Chapter 5

Study of Existence Results for Fractional Functional Differential Equations Involving Riesz-Caputo Derivative

5.1 Introduction

In Chapter 3, we studied the existence and uniqueness of mild solutions of FDEs (3.1) and the controllability results of the fractional order control system (3.2) along with the Caputo fractional derivative. Apart from this, in Chapter 4, we extended this study and discussed the existence and approximate controllability of a fractional control system with delay along with the Caputo fractional derivative. However, in some physical phenomena that started in the past but are also dependent on their evolution in the future, such as geophysics, stock price options, control theory, mechanics, etc., there have been studies to support the Riesz derivatives being more appealing than the right or left derivatives as they can account for the synchronous impacts from either side of the domain[16, 17]. For example, to simulate the occurrence of anomalous diffusion within porous media (a physical phenomenon wherein the rate at which particles spread deviates from that predicted by classical Brownian motion), Riesz derivatives are more suitable. Further, taking the Riesz-Caputo derivative rather than the Riesz derivative in the Riemann-Liouville sense allows us to avoid a number of nonphysical challenges[119]. According to the findings presented in [18], the Caputo definition offers a solution to several challenges encountered using the Riemann-Liouville derivative. It addresses problems such as the non-zero derivative of a constant, mass balance error, hyper-singular improper integral, and the ill-posed nature of the fractional derivative related to the initial condition, thereby providing more accurate solutions. Furthermore, it has been demonstrated that the Caputo derivative, unlike the Riemann-Liouville derivative, exhibits a suitable extension of the extremum principle [19].

Incorporating the Riesz-Caputo derivative in FDEs enables the efficient and accurate representation of nonlocal dependencies and complex temporal behaviors. Now, we pay attention to the differential equations along with Riesz Caputo's fractional derivative. In particular, this chapter is concerned with the existence results for a family of FDDEs employing the Riesz-Caputo fractional derivative in a Banach space by utilizing FC techniques, Kuratowski's measure of noncompactness, Carath*é*odory conditions, and some theorems on fixed-points.

Due to the challenges associated with obtaining analytical and numerical solutions to such equations, particularly those involving both the right and left fractional derivatives at the same time, some researchers are investigating the existence and uniqueness of the solutions of various types of linear and non-linear FDEs. In the past few years, substantial progress has been achieved in establishing the controllability, existence, and uniqueness of various classes of FDEs [120, 121] and cited therein.

In 2021, Toprakseven [122] established existence and uniqueness results for solutions to a fractional BVP involving higher-order Riesz-Caputo derivative, using some fixed-point theorems. Zhang et al. [123] investigated a fractional BVP with $\tau \in [0,1]$ in 2019, and they studied the existence of positive solutions using Krasnoselskii's and Leray-Schauder theorems related to fixed-points. In [124], the authors discussed some existence and uniqueness results under the self-similar form to the space-fractional diffusion equation involving Riesz derivative in the Caputo sense by employing some fixed-point theorems and FC techniques. Further, in [89], the authors explored the existence and uniqueness of solutions to a family fractional BVP consisting of second-order Riesz operators in the Caputo terms. The Banach fixed-point theorem was employed to establish the uniqueness-related theorem, while the demonstration of existence results utilized Krasnoselskii and Schaefer's fixed-point theorems.

Driven by the earlier discussion, in this chapter, we consider the following fractional delay differential equations:

$$\begin{cases} {}_0^{RC}D_t^{\eta}\mathsf{x}(t) = \mathsf{f}\big(t, \mathsf{x}(\mu_1(t)), \mathsf{x}(\mu_2(t)), \ldots, \mathsf{x}(\mu_n(t))\big), t \in [0,T], \\ \mathsf{x}(0) = \mathsf{x}_0, \quad \mathsf{x}(T) = \mathsf{x}_T, \end{cases} \tag{5.1}$$

where ${}_0^{RC}D_t^{\eta}$ is the Riesz-Caputo fractional derivative of order $\eta \in (0,1)$, $J = [0,T]$, and X is a Banach space. The continuous functions $\mu_i : [0,T] \to [0,T]$ are delay terms satisfying the conditions $0 \leq \mu_i(t) \leq t$ for $i = 1, 2, \ldots, n$, $n \in \mathbb{N}$. We note that, $\mathsf{f} : [0, T] \times X^n \to X$ is a continuous function.

We aim to explore a more profound investigation in this area and derive additional results beyond those already established. Specifically, under certain assumptions on f, we analyze the existence of solutions to the equation (5.1). Further, we provide illustrative examples relevant to our findings. We utilized tools from fractional calculus, nonlinear analysis, and fixed-point theory to achieve these results.

This chapter is structured as follows: Section 5.2 begins with a discussion of fundamental definitions, theorems, and lemmas essential for establishing the main results. In Section 5.3, we present three existence theorems for the solutions of the considered differential equation (5.1) based on various sets of assumptions on f. In Section 5.4., we discuss some examples that illustrate our findings. Finally, in Section 5.5, we give the conclusion.

5.2 Preliminary Results

This section introduces some definitions and basic results which will be used throughout this chapter.

Definition 5.1. [125] **(Carath*é*odory conditions)** A map ψ from $[0, T] \times X^n$ to X fulfill the Carath*é*odory conditions if:

(a) $\psi(\cdot, \mu_1, \mu_2, \cdot, \mu_n)$ is measurable $\forall$ $\mu_i \in X$, $(i = 1, 2, \ldots, n)$.

(b) $\psi(t, \cdot, \cdot, \ldots, \cdot)$ is continuous for *a.e.* $t \in [0, T]$, .

(c) Let $B_{\mathsf{r}} = \{\mu \in X : \|\mu\| \leq \mathsf{r}\}$, then for any $\mathsf{r} > 0$, $\exists\ \mathsf{g}_{\mathsf{r}} \in L^1[0,T]$ such that

$$\sup_{\mu_i \in \mathcal{B}_{\mathsf{r}}} \|\psi(\tau, \mu_1, \mu_2, \ldots, \mu_n)\| \leq \mathsf{g}_{\mathsf{r}}(t),\ \ a.e.\ \tau \in [0,T].$$

Definition 5.2. [126, 127, 128] **(Kuratowski measure of noncompactness)** Let $\nu : \mathcal{K} \to \mathbb{R}^+$, where $\mathcal{K}$ be a subset of Banach space X, and $\mathcal{K}$ is bounded. Then, the Kuratowski measure of noncompactness ν is given by:

$$\begin{aligned}\nu(\mathcal{K}) &= \inf\{\mathsf{d} > 0 : \mathcal{K} \text{ can be covered by a finite number of sets of diameter} \leq \mathsf{d}\}\\ &= \inf\{\mathsf{d} > 0 : \mathcal{K} = \cup_{i=1}^{n}\mathcal{K}_i \text{ and diameter}(\mathcal{K}_i) \leq \mathsf{d} \text{ for } i = 1, 2, \ldots, n\}.\end{aligned}$$

Lemma 5.3. *[126, 127, 128] Given a Banach space X and bounded subsets $\mathcal{G}_1$ and $\mathcal{G}_2$ of X, the Kuratowski measure of noncompactness possesses the following characteristics:*

(1) $\mathcal{G}_1$ is precompact iff $\nu(\mathcal{G}_1) = 0$.

(2) For $\mathrm{c} \in \mathbb{R}$, $\nu(\mathrm{c}\mathcal{G}_1) = |\mathrm{c}|\nu(\mathcal{G}_1)$.

(3) $\nu(\mathcal{G}_1) = \nu(\overline{\mathcal{G}_1}) = \nu(\overline{\mathcal{C}o\mathcal{G}_1})$.

(4) For $\mathcal{G}_1 \subset \mathcal{G}_2$, $\nu(\mathcal{G}_1) \leq \nu(\mathcal{G}_2)$.

(5) $\nu(\mathcal{G}_1 \cup \mathcal{G}_2) \leq \max\big\{\nu(\mathcal{G}_1), \nu(\mathcal{G}_2)\big\}$.

(6) Let $\mathcal{G}_1 + \mathcal{G}_2 = \{\mathsf{g} : \mathsf{g} = \mathsf{g}_1 + \mathsf{g}_2,\ \mathsf{g}_1 \in \mathcal{G}_1, \mathsf{g}_2 \in \mathcal{G}_2\}$. Then, $\nu((\mathcal{G}_1 + \mathcal{G}_2)) \leq \nu(\mathcal{G}_1) + \nu(\mathcal{G}_2)$.

(7) Suppose U is a Banach space, and the function ψ from $D(\psi) \subseteq X$ to U holds the Lipschitz condition with the Lipschitz constant q. Then, for any $\mathcal{G} \subset D(\psi)$, where $\mathcal{G}$ is bounded, $\nu(\psi(\mathcal{G})) \leq \mathrm{q}\nu(\mathcal{G})$.

(8) Consider a decreasing sequence of nonempty, closed, and bounded subsets of a Banach space X denoted by $\{\mathcal{G}_i\}_{i=1}^{\infty}$, and $\lim_{i\to\infty}\nu(\mathcal{G}_i) = 0$, then $\cap_{i=1}^{\infty}\mathcal{G}i$ is nonempty and compact in X.

We refer [126, 127, 128] to gain a more profound understanding of the Kuratowski measure of noncompactness and its associated characteristics.

Lemma 5.4. *[129] Suppose $\mathcal{G}$ is an equicontinuous and bounded subset of $C(J, X)$, then so is $\overline{\mathcal{C}o\mathcal{G}}$.*

Lemma 5.5. *[2, 130] Let $\mathcal{G} \subset X$ be bounded, where X is Banach space. So for a countable subset $\mathcal{G}_0$ of $\mathcal{G}$, $\nu(\mathcal{G}) \leq 2\nu(\mathcal{G}_0)$.*

Lemma 5.6. *[131, 132] Suppose $\mathcal{G} \subset C(J, X)$ is equicontinuous and bounded, then $\nu(\mathcal{G}(\tau))$ is continuous on J and satisfies the following inequlity:*

$$\nu\left(\int_J \mathcal{G}(p)dp\right) \leq \int_J \nu(\mathcal{G}(p))dp.$$

Lemma 5.7. *[127] Consider $\mathcal{G} \subset C(J, X)$, where $\mathcal{G}$ is equicontinuous and bounded. Then, $\nu(\mathcal{G}(t))$ is continuous for $t \in J$ along with $\nu(\mathcal{G}) = \max_{t\in J}\nu(\mathcal{G}(t))$.*

Definition 5.8. [128] Let $D \subset X$, where X is a Banach space. Then, a continuous function ψ from D to X is called strict set contraction if $\nu(\psi(\mathcal{G})) \leq \mathsf{q}\nu(\mathcal{G})$ for some $0 \leq \mathsf{q} < 1$ and all bounded sets $\mathcal{G} \subset D$.

Lemma 5.9. *[128] Suppose $\mathcal{G}$ is a bounded, closed, and convex set contained in a Banach space X. If ψ is a strict set contraction mapping from $\mathcal{G}$ to itself, then it possesses a fixed-point in $\mathcal{G}$.*

Lemma 5.10. *[82] Suppose ψ is n times continuously differentiable function over the interval $[0, T]$, then*

$${}_0I^{\eta}_{t}\,{}^{\mathcal{RC}}_{0}D^{\eta}_{t}\psi(t) = \psi(t) - \sum_{k=0}^{n-1} \frac{\psi^{(k)}(0)}{k!}(t-0)^k,$$

and

$${}_{\tau}I^{\eta}_{T}\,{}^{\mathcal{RC}}_{\tau}D^{\eta}_{T}\psi(t) = (-1)^n\left[\psi(t) - \sum_{k=0}^{n-1} \frac{(-1)^k\psi^{(k)}(T)}{k!}(T-t)^k\right].$$

5.3 Existence Results

In this section, we introduce some pertinent findings regarding the existence results to the system (5.1) under various set assumptions on f.

Lemma 5.11. *The boundary value problem (5.1) can be reformulated as the following integral equation:*

$$\begin{aligned}\mathsf{x}(t) &= \frac{1}{2}(\mathsf{x}_0 + \mathsf{x}_T)\\ &+ \frac{1}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)\,dp\\ &+ \frac{1}{\Gamma(\eta)}\int_t^T (p-t)^{\eta-1}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)\,dp. \qquad (5.2)\end{aligned}$$

Proof. By definition (1.11), we have

$${}_0I^{\eta}_{T}\,{}^{\mathcal{RC}}_{0}D^{\eta}_{T}\mathsf{x}(t) = \frac{1}{2}\big({}_0I^{\eta}_{t}\,{}^{\mathcal{C}}_{0}D^{\eta}_{t} + (-1)^n\,{}_tI^{\eta}_{T}\,{}^{\mathcal{C}}_{t}D^{\eta}_{T}\big)\mathsf{x}(t)$$

where ${}_0I_t^{\eta}$ and ${}_tI_T^{\eta}$ are the left, right Rieamann-Liouville integrals respectively. Now, applying Lemma 5.10 for $0 < \eta \leq 1$, we have

$$ {}_0I_T^{\eta}\, {}_0^{\mathcal{RC}}D_T^{\eta}\mathsf{x}(t) = \mathsf{x}(t) - \frac{1}{2}\big[\mathsf{x}(0) + \mathsf{x}(T)\big]. $$

This gives

$$ \mathsf{x}(t) = \frac{1}{2}\big[\mathsf{x}(0) + \mathsf{x}(T)\big] +_0 I_T^{\eta}\, {}_0^{\mathcal{RC}}D_T^{\eta}\mathsf{x}(t). $$

Now, using equation (5.1), in the above equation we have

$$ \begin{aligned} \mathsf{x}(t) = {} & \frac{1}{2}(\mathsf{x}_0 + \mathsf{x}_T) \\ & + \frac{1}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)\,dp \\ & + \frac{1}{\Gamma(\eta)}\int_t^T (p-t)^{\eta-1}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)\,dp. \end{aligned} \tag{5.3} $$

This completes the proof. □

Theorem 5.12. *Let us assume the following conditions:*

(R_1) *Suppose* f *from* $[0,T]\times X^n$ *to* X *is a continuous function and there exist non-negative constants* $\mathcal{P}_i$ *such that*

$$ \begin{aligned} \Big\|\mathsf{f}\big(t, \mathsf{x}(\mu_1(t)), \mathsf{x}(\mu_2(t)), \ldots, \mathsf{x}(\mu_n(t))\big) - \mathsf{f}\big(t, \tilde{\mathsf{x}}(\mu_1(t)), \tilde{\mathsf{x}}(\mu_2(t)), \ldots, \tilde{\mathsf{x}}(\mu_n(t))\big)\Big\| \\ \leq \sum_{i=1}^{n}\mathcal{P}_i\big\|\mathsf{x}(\mu_i(t)) - \tilde{\mathsf{x}}(\mu_i(t))\big\|, \end{aligned} $$

for all $\mathsf{x}\big(\mu_i(t)\big), \tilde{\mathsf{x}}\big(\mu_i(t)\big) \in X$ *and* $\mathcal{P} = \max_{t\in J}\big\|\mathsf{f}(t, 0, \ldots, 0)\big\|$.

(R_2) *Let* $\frac{1}{2}||\mathsf{x}_0 + \mathsf{x}_T|| + \gamma\mathcal{N}_1 \leq \mathfrak{m}$ *for some* $\mathfrak{m} > 0$*, where* $\mathcal{N}_1 = \sum_{i=1}^{n} \mathfrak{m}\mathcal{P}_i + \mathcal{P}$, $\gamma = \frac{2T^\eta}{\Gamma(\eta+1)}$ *and* $\rho = \gamma \sum_{i=1}^{n} \mathcal{P}_i$ *be such that* $0 \leq \rho < 1$.

Then, for any $\mathsf{x}_0, \mathsf{x}_T \in X$*, there exists a solution* $\mathsf{x}^* \in C(J, X)$ *to the system* (5.1).

Proof. Suppose $\mathcal{H}_{\mathsf{m}}$=$\{\mathsf{x} : \mathsf{x} \in X$ with $||\mathsf{x}|| \leq \mathsf{m}$ for $t \in J$, $\mathsf{x}(0) = \mathsf{x}_0$ and $\mathsf{x}(T) = \mathsf{x}_T\}$. Now, let us define the operator $\Phi : \mathcal{H}_{\mathsf{m}} \to \mathcal{H}_{\mathsf{m}}$ as follows:

$$\begin{aligned}\Phi\mathsf{x}(t) = &\frac{1}{2}(\mathsf{x}_0 + \mathsf{x}_T)\\ &+ \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{\eta-1} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)\, dp\\ &+ \frac{1}{\Gamma(\eta)} \int_t^T (p-t)^{\eta-1} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)\, dp.\end{aligned}$$

First, we prove that Φ maps $\mathcal{H}_{\mathsf{m}}$ to $\mathcal{H}_{\mathsf{m}}$. Now, for all $\mathsf{x} \in \mathcal{H}_{\mathsf{m}}$, we have

$$\begin{aligned}\|\Phi\mathsf{x}(t)\| \leq &\frac{1}{2}\|\mathsf{x}_0 + \mathsf{x}_T\|\\ &+ \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{(\eta-1)} \big\|\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)\big\| dp\\ &+ \frac{1}{\Gamma(\eta)} \int_t^T (p-t)^{(\eta-1)} \big\|\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)\big\| dp\\ \leq &\frac{1}{2}\|\mathsf{x}_0 + \mathsf{x}_T\|\\ &+ \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{(\eta-1)} \Big[\big\|\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)\\ &- \mathsf{f}(p, 0, \ldots, 0)\big\| + \|\mathsf{f}(p, 0, \ldots, 0)\|\Big] dp\\ &+ \frac{1}{\Gamma(\eta)} \int_t^T (p-t)^{(\eta-1)} \Big[\big\|\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)\\ &- \mathsf{f}(p, 0, \ldots, 0)\big\| + \|\mathsf{f}(p, 0, \ldots, 0)\|\Big] dp\\ \leq &\frac{1}{2}\|(\mathsf{x}_0 + \mathsf{x}_T)\|\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{(\eta-1)} \Bigg[\sum_{i=1}^{n} \mathcal{P}_i \|\mathsf{x}(\mu_i(p))\| + \mathcal{P} \Bigg] dp \\
&+ \frac{1}{\Gamma(\eta)} \int_t^T (p-t)^{(\eta-1)} \Bigg[\sum_{i=1}^{n} \mathcal{P}_i \|\mathsf{x}(\mu_i(p))\| + \mathcal{P} \Bigg] dp \\
\leq\ & \frac{1}{2} \|\mathsf{x}_0 + \mathsf{x}_T\| \\
&+ \frac{T^\eta}{\Gamma(\eta+1)} \Bigg[\sum_{i=1}^{n} \mathcal{P}_i \|\mathsf{x}(\mu_i(p))\| + \mathcal{P} \Bigg] + \frac{T^\eta}{\Gamma(\eta+1)} \Bigg[\sum_{i=1}^{n} \mathcal{P}_i \|\mathsf{x}(\mu_i(p))\| + \mathcal{P} \Bigg] \\
\leq\ & \frac{1}{2} \|\mathsf{x}_0 + \mathsf{x}_T\| + \frac{2T^\eta}{\Gamma(\eta+1)} \Bigg[\sum_{i=1}^{n} \mathcal{P}_i \mathsf{m} + \mathcal{P} \Bigg] \\
\leq\ & \frac{1}{2} \|\mathsf{x}_0 + \mathsf{x}_T\| + \gamma \mathcal{N}_1 \\
\leq\ & \mathsf{m}.
\end{aligned}
$$

Therefore, Φ maps $\mathcal{H}_{\mathsf{m}}$ into itself. Now, we show that Φ is a contraction mapping on $\mathcal{H}_{\mathsf{m}}$. For all $\mathsf{x}, \tilde{\mathsf{x}} \in \mathcal{H}_{\mathsf{m}}$, we obtain

$$
\begin{aligned}
&\|\Phi\mathsf{x}(t) - \Phi\tilde{\mathsf{x}}(t)\| \\
&\quad \leq \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{(\eta-1)} \big\| \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) \\
&\qquad - \mathsf{f}\big(p, \tilde{\mathsf{x}}(\mu_1(p)), \tilde{\mathsf{x}}(\mu_2(p)), \ldots, \tilde{\mathsf{x}}(\mu_n(p))\big) \big\| dp \\
&\qquad + \frac{1}{\Gamma(\eta)} \int_t^T (p-t)^{(\eta-1)} \big\| \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) \\
&\qquad - \mathsf{f}\big(p, \tilde{\mathsf{x}}(\mu_1(p)), \tilde{\mathsf{x}}(\mu_2(p)), \ldots, \tilde{\mathsf{x}}(\mu_n(p))\big) \big\| dp \\
&\quad \leq \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{(\eta-1)} \sum_{i=1}^{n} \mathcal{P}_i \big\| \mathsf{x}(\mu_i(p)) - \tilde{\mathsf{x}}(\mu_i(p)) \big\| dp \\
&\qquad + \frac{1}{\Gamma(\eta)} \int_t^T (p-t)^{(\eta-1)} \sum_{i=1}^{n} \mathcal{P}_i \big\| \mathsf{x}(\mu_i(p)) - \tilde{\mathsf{x}}(\mu_i(p)) \big\| dp \\
&\quad \leq \frac{T^\eta}{\Gamma(\eta+1)} \sum_{i=1}^{n} \mathcal{P}_i \ \|\mathsf{x} - \tilde{\mathsf{x}}\| + \frac{(T-t)^\eta}{\Gamma(\eta+1)} \sum_{i=1}^{n} \mathcal{P}_i \|\mathsf{x} - \tilde{\mathsf{x}}\| \\
&\quad \leq \frac{2T^\eta}{\Gamma(\eta+1)} \sum_{i=1}^{n} \mathcal{P}_i \|\mathsf{x} - \tilde{\mathsf{x}}\|
\end{aligned}
$$

$$\leq \left[\gamma \sum_{i=1}^{n} \mathcal{P}_i\right] \|\mathsf{x} - \tilde{\mathsf{x}}\|$$
$$\leq \rho \|\mathsf{x} - \tilde{\mathsf{x}}\|,$$

which shows that

$$\|\Phi\mathsf{x}(t) - \Phi\tilde{\mathsf{x}}(t)\| \leq \rho \|\mathsf{x} - \tilde{\mathsf{x}}\|.$$

From the above discussion, we deduce that Φ is a contraction mapping from $\mathcal{H}_{\mathsf{m}}$ to $\mathcal{H}_{\mathsf{m}}$. Consequently, the system (5.1) has a solution by the generalized Banach contraction principle. With this, we conclude the proof. □

In the subsequent Theorems 5.13 and 5.14, we demonstrate the existence result for the solution to the system (5.1) by using different set of assumptions on f.

Theorem 5.13. *Let the function* f *from* $J \times X^n$ *to* X *satisfy the Carathéodory conditions. Then,* $\forall\ \mathsf{x}_0, \mathsf{x}_T \in X$*, the boundary value problem* (5.1) *has a solution* $\mathsf{x}^* \in C(J, X)$.

Proof. Let us assume that $\Phi : C(J,X) \to C(J,X)$ defined by

$$\begin{aligned}\Phi\mathsf{x}(t) &= \frac{1}{2}(\mathsf{x}_0 + \mathsf{x}_T) \\ &+ \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{\eta-1} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)\, dp \\ &+ \frac{1}{\Gamma(\eta)} \int_t^T (p-t)^{\eta-1} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)\, dp. \end{aligned} \tag{5.4}$$

The proof consists of multiple steps.

Step 1: First, we show the continuity of $\Phi : C(J,X) \to C(J,X)$

Let the sequence $\{\mathsf{x}_m\}_{m\in\mathbb{N}} \subset C(J,X)$ such that $\mathsf{x}_m \to \mathsf{x}$ in $C(J,X)$. Then, there exists a positive integer $\mathsf{r}(\mathsf{r} > 1)$ in such a way that for all $m \in \mathbb{N}$, $\|\mathsf{x}_m\| \leq \mathsf{r}$. Define the set $\mathcal{B}(\mathsf{x},\mathsf{r})$ as follows:

$$\mathcal{B}(\mathsf{x},\mathsf{r}) = \big\{\mathsf{x} \in C\left(J, X\right) : \|\mathsf{x}\| \leq \mathsf{r}\big\}.$$

Since for each $t \in J$, we have $0 \leq \mu_i(t) \leq t$; $(i = 1, 2, \ldots, n, n+1)$. Thus, for $\mathsf{x}_m, \mathsf{x} \in \mathcal{B}(\mathsf{x},\mathsf{r})$, we get

$$\big\|\mathsf{x}_m\big(\mu_i(t)\big) - \mathsf{x}\big(\mu_i(t)\big)\big\| \leq \|\mathsf{x}_m - \mathsf{x}\| \to 0 \text{ as } m \to \infty.$$

Since, f satisfy the Carath*é*odory conditions. So, for *a.e.* $t \in [0, T]$, we have

$$\begin{aligned} &\mathsf{f}\big(t, \mathsf{x}_m(\mu_1(t)), \mathsf{x}_m(\mu_2(t)), \ldots, \mathsf{x}_m(\mu_n(t))\big) \\ &\qquad \to \mathsf{f}\big(t, \mathsf{x}(\mu_1(t)), \mathsf{x}(\mu_2(t)), \ldots, \mathsf{x}(\mu_n(t))\big) \text{ as } m \to \infty. \end{aligned}$$

For every $m \in \mathbb{N}$, we have

$$\begin{aligned} &\mathsf{f}\big(t, \mathsf{x}_m(\mu_1(t)), \mathsf{x}_m(\mu_2(t)), \ldots, \mathsf{x}_m(\mu_n(t))\big) \\ &\qquad \to \mathsf{f}\big(t, \mathsf{x}(\mu_1(t)), \mathsf{x}(\mu_2(t)), \ldots, \mathsf{x}(\mu_n(t))\big) \leq 2\mathsf{g}_\mathsf{r}(t). \end{aligned}$$

Further,

$$\begin{aligned} &\|\Phi\mathsf{x}_m(t) - \Phi\mathsf{x}(t)\| \\ &\qquad \leq \bigg\|\frac{1}{2}\big(\mathsf{x}_0 + \mathsf{x}_T\big) - \frac{1}{2}\big(\mathsf{x}_0 + \mathsf{x}_T\big) \\ &\qquad\quad + \frac{1}{\Gamma(\eta)}\int_0^t (t-p)^{(\eta-1)}\mathsf{f}\big(p, \mathsf{x}_m(\mu_1(p)), \mathsf{x}_m(\mu_2(p)), \ldots, \mathsf{x}_m(\mu_n(p))\big) \\ &\qquad\quad - \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)dp \\ &\qquad\quad + \frac{1}{\Gamma(\eta)}\int_t^T (p-t)^{(\eta-1)}\mathsf{f}\big(p, \mathsf{x}_m(\mu_1(p)), \mathsf{x}_m(\mu_2(p)), \ldots, \mathsf{x}_m(\mu_n(p))\big) \end{aligned}$$

$$
\begin{aligned}
&- \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp \Big\| \\
&\leq \Big\| \frac{1}{2}\big(\mathsf{x}_0 + \mathsf{x}_T\big) - \frac{1}{2}\big(\mathsf{x}_0 + \mathsf{x}_T\big) \Big\| \\
&+ \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{(\eta-1)} \Big\| \mathsf{f}\big(p, \mathsf{x}_m(\mu_1(p)), \mathsf{x}_m(\mu_2(p)), \ldots, \mathsf{x}_m(\mu_n(p))\big) \\
&- \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) \Big\| dp \\
&+ \frac{1}{\Gamma(\eta)} \int_t^T (p-t)^{(\eta-1)} \Big\| \mathsf{f}\big(p, \mathsf{x}_m(\mu_1(p)), \mathsf{x}_m(\mu_2(p)), \ldots, \mathsf{x}_m(\mu_n(p))\big) \\
&- \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) \Big\| dp \to 0 \quad \text{as} \quad m \to \infty.
\end{aligned}
$$

Therefore, Φ is continuous.

Step 2: Next, we show that$\Phi\big(\mathcal{B}(\mathsf{x}, \mathsf{r})\big)$ is equicontinuous for $\mathsf{x} \in \mathcal{B}(\mathsf{x}, \mathsf{r})$ and $t_1, t_2 \in J$ such that $t_1 < t_2$. Then,

$$
\begin{aligned}
&\|\Phi\mathsf{x}(t_2) - \Phi\mathsf{x}(t_1)\| \\
&= \Big\| \Big[-\frac{1}{\Gamma(\eta)} \int_0^{t_1} (t_1 - p)^{(\eta-1)} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp \\
&\quad - \frac{1}{\Gamma(\eta)} \int_{t_1}^T (p - t_1)^{(\eta-1)} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp \Big] \\
&\quad + \Big[\frac{1}{\Gamma(\eta)} \int_0^{t_2} (t_2 - p)^{(\eta-1)} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp \\
&\quad + \frac{1}{\Gamma(\eta)} \int_{t_2}^T (p - t_2)^{(\eta-1)} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp \Big] \Big\| \\
&\leq \Big\| \Big[\frac{1}{\Gamma(\eta)} \int_0^{t_1} (t_1 - p)^{(\eta-1)} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp \\
&\quad - \frac{1}{\Gamma(\eta)} \int_0^{t_2} (t_2 - p)^{(\eta-1)} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp \Big] \Big\| \\
&\quad + \Big\| \Big[\frac{1}{\Gamma(\eta)} \int_{t_1}^T (p - t_1)^{(\eta-1)} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp \\
&\quad - \frac{1}{\Gamma(\eta)} \int_{t_2}^T (p - t_2)^{(\eta-1)} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp \Big] \Big\| \\
&= I_1 + I_2.
\end{aligned}
$$

We have

$$\begin{aligned}
I_1 = \Bigg\| & \frac{1}{\Gamma(\eta)} \int_0^{t_1} (t_1 - p)^{(\eta-1)} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp \\
& - \frac{1}{\Gamma(\eta)} \int_0^{t_1} (t_2 - p)^{(\eta-1)} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp \\
& - \frac{1}{\Gamma(\eta)} \int_{t_1}^{t_2} (t_2 - p)^{(\eta-1)} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp \Bigg\| \\
\leq & \frac{1}{\Gamma(\eta)} \int_0^{t_1} \big[(t_1 - p)^{\eta-1} - (t_2 - p)^{\eta-1}\big] \mathsf{g}_{\mathsf{r}}(p) dp \\
& + \frac{1}{\Gamma(\eta)} \int_{t_1}^{t_2} (t_2 - p)^{\eta-1} \mathsf{g}_{\mathsf{r}}(p) dp \to 0 \quad \text{as } t_1 \to t_2.
\end{aligned}$$

Similarly, we can say that $I_2 \to 0$ as $t_1 \to t_2$, and hence $\|\Phi\mathsf{x}(t_2) - \Phi\mathsf{x}(t_1)\| \to 0$ as $t_1 \to t_2$.
Therefore $\Phi(\mathcal{B}(\mathsf{x}, \mathsf{r}))$ is equicontinuous.

Step 3: Finally, we show that $\mathcal{Z}(t) = \{\Phi\big(\mathcal{B}(\mathsf{x}, \mathsf{r})\big) : \mathsf{x} \in \mathcal{B}(\mathsf{x}, \mathsf{r})\}$ is relatively compact in X. Note that $\mathcal{Z}(0)$ and $\mathcal{Z}(T)$ are relatively compact in X. Fixed $t \in (0, T)$, and for each $\mathsf{x} \in \mathcal{B}(\mathsf{x}, \mathsf{r})$, and $\delta \in (0, t)$ define

$$\begin{aligned}
\Phi_\delta \mathsf{x}(t) = & \frac{1}{2}\big(\mathsf{x}_0 + \mathsf{x}_T\big) \\
& + \frac{1}{\Gamma(\eta)} \int_0^{t-\delta} (t - p)^{\eta-1} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp \\
& + \frac{1}{\Gamma(\eta)} \int_{t-\delta}^{T} (p - t)^{\eta-1} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp
\end{aligned}$$

Here,

$$\begin{aligned}
\Bigg\| \int_0^{t-\delta} & (t - p)^{\eta-1} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp \Bigg\| \\
& \leq \int_0^{t-\delta} (t - p)^{\eta-1} \mathsf{g}_{\mathsf{r}}(p) dp,
\end{aligned}$$

and

$$\begin{aligned}\bigg\| \int_{t-\delta}^{T} &(p-t)^{\eta-1}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)dp \bigg\| \\ &\leq \int_{t-\delta}^{T} (p-t)^{\eta-1}\mathsf{g}_{\mathsf{r}}(p)dp,\end{aligned}$$

Since $(t-p)^{\eta-1}\mathsf{g}_{\mathsf{r}}(p)$ and $(p-t)^{\eta-1}\mathsf{g}_{\mathsf{r}}(p) \in L^1[0,T]$. Then, the set $\mathscr{Z}_\delta(t) = \{\Phi_\delta \mathsf{x}(t) : \mathsf{x} \in \mathcal{B}(\mathsf{x}, \mathsf{r})$ is relatively compact in X. For any $\mathsf{x} \in \mathcal{B}(\mathsf{x}, \mathsf{r})$, we get

$$\begin{aligned}&\|\Phi\mathsf{x}(t) - \Phi_\delta\mathsf{x}(t)\| \\ &\quad\leq \frac{1}{\Gamma(\eta)} \bigg\| \int_{t-\delta}^{t} (t-p)^{\eta-1}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)dp \\ &\qquad - \int_{t-\delta}^{t} (p-t)^{\eta-1}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)dp \bigg\| \\ &\quad\leq \frac{1}{\Gamma(\eta)} \int_{t-\delta}^{t} \Big[(t-p)^{\eta-1} - (p-t)^{\eta-1}\Big] \\ &\qquad \times \Big\| \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)dp \Big\| \\ &\quad\leq \frac{1}{\Gamma(\eta)} \int_{t-\delta}^{t} \big[(t-p)^{\eta-1} - (p-t)^{\eta-1}\big]\mathsf{g}_{\mathsf{r}}(p)dp.\end{aligned}$$

Hence, we have relatively compact sets $\{\Phi_\delta\mathsf{x}(t) : \mathsf{x} \in \mathcal{B}(\mathsf{x}, \mathsf{r})\}$ which are arbitrarily close to the set $\{\Phi\mathsf{x}(t) : \mathsf{x} \in \mathcal{B}(\mathsf{x}, \mathsf{r})\}$, for $t \in (0, T)$. Thus, $\{\Phi\mathsf{x}(t) : \mathsf{x} \in \mathcal{B}(\mathsf{x}, \mathsf{r})\}$ is relatively compact in X, $\forall\ t \in (0, T)$. Consequently, it is compact in X, $\forall\ t \in J$ as it is already compact at $t = 0, T$. Therefore, $\Phi : C(J, X) \to (J, X)$ is completely continuous as an implication of the Arzelá-Ascoli theorem.

Let $\mathsf{x} = \zeta(\Phi\mathsf{x})$. Then, for $0 < \zeta < 1$, we get

$$\begin{aligned}\mathsf{x}(t) = &\frac{\zeta}{2}\big(\mathsf{x}_0 + \mathsf{x}_T\big) \\ &+ \frac{\zeta}{\Gamma(\eta)} \int_{0}^{t} (t-p)^{\eta-1}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)dp\end{aligned}$$

$$+\frac{\zeta}{\Gamma(\eta)}\int_t^T (p-t)^{\eta-1}\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}(\mu_n(p))\big)dp,$$

and for $t\in J$, we get

$$\begin{aligned}\|\mathsf{x}(t)\| &= \frac{\zeta}{2}\big\|\mathsf{x}_0+\mathsf{x}_T\big\|\\ &\quad+\frac{\zeta}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1}\|\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}(\mu_n(p))\big)\|dp\\ &\quad+\frac{\zeta}{\Gamma(\eta)}\int_t^T (p-t)^{\eta-1}\|\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}(\mu_n(p))\big)\|dp\\ &\le \frac{\zeta}{2}\big\|\mathsf{x}_0+\mathsf{x}_T\big\|\\ &\quad+\frac{\zeta}{\Gamma(\eta)}\bigg[\int_0^t (t-p)^{\eta-1}\mathsf{g}_\mathsf{r}(p)dp+\int_t^T (p-t)^{\eta-1}\mathsf{g}_\mathsf{r}(p)dp\bigg]\\ &= \mathcal{M}\quad \text{(say)}.\end{aligned}$$

Thus, there exists a constant $\beta>\mathcal{M}$ satisfying $\|\mathsf{x}\|\le\beta$. Consider the set $\mathcal{B}=\{\mathsf{x}\in C\,(J,X):\|\mathsf{x}\|<\beta\}$. Clearly, there is no $\mathsf{x}\in\partial\mathcal{B}$ fulfilling the condition $\mathsf{x}=\zeta(\Phi\mathsf{x})$ for $\zeta\in(0,1)$. As a result of the Lemma 1.3, there exists a fixed-point $\mathsf{x}\in\overline{\mathcal{B}}$ for Φ, thus (5.1) has a solution. This completes the proof. □

Theorem 5.14. *Assume that the following hypotheses are met:*

(R_3) *Let the nonlinear function* f *from* $J\times T_{\mathcal{R}}{}^n$ *to* X *be continuous and bounded such that*

$$\lim_{\mathcal{R}\to\infty}\frac{\mathcal{M}(\mathcal{R})}{\mathcal{R}}<\frac{1}{T},\tag{5.5}$$

where $\mathcal{M}(\mathcal{R})=\sup\{\|\mathsf{f}(t,\mathsf{x}_1,\mathsf{x}_2,\ldots,\mathsf{x}_n)\|:(t,\mathsf{x}_1,\mathsf{x}_2,\ldots,\mathsf{x}_n)\in J\times T_{\mathcal{R}}^n\}$,

$T_{\mathcal{R}}=\{\mathsf{x}\in C\,(J,X):\|\mathsf{x}\|\le\mathcal{R}\}$.

(R_4) *For any equicontinuous and countable sets* $\mathcal{G}_i\subset X$ $(i=1,2,\ldots,n)$*, there are non-negative Lebesgue integrable functions* $\mathsf{g}_i\in L^1(J,\mathbb{R}^+)$*,* $(i=1,2,\ldots,n)$

such that for $t \in J$,

$$\nu\big(\mathsf{f}(t, \mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n)\big) \leq \sum_{i=1}^{n} \mathsf{g}_i(t)\nu(\mathcal{G}_i), \tag{5.6}$$

with

$$\frac{1}{\Gamma(\eta)} \sup_{t\in J} \left[\int_0^t (t-p)^{\eta-1} \sum_{i=1}^{n} \mathsf{g}_i(p)dp + \int_t^T (p-t)^{\eta-1} \sum_{i=1}^{n} \mathsf{g}_i(p)dp \right] < 1. \tag{5.7}$$

Then, for any $\mathsf{x}_0, \mathsf{x}_T \in X$, *there exists a solution* $\mathsf{x}^* \in C(J, X)$ *to the system* (5.1) .

Proof. Let $\Phi : C(J, X) \to C(J, X)$ defined by

$$\begin{aligned}\Phi\mathsf{x}(t) = {} & \frac{1}{2}(\mathsf{x}_0 + \mathsf{x}_T) \\ & + \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{\eta-1} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)dp \\ & + \frac{1}{\Gamma(\eta)} \int_t^T (p-t)^{\eta-1} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)dp. \end{aligned} \tag{5.8}$$

Here Φ is a mapping from $C(J, X)$ to $C(J, X)$, which is continuous due to the continuity of f in $J \times {T_{\mathcal{R}}}^n$. Now, let us consider a positive constant $\mathsf{r} > 0$ such that $\lim\limits_{\mathcal{R}\to\infty} \dfrac{\mathcal{M}(\mathcal{R})}{\mathcal{R}} < \mathsf{r} < \dfrac{1}{T}$. Based on condition (5.5), we can find another constant $\mathcal{R}_0 > 0$ such that

$$\mathcal{M}(\mathcal{R}) < \mathsf{r}\mathcal{R}_0, \quad \forall\ \mathcal{R} > \mathcal{R}_0. \tag{5.9}$$

Let $\mathcal{R}^* = \max\left\{\mathcal{R}_0, \frac{1}{2}(\mathsf{x}_0 + \mathsf{x}_T)\left[1 - \frac{2\mathsf{r}T^\eta}{\Gamma(\eta+1)}\right]^{-1}\right\}$, and $T_{\mathcal{R}^*}$ is a set defined as follows:

$$T_{\mathcal{R}^*} = \{\mathsf{x} \in C(J, X) : \|\mathsf{x}\| \leq \mathcal{R}^*\}.$$

Using equations (5.8) and (5.9), we can deduce the following:

$$
\begin{aligned}
\|\Phi \mathsf{x}\| \leq \frac{1}{2}\|\mathsf{x}_0 + \mathsf{x}_T\| \\
&+ \frac{1}{\Gamma(\eta)}\int_0^t (t-p)^{(\eta-1)}\big\|\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)\big\| dp \\
&+ \frac{1}{\Gamma(\eta)}\int_t^T (p-t)^{(\eta-1)}\big\|\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big)\big\| dp \\
&\leq \frac{1}{2}\|\mathsf{x}_0 + \mathsf{x}_T\| + \frac{1}{\Gamma(\eta)}\int_0^t (t-p)^{(\eta-1)}\mathcal{M}(\mathcal{R}) dp \\
&+ \frac{1}{\Gamma(\eta)}\int_t^T (p-t)^{(\eta-1)}\mathcal{M}(\mathcal{R}) dp \\
&\leq \frac{1}{2}\|\mathsf{x}_0 + \mathsf{x}_T\| + \frac{2\mathsf{r}T^\eta\mathcal{R}^*}{\Gamma(\eta+1)} \leq \mathcal{R}^*.
\end{aligned}
$$

This means $\Phi : T_{\mathcal{R}^*} \to T_{\mathcal{R}^*}$ is continuous and bounded.

Next, we show that $\Phi(T_{\mathcal{R}^*})$ is equicontinuous for $t_1, t_2 \in J$ such that $t_1 < t_2$:

$$
\begin{aligned}
&\|\Phi \mathsf{x}(t_2) - \Phi \mathsf{x}(t_1)\| \\
&= \Bigg\|\Bigg[- \frac{1}{\Gamma(\eta)}\int_0^{t_1} (t_1-p)^{(\eta-1)}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp \\
&- \frac{1}{\Gamma(\eta)}\int_{t_1}^T (p-t_2)^{(\eta-1)}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp\Bigg] \\
&+ \Bigg[\frac{1}{\Gamma(\eta)}\int_0^{t_2} (p-t_2)^{(\eta-1)}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp \\
&+ \frac{1}{\Gamma(\eta)}\int_{t_2}^T (p-t_2)^{(\eta-1)}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp\Bigg]\Bigg\| \\
&\leq \Bigg\|\Bigg[\frac{1}{\Gamma(\eta)}\int_0^{t_1} (t_1-p)^{(\eta-1)}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp \\
&- \frac{1}{\Gamma(\eta)}\int_0^{t_2} (t_2-p)^{(\eta-1)}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp\Bigg]\Bigg\| \\
&+ \Bigg\|\Bigg[\frac{1}{\Gamma(\eta)}\int_{t_1}^T (p-t_1)^{(\eta-1)}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)))\big) dp \\
&- \frac{1}{\Gamma(\eta)}\int_{t_2}^T (p-t_2)^{(\eta-1)}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp\Bigg]\Bigg\| \\
&= I_1 + I_2.
\end{aligned}
$$

We have

$$\begin{aligned}
I_1 = \Bigg\| & \frac{1}{\Gamma(\eta)} \int_0^{t_1} (t_1 - p)^{(\eta-1)} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp \\
& - \frac{1}{\Gamma(\eta)} \int_0^{t_1} (t_2 - p)^{(\eta-1)} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp \\
& - \frac{1}{\Gamma(\eta)} \int_{t_1}^{t_2} (t_2 - p)^{(\eta-1)} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p))\big) dp \Bigg\| \\
\leq & \frac{1}{\Gamma(\eta)} \int_0^{t_1} \big[(t_1 - p)^{\eta-1} - (t_2 - p)^{\eta-1}\big] \mathcal{M}(\mathcal{R}^*) dp \\
& + \frac{1}{\Gamma(\eta)} \int_{t_1}^{t_2} (t_2 - p)^{\eta-1} \mathcal{M}(\mathcal{R}^*) dp \to 0 \quad \text{as} \quad t_1 \to t_2.
\end{aligned}$$

Similarly, we can say that $I_2 \to 0$ as $t_1 \to t_2$, and hence $\|\Phi\mathsf{x}(t_2) - \Phi\mathsf{x}(t_1)\| \to 0$ as $t_1 \to t_2$, which shows $\Phi(\mathcal{B}_{\mathcal{R}^*})$ is equicontinuous. So, using Lemma 5.4 $\overline{\mathcal{C}o}\Phi(\mathcal{B}_{\mathcal{R}^*})$ is bounded and equicontinuous subset of $\Phi(\mathcal{B}_{\mathcal{R}^*})$.

Finally, we show that $\Phi : \overline{\mathcal{C}o}\Phi(\mathcal{B}_{\mathcal{R}^*}) \to \overline{\mathcal{C}o}\Phi(\mathcal{B}_{\mathcal{R}^*})$ is a strict set contraction mapping. Now, by means of Lemma 5.5, for any $\mathcal{G} \subset \overline{\mathcal{C}o\mathcal{G}}$, there is a countable set $\mathcal{G}_0 = \{\mathsf{x}_n\} \subset \mathcal{G}$ such that

$$\nu\big(\Phi(\mathcal{G})\big) \leq 2\nu\big(\Phi(\mathcal{G}_0)\big). \tag{5.10}$$

Since $\overline{\mathcal{C}o}\Phi(\mathcal{B}_{\mathcal{R}^*})$ is equicontinuous and bounded therefore $\mathcal{G}_0 \subset \overline{\mathcal{C}o}\Phi(\mathcal{B}_{\mathcal{R}^*})$ is equicontinuous and bounded.

Now, as the consequence of Lemma 5.6, we have

$$\begin{aligned}
\nu(\Phi(\mathcal{G}_0(t))) = \nu\Bigg[& \frac{1}{\Gamma(\eta)} \int_0^t (t - p)^{\eta-1} \mathsf{f}\big(p, \mathcal{G}_0(\mu_1(p)), \mathcal{G}_0(\mu_2(p)), \ldots, \mathcal{G}_0(\mu_n(p))\big) dp \\
& + \frac{1}{\Gamma(\eta)} \int_t^T (p - t)^{\eta-1} \mathsf{f}\big(p, \mathcal{G}_0(\mu_1(p)), \mathcal{G}_0(\mu_2(p)), \ldots, \mathcal{G}_0(\mu_n(p))\big) dp \Bigg] \\
\leq & \frac{1}{\Gamma(\eta)} \Bigg[\int_0^t (t - p)^{\eta-1} \nu\Big[\mathsf{f}\big(p, \mathcal{G}_0(\mu_1(p)), \mathcal{G}_0(\mu_2(p)), \ldots, \mathcal{G}_0(\mu_n(p))\big)\Big] dp \\
& + \int_t^T (p - t)^{\eta-1} \nu\Big[\mathsf{f}\big(p, \mathcal{G}_0(\mu_1(p)), \mathcal{G}_0(\mu_2(p)), \ldots, \mathcal{G}_0(\mu_n(p))\big) dp\Big]
\end{aligned}$$

$$\leq \frac{1}{\Gamma(\eta)}\Bigg[\int_0^t (t-p)^{\eta-1}\sum_{i=1}^{n}\mathsf{g}_i(p)\nu\big(\mathcal{G}_i(p)\big)\,dp + \int_t^T (p-t)^{\eta-1}\sum_{i=1}^{n}\mathsf{g}_i(p)\nu\big(\mathcal{G}_i(p)\big)\,dp\Bigg].$$

Now, through Lemma 5.7, we obtain $\nu(\Phi(\mathcal{G}_0)) = \max_{t\in J}\nu\big(\Phi(\mathcal{G}_0(t))\big)$. Therefore,

$$\begin{aligned}
\nu(\Phi(\mathcal{G})) &\leq 2\nu(\Phi(\mathcal{G}_0))\\
&\leq \frac{1}{\Gamma(\eta)}\Bigg[\int_0^t (t-p)^{\eta-1}\sum_{i=1}^{n}\mathsf{g}_i(p)\nu\big(\mathcal{G}_i(p)\big)\,dp\\
&\quad + \int_t^T (p-t)^{\eta-1}\sum_{i=1}^{n}\mathsf{g}_i(p)\nu\big(\mathcal{G}_i(p)\big)\,dp\Bigg]\\
&\leq \frac{1}{\Gamma(\eta)}\Bigg[\int_0^t (t-p)^{\eta-1}\sum_{i=1}^{n}\mathsf{g}_i(p)dp + \int_t^T (p-t)^{\eta-1}\sum_{i=1}^{n}\mathsf{g}_i(p)dp\Bigg]\nu(\mathcal{G}).
\end{aligned}$$

By equation (5.7), we obtain

$$\nu(\Phi(\mathcal{G})) \leq \nu(\mathcal{G}).$$

Hence, the mapping $\Phi : \overline{Co}\Phi(\mathcal{B}_{\mathcal{R}^*}) \to \overline{Co}\Phi(\mathcal{B}_{\mathcal{R}^*})$ is a strict set contraction mapping. As a consequence of Lemma 5.9, it follows that Φ possesses a fixed-point $\mathsf{u}^* \in \overline{Co}\Phi(\mathcal{B}_{\mathcal{R}^*})$, which is contained within the space $C(J,X)$. This concludes the proof.

□

5.4 Applications

A few illustrated instances of our suggested results are provided in this section.

Example 5.1. *Let's take a look at the following fractional differential equation:*

$$\begin{cases} {}^{\mathcal{RC}}_{0}D^{\frac{1}{2}}_{t}\mathsf{x}(t) = t + \dfrac{\sin(t)\mathsf{x}(\mu_1(t))}{100} + \dfrac{e^{-t}\mathsf{x}(\mu_2(t))}{50} + \dfrac{\cos(t)\mathsf{x}(\mu_3(t))}{100}, \\ t \in (0,1), \\ \mathsf{x}(0) = 0, \quad \mathsf{x}(1) = 2. \end{cases} \tag{5.11}$$

Here,

$$\mathsf{f}\big(t, \mathsf{x}(\mu_1(t)), \mathsf{x}(\mu_2(t)), \mathsf{x}(\mu_3(t))\big) = t + \frac{\sin(t)\mathsf{x}(\mu_1(t))}{100} + \frac{e^{-t}\mathsf{x}(\mu_2(t))}{50} + \frac{\cos(t)\mathsf{x}(\mu_3(t))}{100}$$

Let $\mathsf{x}\big(\mu_i(t)\big), \tilde{\mathsf{x}}\big(\mu_i(t)\big) \in X,\ i = 1,2,3$. We get

$$\begin{aligned} &\big\|\mathsf{f}\big(t, \mathsf{x}(\mu_1(t)), \mathsf{x}(\mu_2(t)), \mathsf{x}(\mu_3(t))\big) - \mathsf{f}\big(t, \tilde{\mathsf{x}}(\mu_1(t)), \tilde{\mathsf{x}}(\mu_2(t)), \tilde{\mathsf{x}}(\mu_3(t))\big)\big\| \\ &\qquad \leq \frac{1}{100}\big\|\mathsf{x}\big(\mu_1(t)\big) - \tilde{\mathsf{x}}\big(\mu_1(t)\big)\big\| + \frac{1}{50}\big\|\mathsf{x}\big(\mu_2(t)\big) - \tilde{\mathsf{x}}\big(\mu_2(t)\big)\big\| \\ &\qquad + \frac{1}{100}\big\|\mathsf{x}\big(\mu_3(t)\big) - \tilde{\mathsf{x}}\big(\mu_3(t)\big)\big\| \end{aligned}$$

Hence, (R_1) of Theorem 5.12 holds.

Clearly,

$$\mathcal{P} = \max_{t\in[0,1]} ||\mathsf{f}(t,0,0,0)|| = 1.$$

Now, if we take $\mathsf{m} > \left[\dfrac{1 + 2/\Gamma(3/2)}{1 - 2/25\Gamma(3/2)}\right] + 1$, then all the assumptions of Theorem 5.12 are met. So by making use of Theorem 5.12, equation (5.11) has a solution.

Example 5.2. *Let us explore the subsequent fractional differential equation:*

$$\begin{cases} {}^{\mathcal{RC}}_{0}D^{\eta}_{t}\mathsf{x}(t) = t\sin\big(\mathsf{x}(\mu_1(t)\big)\cos\big(\mathsf{x}(\mu_2(t)\big), \quad t \in (0,1), \\ \mathsf{x}(0) = \mathsf{x}_0, \quad \mathsf{x}(T) = \mathsf{x}_T. \end{cases} \tag{5.12}$$

Here, we have

$$\mathsf{f}\big(t, \mathsf{x}(\mu_1(t)), \mathsf{x}(\mu_2(t))\big) = t\sin\big(\mathsf{x}(\mu_1(t)\big)\cos\big(\mathsf{x}(\mu_2(t)\big)$$

Clearly, $\mathsf{f}\big(\cdot, \mathsf{x}(\mu_1(t)), \mathsf{x}(\mu_2(t))\big)$ is measurable, $\forall\ \mathsf{x}(\mu_i(t)) \in X$, $(i = 1, 2)$, and $\mathsf{f}(t, \cdot, \cdot)$ is continuous for all $t \in [0, 1]$.
Now, for any $\mathsf{r} > 0$, $\exists\ \mathsf{g}_\mathsf{r}(t) = e^t \in L^1[0, 1]$ such that

$$\sup_{\mathsf{x}_i \in \mathcal{B}_\mathsf{r}} \|\mathsf{f}\big(t, \mathsf{x}(\mu_1(t)), \mathsf{x}(\mu_1(t))\big)\| \leq e^t \quad \forall\ t \in [0, 1],$$

where $\mathcal{B}_\mathsf{r} = \{\mathsf{x} \in X : \|\mathsf{x}\| \leq \mathsf{r}\}$. So, by Theorem 5.13, equation (5.12) has at least a solution.

Example 5.3. *Let us consider the following differential equations:*

$$\begin{cases} {}_0^{RC}D_t^{\eta}\mathsf{x}(t) = \lambda \displaystyle\sum_{i=1}^{n} \mathsf{x}_i(t), t \in [0, T], \\ \mathsf{x}(0) = \mathsf{x}_0, \quad \mathsf{x}(T) = \mathsf{x}_T, \end{cases} \tag{5.13}$$

where $\lambda > 0$, *and* $\|\mathsf{x}_i(t)\| \leq \mathcal{R}, (i = 1, 2, \ldots, n)$.

Clearly, $\mathsf{f} : J \times T_\mathcal{R} \to X$ such that

$$\big\|\mathsf{f}\big(t, \mathsf{x}_1(t), \mathsf{x}_2(t), \ldots, \mathsf{x}_n(t)\big)\big\| \leq \lambda \sum_{i=1}^{n} \big\|\mathsf{x}_i(t)\big\| \leq n\lambda\mathcal{R}.$$

So, we get $\mathcal{M}(\mathcal{R}) \leq n\lambda\mathcal{R} \implies \dfrac{\mathcal{M}(\mathcal{R})}{\mathcal{R}} \leq n\lambda$ Taking $\mathcal{R} \to \infty$ we get

$$\lim_{\mathcal{R}\to\infty} \frac{\mathcal{M}(\mathcal{R})}{\mathcal{R}} \leq n\lambda$$

If we choose λ such that $n\lambda < \frac{1}{T}$, i.e., $\lambda < \frac{1}{nT}$, then $\big(R_3\big)$ the condition of Theorem 5.14 is satisfied. Let $\mathcal{P}$ be the kernel of Kuratowski's measure of noncompactness on the Banach space $\big(X, \|\cdot\|\big)$, with zero element θ, and $\{\theta\} \in \mathcal{P}$. Now, if we choose λ such that $\lambda \leq \frac{1}{n}$, then according to arguemnt presented in [133] for any equicontinuous and countable sets $\mathcal{G}_i \subset X, (i = 1, 2, \ldots, n)$ we have

$$\nu\big(\mathsf{f}(t, \mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n)\big) \leq \nu\bigg(\lambda \sum_{i=1}^{n} \mathcal{G}_i\bigg) \leq \lambda \sum_{i=1}^{n} \nu(\mathcal{G}_i), \tag{5.14}$$

where λ is non-negative Lebesgue integrable function. Again,

$$\begin{aligned}
&\frac{1}{\Gamma(\eta)} \sup_{t\in J} \left[\int_0^t (t-p)^{\eta-1} \sum_{i=1}^{n} \mathsf{g}_i(p)dp + \int_t^T (p-t)^{\eta-1} \sum_{i=1}^{n} \mathsf{g}_i(p)dp\right] \\
&\leq \frac{1}{\Gamma(\eta)} \sup_{t\in J} \left[\int_0^t (t-p)^{\eta-1} \sum_{i=1}^{n} \lambda dp + \int_t^T (p-t)^{\eta-1} dp\right] \\
&\leq \frac{n\lambda}{\Gamma(\eta)} \sup_{t\in J} \left[\int_0^t (t-p)^{\eta-1} dp + \int_t^T (p-t)^{\eta-1} \sum_{i=1}^{n} \lambda dp\right] \\
&\leq \frac{2n\lambda T^{\eta}}{\Gamma(\eta+1)}.
\end{aligned}$$

Now, for $\lambda < \min\left\{\frac{1}{nT}, \frac{1}{n}, \frac{\Gamma(\eta+1)}{2nT^{\eta}}\right\}$ all the conditions of Theorem 5.14 will be satisfied, and hence equation (5.13) has a solution.

5.5 Conclusion

This chapter investigates the existence of solutions for fractional delay integro-differential equations in Banach space along with the Riesz-Caputo fractional derivative. Employing FC techniques, Kuratowski's measure of noncompactness, Carathéodory conditions, and multiple fixed-point theorems, we establish a few results for the existence of solutions. A few instances are presented at the conclusion to demonstrate the competence of the suggested outcomes.

****