\dim}_B(Gr(f^\alpha)) \leq s^*$, *where* s^* *is satisfying* $\sum_{(i,j)\in\Sigma} C_{ij}^{s^*} = 1$.

Remark 5.2.3. *If* $c_{ij} = C_{ij}$ *for all* (i, j), *that is, all maps* W_{ij} *are similarity transformations, then by a result of Hutchinson [2], we get* $\dim_H(Gr(f^\alpha)) = \dim_B(Gr(f^\alpha)) = s$, *where* s *is a unique solution of* $\sum_{(i,j)\in\Sigma} c_{ij}^s = 1$. *In addition to that we have* $0 < \mathcal{H}^s(Gr(f^\alpha)) < \infty$.

Remark 5.2.4. *As is well-known [5] that the OSC and SOSC are equivalent for similitudes and self-conformal maps. One may ask that whether or not the OSC and SOSC are equivalent for the IFSs generating* α*-fractal functions.*

Remark 5.2.5. *It is known that for a pure self-similar set or self-conformal set* A, $\dim_H(A) = \underline{\dim}_B(A) = \overline{\dim}_B(A)$, *for more details see [5]. Note that the nature of* α*-fractal functions depend on the IFS parameter. In particular, one can obtain pure self-similar or partial self similar* α*-fractal functions by choosing suitable scaling functions and thus for* α*-fractal functions we may or may not get the equal dimensions.*

Remark 5.2.6. *Here we talk about continuity of the Hausdorff dimension. Note that the Hausdorff dimension is not a continuous function. For example,* $A_n := [0, \frac{1}{n}] \to A := \{0\}$ *in Hausdorff metric but* $\dim_H(A_n) = 1$ *does not converge to* $\dim_H(A) = 0$. *In the previous section, we have shown the continuous dependence of* α*-fractal function on the parameters. A natural question on the continuity of the Hausdorff dimension of* α*-fractal function with respect to the parameters involved may arise. However, we feel that the result may not be true in general.*

Next we aim to compute fractal dimension of $Gr(f^{\alpha})$ using the concept of oscillation spaces. We refer the reader to [38, 93, 94] for oscillation spaces. It is worth to emphasize that some proofs regarding oscillation spaces are motivated by a research work [25, 38]. However, we include an expanded version of arguments for completeness and record. Let $Q \subset [0,1] \times [0,1] =: I^2$ dyadic square so that $Q = \left[\frac{i}{2^m}, \frac{i+1}{2^m}\right] \times \left[\frac{j}{2^m}, \frac{j+1}{2^m}\right]$ for some integers $m \geq 0$ and $0 \leq i, j < \frac{1}{2^m}$. For a continuous function $f : I^2 \to \mathbb{R}$,

we define oscillation of f over Q as follows:

$$\begin{aligned} R_f(Q) &= \sup_{\boldsymbol{x},\boldsymbol{y} \in Q} |f(\boldsymbol{x}) - f(\boldsymbol{y})| \\ &= \sup_{\boldsymbol{x} \in Q} f(\boldsymbol{x}) - \inf_{\boldsymbol{x} \in Q} f(\boldsymbol{x}), \end{aligned}$$

and total oscillation of order m,

$$Osc(m, f) = \sum_{|Q| = 2^{-m}} R_f(Q),$$

where the sum ranges over all dyadic squares $Q \subset I^2$ of side-length $|Q| = \frac{1}{2^m}$.
We define oscillation space $\mathcal{V}^{\beta}(I^2)$ by

$$\mathcal{V}^{\beta}(I^2) = \Big\{ f \in \mathcal{C}(I^2) : \sup_{m \in \mathbb{N}} \frac{Osc(m, f)}{2^{m(2-\beta)}} < \infty \Big\}.$$

One can define

$$\mathcal{V}^{\beta-}(I^2) = \{ f \in \mathcal{C}(I^2) : f \in \mathcal{V}^{\beta-\epsilon}(I^2) \ \forall \ \epsilon > 0 \}$$

and

$$\mathcal{V}^{\beta+}(I^2) = \{ f \in \mathcal{C}(I^2) : f \notin \mathcal{V}^{\beta+\epsilon}(I^2) \ \forall \ \epsilon > 0 \}.$$

Theorem 5.2.3 ([93], Theorem 4.1)**.** *Let f be a real-valued continuous function defined on I^2, we have*

$$\overline{\dim}_B(Gr(f)) \leq 3-\gamma \iff f \in \mathcal{V}^{\gamma-}(I^2) \quad \text{if } 0<\gamma\leq 1$$

and

$$\overline{\dim}_B(Gr(f)) \geq 3-\gamma \iff f \in \mathcal{V}^{\gamma+}(I^2) \ \ \text{if } 0\leq\gamma < 1.$$

Lemma 5.2.4. *Let $f, g \in \mathcal{C}(I^2)$ and $\lambda \in \mathbb{R}$. Then for $m \in \mathbb{N}$, we have the following*

1. $Osc(m, \lambda f) = |\lambda| Osc(m, f)$.
2. $Osc(m, f+g) \leq Osc(m, f) + Osc(m, g)$.
3. $Osc(m, fg) \leq \|g\|_\infty Osc(m, f) + \|f\|_\infty Osc(m, g)$.

Proof. 1. For $m \in \mathbb{N}$ and $f, g \in \mathcal{C}(I^2)$, one proceeds as follows

$$\begin{aligned}
Osc(m, \lambda f) &= \sum_{|Q|=2^{-m}} R_{\lambda f}(Q) \\
&= \sum_{|Q|=2^{-m}} \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |(\lambda f)(\boldsymbol{x}) - (\lambda f)(\boldsymbol{y})| \\
&= \sum_{|Q|=2^{-m}} |\lambda| \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |f(\boldsymbol{x}) - f(\boldsymbol{y})| \\
&= |\lambda| \sum_{|Q|=2^{-m}} \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |f(\boldsymbol{x}) - f(\boldsymbol{y})|.
\end{aligned}$$

Finally, we get

$$Osc(m, \lambda f) = |\lambda| \ Osc(m, f).$$

2. Turning to second item, we have

$$\begin{aligned}
Osc(m, f+g) &= \sum_{|Q|=2^{-m}} R_{f+g}(Q) \\
&= \sum_{|Q|=2^{-m}} \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |(f+g)(\boldsymbol{x}) - (f+g)(\boldsymbol{y})| \\
&\leq \sum_{|Q|=2^{-m}} \Big(\sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |f(\boldsymbol{x}) - f(\boldsymbol{y})| \\
&\quad + \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |g(\boldsymbol{x}) - g(\boldsymbol{y})| \Big) \\
&= \sum_{|Q|=2^{-m}} R_f(Q) + \sum_{|Q|=2^{-m}} R_g(Q) \\
&= Osc(m, f) + Osc(m, g).
\end{aligned}$$

3. The third item follows through the following lines.

$$\begin{aligned}
Osc(m, fg) &= \sum_{|Q|=2^{-m}} R_{fg}(Q) \\
&= \sum_{|Q|=2^{-m}} \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |(fg)(\boldsymbol{x}) - (fg)(\boldsymbol{y})| \\
&= \sum_{|Q|=2^{-m}} \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |f(\boldsymbol{x})g(\boldsymbol{x}) - f(\boldsymbol{y})g(\boldsymbol{x}) \\
&\quad + f(\boldsymbol{y})g(\boldsymbol{x}) - f(\boldsymbol{y})g(\boldsymbol{y})| \\
&\leq \sum_{|Q|=2^{-m}} \Big(\sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |g(\boldsymbol{x})||f(\boldsymbol{x}) - f(\boldsymbol{y})| \\
&\quad + \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |f(\boldsymbol{y})||g(\boldsymbol{x}) - g(\boldsymbol{y})| \Big) \\
&\leq \sum_{|Q|=2^{-m}} \Big(\|g\|_\infty \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |f(\boldsymbol{x}) - f(\boldsymbol{y})| \\
&\quad + \|f\|_\infty \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |g(\boldsymbol{x}) - g(\boldsymbol{y})| \Big) \\
&= \|g\|_\infty \sum_{|Q|=2^{-m}} R_f(Q) + \|f\|_\infty \sum_{|Q|=2^{-m}} R_g(Q) \\
&= \|g\|_\infty Osc(m, f) + \|f\|_\infty Osc(m, g).
\end{aligned}$$

Thus we have established the result. □

Proposition 5.2.5. *Let* $f \in \mathcal{V}^{\beta}(I^2)$, *we define* $\|f\|_{\mathcal{V}^{\beta}} := \|f\|_{\infty} + \sup_{m\in\mathbb{N}} \frac{Osc(m,f)}{2^{m(2-\beta)}}$. *Then* $\|.\|_{\mathcal{V}^{\beta}}$ *forms a norm on* $\mathcal{V}^{\beta}(I^2)$.

Proof. Through simple and straightforward calculations, we have

1.
$$\begin{aligned} &\|f\|_{\mathcal{V}^{\beta}} = 0 \\ \iff &\|f\|_{\infty} = 0 \text{ and } \sup_{m\in\mathbb{N}} \frac{Osc(m,f)}{2^{m(2-\beta)}} = 0 \\ \iff &f = 0, \end{aligned}$$

2.
$$\begin{aligned} \|\lambda f\|_{\mathcal{V}^{\beta}} =&\|\lambda f\|_{\infty} + \sup_{m\in\mathbb{N}} \frac{Osc(m,\lambda f)}{2^{m(2-\beta)}} \\ =&|\lambda|\ \|f\|_{\infty} + |\lambda| \sup_{m\in\mathbb{N}} \frac{Osc(m,f)}{2^{m(2-\beta)}} \\ =&|\lambda|\ \|f\|_{\mathcal{V}^{\beta}}, \end{aligned}$$

   and

3.
$$\begin{aligned} \|f+g\|_{\mathcal{V}^{\beta}} =&\|f+g\|_{\infty} + \sup_{m\in\mathbb{N}} \frac{Osc(m,f+g)}{2^{m(2-\beta)}} \\ \leq&\|f\|_{\infty} + \|g\|_{\infty} + \sup_{m\in\mathbb{N}} \frac{Osc(m,f+g)}{2^{m(2-\beta)}} \\ \leq&\|f\|_{\infty} + \|g\|_{\infty} + \sup_{m\in\mathbb{N}} \frac{Osc(m,f)}{2^{m(2-\beta)}} + \sup_{m\in\mathbb{N}} \frac{Osc(m,g)}{2^{m(2-\beta)}} \\ =&\|f\|_{\mathcal{V}^{\beta}} + \|g\|_{\mathcal{V}^{\beta}}. \end{aligned}$$

Thus the proof of proposition is complete. □

Lemma 5.2.6. *Let* $\{f_n\}$ *be a sequence of continuous functions that converges uniformly to some* $f : I^2 \to \mathbb{R}$ *and* $m \in \mathbb{N}$, *then we have*

$$Osc(m, f_n) \to Osc(m, f).$$

Furthermore, let $\{f_n\}$ *be a sequence in* $\mathcal{V}^{\beta}(I^2)$ *that converges uniformly to some* $f : I^2 \to \mathbb{R}$, *then we have*

$$\sup_{m\in\mathbb{N}} \frac{Osc(m,f)}{2^{m(2-\beta)}} \leq \liminf_{n\to\infty} \sup_{m\in\mathbb{N}} \frac{Osc(m,f_n)}{{}^{m(2-\beta)}}.$$

Proof. Let $m \in \mathbb{N}$, then we have

$$\begin{aligned}
\lim_{n\to\infty} Osc(m,f_n) &= \lim_{n\to\infty} \sum_{|Q|=2^{-m}} R_{f_n}(Q)\\
&= \lim_{n\to\infty} \sum_{|Q|=2^{-m}} \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |f_n(\boldsymbol{x}) - f_n(\boldsymbol{y})|\\
&= \sum_{|Q|=2^{-m}} \lim_{n\to\infty} \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |f_n(\boldsymbol{x}) - f_n(\boldsymbol{y})|\\
&= \sum_{|Q|=2^{-m}} \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |f(\boldsymbol{x}) - f(\boldsymbol{y})|\\
&= Osc(m,f).
\end{aligned}$$

Now for $m \in \mathbb{N}$, we get

$$\begin{aligned}
\frac{Osc(m,f)}{2^{m(2-\beta)}} &= \frac{\sum_{|Q|=2^{-m}} \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |f(\boldsymbol{x}) - f(\boldsymbol{y})|}{2^{m(2-\beta)}}\\
&= \frac{\sum_{|Q|=2^{-m}} \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |\lim_{n\to\infty} f_n(\boldsymbol{x}) - \lim_{n\to\infty} f_n(\boldsymbol{y})|}{2^{m(2-\beta)}}\\
&= \lim_{n\to\infty} \frac{\sum_{|Q|=2^{-m}} \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |f_n(\boldsymbol{x}) - f_n(\boldsymbol{y})|}{2^{m(2-\beta)}}\\
&= \lim_{n\to\infty} \frac{Osc(m,f_n)}{2^{m(2-\beta)}}\\
&\leq \liminf_{n\to\infty} \Big(\sup_{m\in\mathbb{N}} \frac{Osc(m,f_n)}{2^{m(2-\beta)}} \Big).
\end{aligned}$$

This proves the assertion. □

Theorem 5.2.7. *The space* $(\mathcal{V}^{\beta}(I^2), \|.\|_{\mathcal{V}^{\beta}})$ *is a Banach space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{V}^\beta(I^2)$ with respect to $\|.\|_{\mathcal{V}^\beta}$. Equivalently, for a given $\epsilon > 0$ we have a natural number n_0 such that

$$\|f_n - f_k\|_{\mathcal{V}^\beta} < \epsilon \ \ \forall \ n, k \geq n_0.$$

By definition of $\|.\|_{\mathcal{V}^\beta}$, one gets $\|f_n - f_k\|_\infty < \epsilon \ \ \forall \ n, k \geq n_0$. Since $(\mathcal{C}(I^2), \|.\|_\infty)$ is a Banach space, we have a continuous function f with $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. We claim that $f \in \mathcal{V}^\beta(I^2)$ and $\|f_n - f\|_{V^\beta} \to 0$ as $n \to \infty$. Let $m \in \mathbb{N}$ and $n \geq n_0$. In view of Lemma 5.2.6, we have

$$\begin{aligned}
\|f_n - f\|_\infty + \frac{Osc(m, f_n - f)}{2^{m(2-\beta)}} &= \lim_{k \to \infty} \left(\|f_n - f_k\|_\infty + \frac{Osc(m, f_n - f_k)}{2^{m(2-\beta)}} \right) \\
&\leq \lim_{k \to \infty} \left(\|f_n - f_k\|_\infty + \sup_{m' \in \mathbb{N}} \frac{Osc(m', f_n - f_k)}{2^{m'(2-\beta)}} \right) \\
&\leq \sup_{k \geq n_0} \left(\|f_n - f_k\|_\infty + \sup_{m' \in \mathbb{N}} \frac{Osc(m', f_n - f_k)}{2^{m'(2-\beta)}} \right) \\
&= \sup_{k \geq n_0} \|f_n - f_k\|_{\mathcal{V}^\beta} \\
&\leq \epsilon.
\end{aligned}$$

The above is true for every $m \in \mathbb{N}$. Therefore, we obtain $f - f_{n_0} \in \mathcal{V}^\beta(I^2)$. Using Lemma 5.2.4 we have $f = f - f_{n_0} + f_{n_0} \in \mathcal{V}^\beta(I^2)$ and $\|f_n - f\|_{\mathcal{V}^\beta} \leq \epsilon \ \forall \ n \geq n_0$, hence the proof. □

Remark 5.2.7. *Let* $N = M = 2^k$ *for some* $k \in \mathbb{N}$ *and* $|u_j(I)| = |v_j(I)| = \frac{1}{2^k}$ *for all* $j \in \Sigma_N$. *Then for* $m > k$, *we have*

$$Osc(m, g) = N^2 Osc(m - k, f),$$

where $g(\boldsymbol{x}) := f(Q_{ij}(\boldsymbol{x}))$ *for* $\boldsymbol{x} \in \square_{ij}$.

Proof. We have

$$\begin{aligned} Osc(m,g) &= \sum_{|Q|=2^{-m}} \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |g(\boldsymbol{x})-g(\boldsymbol{y})| \\ &= \sum_{|Q|=2^{-m}} \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |f(Q_{ij}(\boldsymbol{x}))-f(Q_{ij}(\boldsymbol{y}))| \\ &= \sum_{i,j\in\Sigma_N} \sum_{|Q|=2^{-(m-k)}} \sup_{\boldsymbol{x},\boldsymbol{y}\in Q} |f(\boldsymbol{x})-f(\boldsymbol{y})| \\ &= N^2 Osc(m-k,f), \end{aligned} \tag{5.2.1}$$

completing the proof. □

Theorem 5.2.8. *Let $f,s,\alpha \in \mathcal{V}^\beta(I^2)$ be such that $s(\boldsymbol{x}) = f(\boldsymbol{x}), \forall\ (i,j) \in \partial\Sigma_{N,0} \times \partial\Sigma_{N,0}$. Further, we assume that $|u_j(I)| = |v_j(I)| = \frac{1}{2^k},\ \forall\ j \in \Sigma_N$ for some $k \in \mathbb{N}$ and $N = 2^k$. If $\max\Big\{\|\alpha\|_\infty + N^2 \sup_{m\in\mathbb{N}} \frac{Osc(m,\alpha)}{2^{m(2-\beta)}}, \frac{N^2\|\alpha\|_\infty}{2^{k(2-\beta)}}\Big\} < 1$, then $f^\alpha \in \mathcal{V}^\beta(I^2)$. Furthermore, we have $2 \le \dim_H(Gr(f^\alpha)) \le \overline{\dim}_B\big(Gr(f^\alpha)\big) \le 3-\beta$.*

Proof. Let $\mathcal{V}^\beta_f(I^2) = \{g \in \mathcal{V}^\beta(I^2) : g(\boldsymbol{x}) = f(\boldsymbol{x}),\ \ \forall\ \boldsymbol{x} \in \partial\Delta\}$. We observe that the space $\mathcal{V}^\beta_f(I^2)$ is a closed subset of $\mathcal{V}^\beta(I^2)$. It follows that $\mathcal{V}^\beta_f(I^2)$ is a complete metric space with respect to the metric induced by norm $\|.\|_{\mathcal{V}^\beta}$. We define a map $T : \mathcal{V}^\beta_f(I^2) \to \mathcal{V}^\beta_f(I^2)$ by

$$(Tg)(\boldsymbol{x}) = f(\boldsymbol{x}) + \alpha(\boldsymbol{x})\ (g-s)(Q_{ij}(\boldsymbol{x})),$$

for all $\boldsymbol{x} \in \square_{ij}$, where $(i,j) \in \Sigma_N \times \Sigma_N$. Lemma 5.2.4 produces the well-definedness of T. Using Remark 5.2.7 for $g,h \in \mathcal{V}^\beta_f(I^2)$, we have

$$\begin{aligned}
\|Tg - Th\|_{\mathcal{V}^\beta} =& \|Tg - Th\|_\infty + \sup_{m\in\mathbb{N}} \frac{Osc(m, Tg - Th)}{2^{m(2-\beta)}} \\
\leq& \|\alpha\|_\infty \|g - h\|_\infty + \sum_{(i,j)\in\Sigma_N\times\Sigma_N} \frac{\|\alpha\|_\infty}{2^{k(2-\beta)}} \sup_{m\in\mathbb{N}, m>k} \frac{Osc(m-k, g-h)}{2^{(m-k)(2-\beta)}} \\
&+ \sum_{(i,j)\in\Sigma_N\times\Sigma_N} \|g - h\|_\infty \sup_{m\in\mathbb{N}} \frac{Osc(m,\alpha)}{2^{m(2-\beta)}} \\
\leq& \left(\|\alpha\|_\infty + \sum_{(i,j)\in\Sigma_N\times\Sigma_N} \sup_{m\in\mathbb{N}} \frac{Osc(m,\alpha)}{2^{m(2-\beta)}} \right) \|g - h\|_\infty \\
&+ \Big(\sum_{(i,j)\in\Sigma_N\times\Sigma_N} \frac{\|\alpha\|_\infty}{2^{k(2-\beta)}} \Big) \sup_{m\in\mathbb{N}} \frac{Osc(m, g-h)}{2^{m(2-\beta)}} \\
\leq& \max\left\{ \|\alpha\|_\infty + N^2 \sup_{m\in\mathbb{N}} \frac{Osc(m,\alpha)}{2^{m(2-\beta)}}, \frac{N^2\|\alpha\|_\infty}{2^{k(2-\beta)}} \right\} \|g - h\|_{\mathcal{V}^\beta}.
\end{aligned}$$

Since $\max\left\{ \|\alpha\|_\infty + N^2 \sup_{m\in\mathbb{N}} \frac{Osc(m,\alpha)}{2^{m(2-\beta)}}, \frac{N^2\|\alpha\|_\infty}{2^{k(2-\beta)}} \right\} < 1$, Banach fixed point theorem yields that T has a unique fixed point $f^\alpha \in \mathcal{V}_f^\sigma(I^2)$. Furthermore, since $T(f^\alpha) = f^\alpha$, we write the following functional equation:

$$f^\alpha(P_{ij}(\boldsymbol{x})) = f(P_{ij}(\boldsymbol{x})) + \alpha(P_{ij}(\boldsymbol{x}))\ (f^\alpha - s)(\boldsymbol{x})$$

for every $\boldsymbol{x} \in I^2$ and $(i,j) \in \Sigma_N \times \Sigma_N$. Now we define functions $W_{ij} : I^2 \times \mathbb{R} \to I^2 \times \mathbb{R}$ for $(i,j) \in \Sigma_N \times \Sigma_N$ by

$$W_{ij}(x,y,z) = \Big(P_{ij}(\boldsymbol{x}), \alpha\big(P_{ij}(\boldsymbol{x})\big)z + f\big(P_{ij}(\boldsymbol{x})\big) - \alpha\big(P_{ij}(\boldsymbol{x})\big)s(\boldsymbol{x}) \Big).$$

We show in the last part of the proof that graph of the associated fractal function f^α is an attractor of the IFS $\{I^2 \times \mathbb{R}; W_{ij},\ (i,j) \in \Sigma_N \times \Sigma_N\}$. Following the proof of Theorem 1 appeared in [8] we may prove that attractor of the above IFS is graph of a function. It remains to show that it is actually graph of fractal perturbation f^α. To see that we use the functional equation and $I^2 = \cup_{(i,j)\in\Sigma_N\times\Sigma_N} u_i(I)\times v_j(I)$ and get

$$\begin{aligned}
&\cup_{(i,j)\in\Sigma_N\times\Sigma_N} W_{ij}(Gr(f^\alpha))\\
&\qquad\qquad =\cup_{(i,j)\in\Sigma_N\times\Sigma_N}\left\{W_{ij}(x,y,f^\alpha(\boldsymbol{x})) : \boldsymbol{x}\in I^2\right\}\\
&\qquad\qquad =\cup_{(i,j)\in\Sigma_N\times\Sigma_N}\Big\{\Big(P_{ij}(\boldsymbol{x}),\alpha\big(P_{ij}(\boldsymbol{x})\big)f^\alpha(\boldsymbol{x})\\
&\qquad\qquad\quad + f\big(P_{ij}(\boldsymbol{x})\big)-\alpha\big(P_{ij}(\boldsymbol{x})\big)s(\boldsymbol{x})\Big) : \boldsymbol{x}\in I^2\Big\}\\
&\qquad\qquad =\cup_{(i,j)\in\Sigma_N\times\Sigma_N}\Big\{\big(P_{ij}(\boldsymbol{x}), f^\alpha(P_{ij}(\boldsymbol{x})\big) : \boldsymbol{x}\in I^2\Big\}\\
&\qquad\qquad =\cup_{(i,j)\in\Sigma_N\times\Sigma_N}\big\{(x,y,f^\alpha(\boldsymbol{x})) : \boldsymbol{x}\in u_i(I)\times v_j(I)\big\}\\
&\qquad\qquad = Gr(f^\alpha).
\end{aligned}$$

From Theorem 5.2.3, we have $\overline{\dim}_B\big(Gr(f^\alpha)\big)\le 3-\beta$, completing the proof. □

Remark 5.2.8. *As mentioned in [94], the oscillation spaces can be treated as refinements of Hölder spaces. From which, we indicate that the above theorem is very interesting, and seems to be important when compared with [58, Theorem 5.19] and [11, Theorem 4.2].*

A function $f : \square \to \mathbb{R}$ is said to be Hölder continuous with exponent σ if

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| \le k_f\|\boldsymbol{x}-\boldsymbol{y}\|_2, \quad \forall\ \boldsymbol{x},\boldsymbol{y}\in\square,$$

and for some $k_f > 0$.

For Hölder continuous functions f with exponent σ, let us define σth Hölder seminorm as

$$[f]_\sigma = \sup_{\boldsymbol{x}\neq\boldsymbol{y}}\frac{|f(\boldsymbol{x})-f(\boldsymbol{y})|}{\|\boldsymbol{x}-\boldsymbol{y}\|_2^\sigma}$$

and consider the Hölder space

$$\mathcal{H}^\sigma(\square) := \{g : I\times J\to\mathbb{R} :\ \text{g is Hölder continuous with exponent } \sigma\}.$$

The space $\mathcal{H}^\sigma(\square)$ is a Banach space when endowed with the norm $\|g\|_\sigma := \|g\|_\infty + [g]_\sigma$.

We refer the reader to [5] for definitions and some properties of box dimension.

The next theorem is a modification of [58, Theorem 5.19]. Theorem 5.19 in [58] is proven with a constant scaling factor, but here we prove with a function scaling. We include it here with a detailed proof for reader's convenience.

Remark 5.2.9. *If* $f \in \mathcal{H}^{\sigma_0}(\square)$ *then* $f \in \mathcal{H}^\sigma(\square)$ *for each* $0 < \sigma < \sigma_0$.

Theorem 5.2.9. *Let* f *and* α *be Hölder continuous with exponent* σ_1 *and* σ_2 *respectively. Let* s *be Hölder continuous with exponent* σ_3 *satisfying* $s(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \forall\ \boldsymbol{x} \in \partial\Delta$. *If* $\max\left\{\|\alpha\|_\sigma, \frac{\|\alpha\|_\infty}{(\min\{|a_i|,|c_j|\})^\sigma}\right\} < 1$ *then* $2 \leq \dim_H\left(Gr(f^\alpha)\right) \leq \underline{\dim}_B\big(Gr(f^\alpha)\big) \leq \overline{\dim}_B\big(Gr(f^\alpha)\big) \leq 3 - \sigma$, *where* $\sigma = \min\{\sigma_1, \sigma_2, \sigma_3\}$.

Proof. From Remark 5.2.9, we say that f, α and s are elements of $\mathcal{H}^\sigma(\square)$, where $\sigma = \min\{\sigma_1, \sigma_2, \sigma_3\}$. Let us define $\mathcal{H}^\sigma_f(\square) = \{g \in \mathcal{H}^\sigma(\square) : g(\boldsymbol{x}) = f(\boldsymbol{x}), \quad \forall\ \boldsymbol{x} \in \partial\Delta\}$. One can check that the space $\mathcal{H}^\sigma_f(\square)$ is a closed subset of $\mathcal{H}^\sigma(\square)$. It follows that $\mathcal{H}^\sigma_f(\square)$ equipped with the obvious metric is a complete metric space. Now we proceed by defining a map $T : \mathcal{H}^\sigma_f(\square) \to \mathcal{H}^\sigma_f(\square)$ by

$$(Tg)(\boldsymbol{x}) = f(\boldsymbol{x}) + \alpha(\boldsymbol{x})\ (g - s)(Q_{ij}(\boldsymbol{x}))$$

for all $\boldsymbol{x} \in \square_{ij}$, where $(i,j) \in \Sigma$. Let $g, h \in \mathcal{H}^\sigma_f(\square)$. We have

$$\|Tg - Th\|_\sigma = \|Tg - Th\|_\infty + [Tg - Th]_\sigma \leq \max\left\{\|\alpha\|_\sigma, \frac{\|\alpha\|_\infty}{(\min\{|a_i|, |c_j|\})^\sigma}\right\}\|g - h\|_\sigma.$$

This with the condition $\max\left\{\|\alpha\|_\sigma, \frac{\|\alpha\|_\infty}{(\min\{|a_i|,|c_j|\})^\sigma}\right\} < 1$ yields that T is a contraction. From Banach fixed point theorem, T has a unique fixed point f^α in $\mathcal{H}^\sigma_f(\square)$. Since $f^\alpha \in \mathcal{H}^\sigma_f(\square)$, [55, Corollary 3.11] it deduces $\overline{\dim}\big(Gr(f^\alpha)\big) \leq 3 - \sigma$. This proves the result. $\square$

Remark 5.2.10. *The above theorem can be compared with [11, Theorem 4.2].*

5.3 Conclusion

This chapter has focused on the continuous dependence of the bivariate α-fractal function on parameters such as the scaling function α, the net $\triangle$ of the rectangular grid, and the base function s involved in its construction. We have also established results regarding its dimension. This chapter has also rigorously examined the dimensions of the graph of fractal interpolation functions defined on a rectangular grid under suitable hypotheses on IFS.

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