Chapter 3

Two-Dimensional Wavelets Collocation Method for Electromagnetic Waves in Dielectric Media

3.1 Introduction

Wavelets are powerful tool which have been used in numerical techniques. Nowadays, wavelets theory are mostly used in the field of applied science and engineering. Also, this allow the accurate representation of a variety of functions and operators. Recently, wavelets have been found their location in many applications (see for instant [111; 112; 113]). Particularly, wavelets are very successfully used in signal analysis [114]. It is proved that wavelets are powerful tool to explore new direction in solving partial differential equations. Wavelets are localized functions [115], which are the basis for energy-bounded functions [116] and in particular for $L^2(R)$. So, we have implemented orthogonal wavelet functions in our proposed method. The most frequently used orthogonal functions are Legendre function [117], Chebyshev [118], Laguerre polynomials [119], etc.. The main notion of using an orthogonal basis is that the problem under consideration reduces to a system of linear or nonlinear algebraic equations. This can be done by truncated series of orthogonal basis function for the solution of the problem using the operational matrices (see for instant [120; 121; 122]). It is noted that wavelets operational matrix method is not only simplifies the problem but also speedup the computation. Therefore, in the last two decades different families of wavelets have been widely for solving fractional partial differential equations (FPDEs).

FPDEs have been one of the essential tools for various areas of applied Mathematics (see for instant [111; 112; 113]). FPDEs occur naturally in many field of science and engineering. In recent years fractional derivatives have found numerous applications in many fields of physics, finance and hydrology [123]. Also, fractional analysis has established so many applications in recent studies in mechanics, and physical sciences phenomena in area like diffusion process [124], electrochemistry [125], arterial sciences [126], the theory of ultra-slow processes [127], etc. Fractional derivatives provide an excellent instruments for the description of memory and hereditary properties of various materials and processes. A great deal of effort has been expanded over the last 15 years or so in attempting to find robust and stable numerical and analytical methods for solving FPDEs of physical interest.

In this chapter, we present a numerical wavelet collocation method (NWCM) by using two wavelets to solving FPDEs arising from electromagnetic wave in dielectric media (EMWDM) (see [128]) as follows:

$$({}_0D_t^{\alpha}u)(t,x) - \lambda_1({}_0D^{\beta}u)(t,x) - \lambda_2\nabla^2 u(t,x) = f(t,x), \tag{3.1}$$

with initial condition

$$u(t,x) = 0, \forall\ t \leqslant 0, \quad u(t,x) \neq 0, \forall\ 0 < t < 1, 0 < x < 1,$$

where, ∇^2 is Laplace operator, λ_1 and λ_2 are known constant coefficients depend on the frequency independent properties of medium and $1 \leq \beta < \alpha < 3$. Also, both the fractional derivative present in $Eq.$(3.1) are defined in the Riemann-Liouville derivative sense. In $Eq.$(3.1), $u(t,x)$ is unknown function which will be determined in this chapter.

3.2 Function approximation

Suppose that $f(t,x)$ is an arbitrary function in $L^2(\Omega)$, then it can be approximated as follows:

$$f(t,x)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{\infty}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=0}^{\infty} f_{nmn'm'}\Phi_{nmn'm'}(t,x). \tag{3.2}$$

If the infinite series (3.2) is truncated for $m=M-1, m'=M'-1$, then approximation of (3.2) can be represented as in the following form

$$f(t,x)\approx\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sum_{n'=1}^{2^{k'}-1}\sum_{m'=0}^{M'-1} f_{nmn'm'}\Phi_{nmn'm'}(t,x)=F^T\Phi(t,x). \tag{3.3}$$

Where, F and Φ are $2^{k-1}2^{k'-1}MM'\times 1$, vector given as follows:

$$\begin{aligned}
F=[&f_{1,0,1,0},\cdots,f_{1,0,1,M'-1},f_{1,0,2,0}\cdots,f_{1,0,2,M'-1},\cdots,f_{1,0,2^{k'-1},0},\cdots,\\
&f_{1,0,2^{k'-1},M'-1},\cdots,f_{1,M-1,1,0},\cdots,f_{1,M-1,1,M'-1},f_{1,M-1,2,0},\cdots,f_{1,M-1,2,M'-1},\\
&f_{1,M-1,2^{k'-1},0}\cdots,f_{1,M-1,2^{k'-1},M'-1},f_{2,0,1,0},\cdots,f_{2,0,1,M'-1},f_{2,0,2,0},\cdots,\\
&f_{2,0,2,M'-1},\cdots,f_{2,0,2^{k'-1},0},\cdots,f_{2,0,2^{k'-1},M'-1},\cdots,f_{2,M-1,1,0},\cdots,f_{2,M-1,1,M'-1},\\
&f_{2,M-1,2,0},\cdots,f_{2,M-1,2,M'-1},f_{2,M-1,2^{k'-1},0}\cdots,f_{2,M-1,2^{k'-1},M'-1},\cdots,\\
&f_{2^{k-1},0,1,0},\cdots,f_{2^{k-1},0,1,M'-1},f_{2^{k-1},0,2,0},\cdots,f_{2^{k-1},M-1,2^{k'-1},M'-1}]^T.
\end{aligned} \tag{3.4}$$

And

$$\Phi = [\Phi_{1,0,1,0}, \cdots, \Phi_{1,0,1,M'-1}, \Phi_{1,0,2,0} \cdots, \Phi_{1,0,2,M'-1}, \cdots, \Phi_{1,0,2^{k'-1},0}, \cdots,$$
$$\Phi_{1,0,2^{k'-1},M'-1}, \cdots, \Phi_{1,M-1,1,0}, \cdots, \Phi_{1,M-1,1,M'-1}, \Phi_{1,M-1,2,0}, \cdots, \Phi_{1,M-1,2,M'-1}$$
$$\Phi_{1,M-1,2^{k'-1},0} \cdots, \Phi_{1,M-1,2^{k'-1},M'-1}, \Phi_{2,0,1,0}, \cdots, \Phi_{2,0,1,M'-1}, \Phi_{2,0,2,0} \cdots,$$
$$\Phi_{2,0,2,M'-1}, \cdots, \Phi_{2,0,2^{k'-1},0}, \cdots, \Phi_{2,0,2^{k'-1},M'-1}, \cdots, \Phi_{2,M-1,1,0}, \cdots, \Phi_{2,M-1,1,M'-1},$$
$$\Phi_{2,M-1,2,0}, \cdots, \Phi_{2,M-1,2,M'-1} \Phi_{2,M-1,2^{k'-1},0} \cdots, \Phi_{2,M-1,2^{k'-1},M'-1}, \cdots, \Phi_{2^{k-1},0,1,0}, \cdots,$$
$$\Phi_{2^{k-1},0,1,M'-1}, \cdots, \Phi_{2^{k-1},0,2^{k'-1},0}, \cdots, \Phi_{2^{k-1},0,2^{k'-1},0}, \cdots, \Phi_{2^{k-1},M-1,2^{k'-1},M'-1}]^T. \tag{3.5}$$

Theorem 3.2.1 *The series solution* $f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} f_{nm} \Phi_{nm}^{L}(t)$ *define in Eq.(1.7) using LWA method converges to* $f(t)$.

Proof See [112].

Theorem 3.2.2 *A function* $f(t)$ *defined on* $[0, 1)$ *with bounded second derivative* $|f''(t)| < B$, *can be expanded as an infinite sum of CWA and the series converges uniformly to the function* $f(t)$, *that is,*

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} f_{nm} \Phi_{nm}^{C}(t).$$

Proof See [37].

3.3 Operational matrices

3.3.1 Legendre wavelet operational matrix of differentiation with respect to variable t

Let $\Phi^L(t,x)$ be two-dimensional Legendre wavelet defined in $Eq.$(1.8) then derivative matrix as follows:

$$\begin{aligned}\frac{\partial}{\partial t}\Phi^L(t,x) &= \frac{\partial}{\partial t}(\Phi^L(t)\otimes\Phi^L(x))\\ &= (D^L\Phi^L(t))\otimes(I\Phi^L(x))\\ &= (D^L\otimes I)(\Phi^L(t)\otimes\Phi^L(x))\\ &= D_t^L\Phi^L(t,x),\end{aligned}$$

so,

$$\frac{\partial}{\partial t}\Phi^L(t,x) = D_t^L\Phi^L(t,x), \tag{3.6}$$

where, $D_t^L = D^L\otimes I$ is the matrix of order $2^{k-1}2^{k'-1}MM'$ (see [131]) and also I is the identity matrix.

3.3.2 Legendre wavelet operational matrix of differentiation with respect to variable x

Let $\Phi^L(t,x)$ be two-dimensional Legendre wavelet defined in $Eq.$(1.8) then derivative matrix as follows:

$$\begin{aligned}\frac{\partial}{\partial x}\Phi^L(t,x) &= \frac{\partial}{\partial x}(\Phi^L(t)\otimes\Phi^L(x))\\ &= \Phi^L(t)\otimes\frac{\partial}{\partial x}\Phi^L(x)\end{aligned}$$

$$= (I\Phi^L(t)) \otimes (D^L\Phi^L(x))$$

$$= (I \otimes D^L)(\Phi^L(t) \otimes \Phi^L(x))$$

$$= D_x\Phi^L(t,x),$$

so,

$$\frac{\partial}{\partial x}\Phi^L(t,x) = D_x^L\Phi^L(t,x), \tag{3.7}$$

where, $D_x^L = I \otimes D^L$ is a $2^{k-1}2^{k'-1}MM' \times 2^{k-1}2^{k'-1}MM'$ matrix (see [131]) and also I is the identity matrix.

3.3.3 Chebyshev wavelet operational matrix of differentiation with respect to variable t

Let $\Phi^C(t,x)$ be two-dimensional Chebyshev wavelet defined in $Eq.$(1.12) then derivative matrix as follows:

$$\frac{\partial}{\partial t}\Phi^C(t,x) = \frac{\partial}{\partial t}(\Phi^C(t) \otimes \Phi^C(x))$$

$$= (D^C\Phi^C(t)) \otimes (I\Phi^C(x))$$

$$= (D^C \otimes I)(\Phi^C(t) \otimes \Phi^C(x))$$

$$= D_t^C\Phi^C(t,x),$$

so,

$$\frac{\partial}{\partial t}\Phi^C(t,x) = D_t^C\Phi^C(t,x), \tag{3.8}$$

where, $D_t^C = D^C \otimes I$ is the matrix of order $2^{k-1}2^{k'-1}MM'$ (see [37]) and also I is the identity matrix.

3.3.4 Chebyshev wavelet operational matrix of differentiation with respect to variable x

Let $\Phi^C(t,x)$ be two-dimensional Chebyshev wavelet defined in $Eq.(1.12)$ then derivative matrix as follows:

$$\begin{aligned}\frac{\partial}{\partial x}\Phi^C(t,x) &= \frac{\partial}{\partial x}(\Phi^C(t)\otimes\Phi^C(x))\\ &= \Phi^C(t)\otimes\frac{\partial}{\partial x}\Phi^C(x)\\ &= (I\Phi^C(t))\otimes(D^C\Phi^C(x))\\ &= (I\otimes D^C)(\Phi^C(t)\otimes\Phi^C(x))\\ &= D_x^C\Phi^C(t,x),\end{aligned}$$

so,

$$\frac{\partial}{\partial x}\Phi^C(t,x) = D_x^C\Phi^C(t,x), \tag{3.9}$$

where, $D_x^C = I\otimes D^C$ is a $2^{k-1}2^{k'-1}MM'\times 2^{k-1}2^{k'-1}MM'$ matrix (see [37]) and also I is the identity matrix.

3.3.5 Legendre wavelet almost operational matrix of integration with singularity for variable t

Let $\Phi^L(t,x)$ be two-dimensional Legendre wavelet defined in $Eq.(1.8)$ and $0<\eta<1$ then,we get

$$\begin{aligned}\int_0^t\frac{\Phi^L(s,x)}{(t-s)^\eta}ds &= \int_0^t\frac{\Phi^L(s)\otimes\Phi^L(x)}{(t-s)^\eta}ds\\ &= \int_0^t\frac{\Phi^L(s)}{(t-s)^\eta}ds\otimes\Phi^L(x)\\ &\approx (V^L\Phi^L(t))\otimes(I\Phi^L(x))\end{aligned}$$

$$= (V^L \otimes I)(\Phi^L(t) \otimes (x))$$

$$= H_t^L \Phi^L(t,x),$$

so,

$$\int_0^t \frac{\Phi^L(s,x)}{(t-s)^\eta} ds \approx H_t^L \Phi^L(t,x), \tag{3.10}$$

where, $H_t^L = V^L \otimes I$, I is identity matrix, and for V^L see appendix subsection 6.1.1.

Now, we find operational matrix for $0 < \nu < 1$ as follows:

$$\int_0^t \frac{\Phi^L(s,x)}{(t-s)^\nu} ds = \int_0^t \frac{\Phi^L(s) \otimes \Phi^L(x)}{(t-s)^\nu} ds$$

$$= \int_0^t \frac{\Phi^L(s)}{(t-s)^\nu} ds \otimes \Phi^L(x)$$

$$\approx (W^L \Phi^L(t)) \otimes (I \Phi^L(x))$$

$$= (W^L \otimes I)(\Phi^L(t) \otimes (x))$$

$$= Q_t^L \Phi^L(t,x),$$

so,

$$\int_0^t \frac{\Phi^L(s,x)}{(t-s)^\nu} ds \approx Q_t^L \Phi^L(t,x), \tag{3.11}$$

where, $Q_t^L = W^L \otimes I$, I is identity matrix, and for W^L see in appendix subsection 6.1.1.

3.3.6 Chebyshev wavelet almost operational matrix of integration with singularity for variable t

Let $\Phi^C(t,x)$ be two-dimensional Chebyshev wavelet defined in $Eq.(1.12)$ and $0 < \varsigma < 1$ then,we get

$$\int_0^t \frac{\Phi^C(s,x)}{(t-s)^\varsigma} ds = \int_0^t \frac{\Phi^C(s) \otimes \Phi^C(x)}{(t-s)^\varsigma} ds,$$

using similar process as we have been done for Legendre wavelets, we get

$$\int_0^t \frac{\Phi^C(s,x)}{(t-s)^\varsigma} ds = H_t^C \Phi^C(t,x), \tag{3.12}$$

where, $H_t^C = V^C \otimes I$, I is identity matrix, and for V^C see appendix subsection 6.1.2.

and also for $0 < \delta < 1$ as follows:,

$$\int_0^t \frac{\Phi^C(s,x)}{(t-s)^\delta} ds = Q_t^C \Phi^C(t,x), \tag{3.13}$$

where, $Q_t^C = W^C \otimes I$, I is identity matrix, and for W^C see in appendix subsection 6.1.2.

3.4 Method of the solution

Representing the $Eq.(3.1)$ via expansion

Table 3.1: List of denotation

General symbol	**Uses of LWA**	**Uses of CWA**
$\Phi(t,x)$	$\Phi^L(t,x)$	$\Phi^C(t,x)$
D	D^L	D^C
V	V^L	V^C
W	W^L	W^C
D_t	D_t^L	D_t^C
D_x	D_x^L	D_x^C
H_t	H_t^L	H_t^C
Q_t	Q_t^L	Q_t^C

Let us consider the problem $Eq.(3.1)$

$$({}_0D_x^{\alpha}u)(t,x) - \lambda_1({}_0D_t^{\beta}u)(t,x) - \lambda_2\nabla^2 u(t,x) = f(t,x), \tag{3.14}$$

now, using definition of Riemann- Liouville fractional derivative defined by $Eq.(1.17)$ and Laplace operator in $Eq.(3.14)$ and then $Eq.(3.14)$ transform as follows:

$$\frac{1}{\Gamma(1-\{\alpha\})}(\frac{\partial}{\partial t})^{[\alpha]+1}\int_0^t \frac{u(s,x)}{(t-s)^{\{\alpha\}}}ds - \lambda_1\frac{1}{\Gamma(1-\{\beta\})}(\frac{\partial}{\partial t})^{[\beta]+1}\int_0^t \frac{u(s,x)}{(t-s)^{\{\beta\}}}ds-$$
$$\lambda_2(u_{tt}(t,x)+u_{xx}(t,x)) = f(t,x).$$

(3.15)

To find the solution of $Eq.(3.15)$ by using the collocation method, we will first obtain matrix form of FPIDEs. In this way, we suppose the approximation of $u(t,x)$ as follows:

$$u(t,x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} \sum_{n'=1}^{2^{k'}-1} \sum_{m'=0}^{M'-1} C_{n,m,n',m'} \Phi_{n,m,n',m'}(t,x) = C^T \Phi(t,x), \tag{3.16}$$

where, $C_{n,m,n',m'}$ is unknown for $n = 1,2,...,2^{k-1}$, $m = 0,1,...,M-1$, $n' = 1,2,...,2^{k'-1}$, $m' = 0,1,...,M'-1$.

Now, partially differentiate $Eq.(3.16)$ w. r. t. t as follows:

$$u_t(t,x) \approx C^T D_t \Phi(t,x),$$

again, partially differentiate above equation with respect to t as follows:

$$u_{tt}(t,x) \approx C^T (D_t)^2 \Phi(t,x), \tag{3.17}$$

similarly,

$$u_{xx}(t,x) \approx C^T (D_x)^2 \Phi(t,x). \tag{3.18}$$

Also

$$\left(\frac{\partial}{\partial t}\right)^{[\alpha]+1} \Phi(t,x) \approx D_\alpha \Phi(t,x), \tag{3.19}$$

$$\left(\frac{\partial}{\partial t}\right)^{[\beta]+1} \Phi(t,x) \approx D_\beta \Phi(t,x), \tag{3.20}$$

and

$$f(t,x) \approx F^T\Phi(t,x). \tag{3.21}$$

Using $Eq.$(3.15) and $Eq.$(3.16) as follows:

$$\frac{1}{\Gamma(1-\{\alpha\})}\left(\frac{\partial}{\partial t}\right)^{[\alpha]+1}\int_0^t \frac{C^T\Phi(s,x)}{(t-s)^{\{\alpha\}}}ds - \lambda_1\frac{1}{\Gamma(1-\{\beta\})}\left(\frac{\partial}{\partial t}\right)^{[\beta]+1}\int_0^t \frac{C^T\Phi(s,x)}{(t-s)^{\{\beta\}}}ds - \lambda_2(u_{tt}(t,x)+u_{xx}(t,x)) = f(t,x). \tag{3.22}$$

Putting the approximation $Eq.$(3.17), $Eq.$(3.18), and $Eq.$(3.21) in $Eq.$(3.22) as fallows:

$$\frac{C^T}{\Gamma(1-\{\alpha\})}\left(\frac{\partial}{\partial t}\right)^{[\alpha]+1}\int_0^t \frac{\Phi(s,y)}{(t-s)^{\{\alpha\}}}ds - \lambda_1\frac{C^T}{\Gamma(1-\{\beta\})}\left(\frac{\partial}{\partial t}\right)^{[\beta]+1}\int_0^t \frac{\Phi(s,x)}{(t-s)^{\{\beta\}}}ds - \lambda_2(C^T(D_t)^2\Phi(t,x)+C^T(D_x)^2\Phi(t,x)) = F^T\Phi(t,x). \tag{3.23}$$

Now, grouping $Eq.$(3.10), $Eq.$(3.11)(or $Eq.$(3.12), $Eq.$(3.13)) and $Eq.$(3.23) as follows:

$$\frac{C^T}{\Gamma(1-\{\alpha\})}\left(\frac{\partial}{\partial t}\right)^{[\alpha]+1}H_t\Phi(t,x) - \lambda_1\frac{C^T}{\Gamma(1-\{\beta\})}\left(\frac{\partial}{\partial t}\right)^{[\beta]+1}Q_t\Phi(t,x) - \lambda_2(C^T(D_t)^2+C^T(D_x)^2)\Phi(t,x) = F^T\Phi(t,x), \tag{3.24}$$

or,

$$\frac{C^T}{\Gamma(1-\{\alpha\})}H_t\left(\frac{\partial}{\partial t}\right)^{[\alpha]+1}\Phi(t,x) - \lambda_1\frac{C^T}{\Gamma(1-\{\beta\})}Q_t\left(\frac{\partial}{\partial t}\right)^{[\beta]+1}\Phi(t,x) - \lambda_2C^T((D_t)^2+(D_x)^2)\Phi(t,x) = F^T\Phi(t,x). \tag{3.25}$$

Grouping $Eq.(3.19)$, $Eq.(3.20)$ and $Eq.(3.25)$ as follows:

$$\frac{C^T}{\Gamma(1-\{\alpha\})}H_tD_\alpha - \lambda_1\frac{C^T}{\Gamma(1-\{\beta\})}Q_tD_\beta - \lambda_2C^T(D_t)^2 \\ -\lambda_2C^T(D_x)^2)\Phi(t,x) = F^T\Phi(t,x), \tag{3.26}$$

or,

$$C^T\left(\frac{1}{\Gamma(1-\{\alpha\})}H_tD_\alpha - \lambda_1\frac{1}{\Gamma(1-\{\beta\})}Q_tD_\beta - \lambda_2(D_t)^2 - \lambda_2(D_x)^2\right) = F^T. \tag{3.27}$$

Finally, $Eq.(3.27)$ as follows:

$$C^T = F^T[\frac{1}{\Gamma(1-\{\alpha\})}H_tD_\alpha - \lambda_1\frac{1}{\Gamma(1-\{\beta\})}Q_tD_\beta - \lambda_2(D_t) - \lambda_2(D_x)^2]^{-1}. \tag{3.28}$$

Grouping $Eq.(3.16)$ and $Eq.(3.28)$, we get

$$u(t,x) = F^T[\frac{1}{\Gamma(1-\{\alpha\})}H_tD_\alpha - \lambda_1\frac{1}{\Gamma(1-\{\beta\})}Q_tD_\beta - \lambda_2(D_t)^2 - \lambda_2(D_x)^2]^{-1}\Phi(t,x). \tag{3.29}$$

Now, we use the collocation method for solving $Eq.(3.29)$. For this, we suppose $t = \{t_i\}_{i=1}^N = \frac{i}{N}$ and $x = \{x_j\}_{j=1}^N = \frac{j}{N}$ are the set of (N) nodes. We substitute these nodes in $Eq.(3.29)$ and then we find the numerical solution of $Eq.(3.15)$ and hence $Eq.(3.1)$ solved numerically.

3.5 Convergence analysis:

3.5.1 For Legendre wavelet

Theorem 3.5.1 *Let* $\left(\frac{\partial^\tau u(t,x)}{\partial x^\tau}\right)_N$, $\tau \geq 0$ *be the Legendre wavelet approximation (LWA) of* $\left(\frac{\partial^\tau u(t,x)}{\partial x^\tau}\right)$ *and assume that the mixed second derivative of* $\left(\frac{\partial^\tau u(t,x)}{\partial x^\tau}\right)$ *is bounded by a constant* K_1*.i.e.* $\left|\frac{\partial^{\tau+4} u(t,x)}{\partial t^2 \partial x^{\tau+2}}\right| < K_1$*, then we have the following upper* *bound of error:*

$$\left\|\left(\frac{\partial^{\tau}u(t,x)}{\partial x^{\tau}}\right)-\left(\frac{\partial^{\tau}u(t,x)}{\partial x^{\tau}}\right)_N\right\|_{L^2}^2<\frac{K_1^2\wp^2}{2^{18}},$$

where, $\wp$ *is a polygamma function* ,$\wp=F_3(-3/2+N)$.

Proof Let $\frac{\partial^{\tau}u}{\partial x^{\tau}}=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{\infty}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=0}^{\infty}C_{n,m,n',m'}\Phi^L_{n,m,n',m'}(t,x)$ truncating it upto N-1 level, we get

$$\left(\frac{\partial^{\tau}u}{\partial x^{\tau}}\right)_N\approx\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{N-1}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=0}^{N-1}C_{n,m,n',m'}\Phi^L_{n,m,n',m'}(t,x).$$

Now,

$$\left(\frac{\partial^{\tau}u}{\partial x^{\tau}}\right)-\left(\frac{\partial^{\tau}u}{\partial x^{\tau}}\right)_N=\sum_{n=1}^{2^{k-1}}\sum_{m=N}^{\infty}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=N}^{\infty}C_{n,m,n',m'}\Phi^L_{n,m,n',m'}(t,x),$$

taking L^2 norm both side, we get

$$\left\|\left(\frac{\partial^{\tau}u}{\partial x^{\tau}}\right)-\left(\frac{\partial^{\tau}u}{\partial x^{\tau}}\right)_N\right\|^2=\int_0^1\int_0^1\left(\left(\frac{\partial^{\tau}u}{\partial x^{\tau}}\right)-\left(\frac{\partial^{\tau}u}{\partial x^{\tau}}\right)_N\right)^2dtdx,$$

or,

$$\left\|\left(\frac{\partial^{\tau}u}{\partial x^{\tau}}\right)-\left(\frac{\partial^{\tau}u}{\partial x^{\tau}}\right)_N\right\|^2$$
$$=\int_0^1\int_0^1\left(\sum_{n=1}^{2^{k-1}}\sum_{m=N}^{\infty}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=N}^{\infty}C_{n,m,n',m'}\Phi^L_{n,m,n',m'}(t,x)\right)^2dtdx,$$

or,

$$\left\|\left(\frac{\partial^{\tau}u}{\partial x^{\tau}}\right)-\left(\frac{\partial^{\tau}u}{\partial x^{\tau}}\right)_N\right\|^2=\sum_{n=1}^{2^{k-1}}\sum_{m=N}^{\infty}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=N}^{\infty}C^2_{n,m,n',m'},\tag{3.30}$$

where,

$$C_{n,m,n',m'}=\int_0^1\int_0^1\frac{\partial^{\tau}u(t,x)}{\partial x^{\tau}}\Phi^L_{n,m,n',m'}(t,x)dtdx.\tag{3.31}$$

Using $Eq.(1.9)$ in $Eq.(3.31)$,we get

$$C_{n,m,n',m'} = \int_0^1 \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \frac{\partial^\tau u(t,x)}{\partial x^\tau}(2m+1)^{1/2}2^{(k-1)/2} \\ P_m(2^k t - 2n+1)\Phi^L_{n',m'}(x)dtdx, \tag{3.32}$$

putting $2^k t - 2n + 1 = s$, in $Eq.(3.32)$ as follows:

$$C_{n,m,n',m'} = \int_0^1 \int_{-1}^1 \frac{\partial^\tau u(\frac{s+2n-1}{2^k},x)}{\partial x^\tau}(2m+1)^{1/2}2^{-(k+1)/2} \\ P_m(s)\Phi^L_{n',m'}(x)dsdx,$$

or,

$$C_{n,m,n',m'} = \frac{2^{-(k+1)/2}}{2(2m+1)^{1/2}}\int_0^1 \Phi^L_{n',m'}(x)\int_{-1}^1 \frac{\partial^\tau u(\frac{s+2n-1}{2^k},x)}{\partial x^\tau} \\ d\left(P_{m+1}(s) - P_{m'-1}(s)\right)dsdx,$$

or,

$$C_{n,m,n',m'} = (-1)\frac{2^{-(k+1)/2}}{2(2m+1)^{1/2}}\int_0^1 \Phi^L_{n',m'}(x)\int_{-1}^1 \frac{\partial^{\tau+1} u(\frac{s+2n-1}{2^k},x)}{\partial s\partial x^\tau} \\ \left(P_{m+1}(s) - P_{m'-1}(s)\right)dsdx,$$

or,

$$C_{n,m,n',m'} = \frac{2^{-(k+1)/2}}{4(2m+1)^{1/2}}\int_0^1 \Phi^L_{n',m'}(x)\int_{-1}^1 \frac{\partial^{\tau+1} u(\frac{s+2n-1}{2^k},x)}{\partial s\partial x^\tau} \\ \left(\frac{P_{m+2}(s) - P_m(s)}{2m+3} - \frac{P_m(s) - P_{m-2}(s)}{2m-1}\right)dsdx,$$

or,

$$C_{n,m,n',m'} = \frac{2^{-(k+1)/2}}{4(2m+1)^{1/2}}\int_0^1 \Phi^L_{n',m'}(x)\int_{-1}^1 \frac{\partial^{\tau+2} u(\frac{s+2n-1}{2^k},x)}{\partial s^2\partial x^\tau} \\ \left(\frac{P_{m+2}(s) - P_m(s)}{2m+3} - \frac{P_m(s) - P_{m-2}(s)}{2m-1}\right)dsdx. \tag{3.33}$$

Using $Eq.$(1.9) in $Eq.$(3.33), we get

$$C_{n,m,n',m'} = \frac{2^{-(k+1)/2}}{4(2m+1)^{1/2}} \int_{\frac{n'-1}{2^{k'-1}}}^{\frac{n'}{2^{k'-1}}} (2m'+1)^{1/2} 2^{k'} P_{m'}(2^{k'}x - 2n' - 1)$$
$$\int_{-1}^{1} \frac{\partial^{\tau+2} u(\frac{s+2n-1}{2^k}, x)}{\partial s^2 \partial x} \left(\frac{P_{m+2}(s) - P_m(s)}{2m+3} - \frac{P_m(s) - P_{m-2}(s)}{2m-1} \right) dsdx. \quad (3.34)$$

Putting $2^{k'}x - 2n' - 1 = r$ in $Eq.$(3.34) and processing the same as done above we get,

$$C_{n,m,n',m'} = \frac{2^{-(k+1)/2} 2^{-(k'+1)/2}}{16(2m+1)^{1/2}(2m'+1)^{1/2}} \int_{-1}^{1} \int_{-1}^{1} \frac{\partial^{\tau+4} u(\frac{s+2n-1}{2^k}, \frac{r+2n'-1}{2^{k'}})}{\partial s^2 \partial r^{\tau+2}}$$
$$\sigma_{n,m}(s) \sigma_{n'm'}(t) dsdr,$$

where,

$$\sigma_{n,m}(s) = \frac{P_{m+2}(s) - P_m(s)}{2m+3} - \frac{P_m(s) - P_{m-2}(s)}{2m-1},$$

and,

$$\sigma_{n',m'}(r) = \frac{P_{m'+2}(t) - P_{m'}(r)}{2m'+3} - \frac{P_{m'}(r) - P_{m'-2}(r)}{2m'-1}.$$

Now,

$$\mid C_{n,m,n',m'} \mid^2 = \frac{2^{-(k+k'+2)}}{256(2m+1)(2m'+1)} \mid \int_{-1}^{1} \int_{-1}^{1} \frac{\partial^{\tau+4} u(\frac{s+2n-1}{2^k}, \frac{r+2n'-1}{2^{k'}})}{\partial s^2 \partial r^{\tau+2}}$$
$$\sigma_{n,m}(s) \sigma_{n',m'}(t) dsdr \mid^2,$$

or

$$\mid C_{n,m,n',m'} \mid^2 \leq \frac{2^{-(k+k'+2)}}{256(2m+1)(2m'+1)} \int_{-1}^{1} \int_{-1}^{1} \mid \frac{\partial^{\tau+4} u(\frac{s+2n-1}{2^k}, \frac{r+2n'-1}{2^{k'}})}{\partial s^2 \partial r^{\tau+2}} \mid^2 dsdr$$
$$\int_{-1}^{1} \int_{-1}^{1} \mid \sigma_{n,m}(s) \sigma_{n',m'}(r) \mid^2 dsdr,$$

or,

$$| C_{n,m,n',m'} |^2 \leq \frac{K_1^2 2^{-(k+k'+2)}}{256(2m+1)(2m'+1)} \int_{-1}^{1}\int_{-1}^{1} | \sigma_{n,m}(s)\sigma_{n',m'}(r) |^2 \, dsdr,$$

where,

$$\left| \frac{\partial^{\tau+4} u(\frac{s+2n-1}{2^k}, \frac{r+2n'-1}{2^{k'}})}{\partial s^2 \partial r^{\tau+2}} \right| < K_1,$$

or,

$$| C_{n,m,n',m'} |^2 \leq \frac{K_1^2}{2^{(k+k'+10)}(2m+1)(2m'+1)} \int_{-1}^{1} | \sigma_{n,m}(s) |^2 \, ds \int_{-1}^{1} | \sigma_{n',m'}(r) |^2 \, dr. \tag{3.35}$$

Now,

$$\int_{-1}^{1} | \sigma_{n,m}(s) |^2 \, ds = \int_{-1}^{1} \left(\frac{P_{m+2}(s) - P_m(s)}{2m+3} - \frac{(P_m(s) - P_{m-2}(s)}{2m-1} \right)^2 ds$$

$$= \int_{-1}^{1} \left(\frac{(2m-1)P_{m+2}(s) - (4m+2)P_m(s) + (2m-3)P_{m-2}(s)}{(2m+3)(2m+1)} \right)^2 ds$$

$$< \int_{-1}^{1} \frac{(2m-1)^2 P_{m+2}^2(s) + (4m+2)^2 P_m^2(s) + (2m+3)^2 P_{m-2}^2(s)}{(2m+3)^2(2m-1)^2} ds$$

$$< \frac{12(2m+3)^2}{(2m+3)^2(2m-1)^2(2m-3)}.$$

Thus ,we get

$$\int_{-1}^{1} | \sigma_{n,m}(s) |^2 \, ds < \frac{12}{(2m-1)^2(2m-3)}. \tag{3.36}$$

Similarly,

$$\int_{-1}^{1} | \sigma_{n',m'}(t) |^2 \, dt < \frac{12}{(2m'-1)^2(2m'-3)}. \tag{3.37}$$

Grouping $Eq.$(3.35), $Eq.$(3.36) and $Eq.$(3.37),we get

$$\mid C_{n,m,n',m'} \mid^2 < \frac{9{K_1}^2}{64(2^{k+k'})(2m-3)^4(2m'-3)^4}, \tag{3.38}$$

$Eq.$(3.38) implies,

$$\sum_{n=1}^{2^{k-1}} \sum_{m=N}^{\infty} \sum_{n'=1}^{2^{k'-1}} \sum_{m'=N}^{\infty} \mid C_{n,m,n',m'} \mid^2 < \sum_{n=1}^{2^{k-1}} \sum_{m=N}^{\infty} \sum_{n'=1}^{2^{k'-1}} \sum_{m'=N}^{\infty} \frac{9K_1^2}{64(2^{k+k'})(2m-3)^4(2m'-3)^4} = \sum_{m=N}^{\infty} \sum_{m'=N}^{\infty} \frac{2^{k-1}2^{k'-1}9{K_1}^2}{64(2^{k+k'})(2m-3)^4(2m'-3)^4}. \tag{3.39}$$

Therefore,

$$\sum_{n=1}^{2^{k-1}} \sum_{m=N}^{\infty} \sum_{n'=1}^{2^{k'-1}} \sum_{m'=N}^{\infty} \mid C_{n,m,n',m'} \mid^2 < \frac{9{K_1}^2}{256} \frac{1}{9216} \left(F_3(\frac{-3}{2}+N) \right)^2 < \frac{K_1^2}{(2^{18})} \left(F_3\left(\frac{-3}{2}+N\right) \right)^2. \tag{3.40}$$

$Eqs.$(3.30) and (3.40) together implies :

$$\left\| \left(\frac{\partial u^\tau(t,x)}{\partial x^\tau} \right) - \left(\frac{\partial u^\tau(t,x)}{\partial x^\tau} \right)_N \right\|_{L^2}^2 < \frac{{K_1}^2}{(2^{18})} \left(F_3(\frac{-3}{2}+N) \right)^2,$$

i.e.

$$\left\| \left(\frac{\partial u^\tau(t,x)}{\partial x^\tau} \right) - \left(\frac{\partial u^\tau(t,x)}{\partial x^\tau} \right)_N \right\|_{L^2}^2 < \frac{{K_1}^2}{(2^{18})} \wp^2, \tag{3.41}$$

where $\wp = F_3\left(\frac{-3}{2}+N\right)$.

Lemma 3.5.2 *Let $u(t,x)$ be the sufficiently smooth function in Ω and $(u_{xx})_N(t,x)$ be the LWA of $u_{xx}(t,x)$. Assume that the mixed second derivative of $u(t,x)$ is bounded by a constant G_1.*

i.e.

$$\left|\left(\frac{\partial u^6(t,x)}{\partial t^2 \partial x^4}\right)\right| < G_1,$$

then ,we have the following upper bound of error :

$$\|u_{xx} - (u_{xx})_N\| < \frac{G_1^2 \wp^2}{2^{18}},$$

where, $\wp = F_3(\frac{-3}{2} + N)$.

Proof Proof is similar as *theorem 3.5.1*, if $\tau = 2$.

Lemma 3.5.3 *Let $u(t,x)$ be the sufficiently smooth function in Ω and $(u_{tt})_N(t,x)$ be the LWA of $u_{tt}(t,x)$. Assume that the mixed second derivative of $u(t,x)$ is bounded by a constant G_2. i.e.*

$$\left|\left(\frac{\partial u^6(t,x)}{\partial t^4 \partial x^2}\right)\right| < G_2,$$

then ,we have the following upper bound of error :

$$\| u_{tt} - (u_{tt})_N \|_{L^2}^2 < \frac{G_2^2 \wp^2}{2^{18}},$$

where, $\wp = F_3(\frac{-3}{2} + N)$.

Proof Proof is similar as *theorem 3.5.1*.

3.5.2 For Chebyshev wavelet

Theorem 3.5.4 *Let $\left(\frac{\partial^{\Upsilon} u(t,x)}{\partial x^{\Upsilon}}\right)_N$, $\Upsilon \geq 0$, be the Chebyshev wavelet approximation (CWA) of $\left(\frac{\partial^{\Upsilon} u(t,x)}{\partial x^{\Upsilon}}\right)$ and assume that the mixed derivative of $\left(\frac{\partial^{\Upsilon} u(t,x)}{\partial x^{\Upsilon}}\right)$ is bounded by a constant K_2.i.e.*

$$\left|\frac{\partial^{\Upsilon+2} u(t,x)}{\partial t \partial x^{\Upsilon+1}}\right| < K_2,$$

then we have the following upper bound of error:

$$\left\| \left(\frac{\partial^\Upsilon u(t,x)}{\partial x^\Upsilon} \right) - \left(\frac{\partial^\Upsilon u(t,x)}{\partial x^\Upsilon} \right)_N \right\|_{L^2}^2 < \frac{K_2^2}{2} (F[1,N])^2,$$

where $F[1,N]$ *is polygamma function.*

Proof Proof start as *theorem* 3.5.1 upto *Eq.*(3.30) and then next steps are given below:

$$C_{n,m,n',m'} = \int_0^1 \int_0^1 \omega(t,x) \frac{\partial^\Upsilon u(t,x)}{\partial x^\Upsilon} \Phi^c_{n,m,n',m'}(t,x) dt dx,$$

or,

$$C_{n,m,n',m'} = \int_0^1 \int_0^1 \omega(t)\omega(x) \frac{\partial^\Upsilon u(t,x)}{\partial x^\Upsilon} \Phi^c_{n,m}(t) \Phi^c_{n',m'}(x) dt dx,$$

or,

$$C_{n,m,n',m'} = \int_0^1 \omega(x) \Phi^c_{n',m'}(x) \left(\int_0^1 \omega(t) \frac{\partial^\Upsilon u(t,x)}{\partial x^\Upsilon} \Phi^c_{n,m}(t) dt \right) dx.$$

Using *Eq.*(1.13) and taking $m > 0$, $m' > 0$ (since proof at $m = m' = 0$ is obvious), we get

$$C_{n,m,n',m'} = \int_0^1 \omega(x) \Phi^c_{n',m'}(x) \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \omega_n(2^k t - 2n + 1) \frac{\partial^\Upsilon u(t,x)}{\partial x^\Upsilon} \frac{2^{(k+1)/2}}{\sqrt{\pi}} T_m(2^k t - 2n + 1) dt dx, \tag{3.42}$$

putting $2^k t - 2n + 1 = \xi$, in *Eq.*(3.42) as follows:

$$C_{n,m,n'm'} = \frac{2^{(-k+1)/2}}{\sqrt{\pi}} \int_0^1 \omega(x) \Phi^c_{n',m'}(x) \int_{-1}^1 \omega_n(\xi) \frac{\partial^\Upsilon u(\frac{\xi+2n-1}{2^k}, x)}{\partial x^\Upsilon} T_m(\xi) d\xi dx,$$

$$C_{n,m,n'm'} = \frac{2^{(-k+1)/2}}{\sqrt{\pi}} \int_0^1 \omega(x)\Phi^c_{n',m'}(x) \int_{-1}^1 \frac{1}{\sqrt{1-\xi^2}} \frac{\partial^\Upsilon u(\frac{\xi+2n-1}{2^k},x)}{\partial x^\Upsilon} T_m(\xi) d\xi dx. \tag{3.43}$$

Putting $\xi = cos\theta$ in $Eq.(3.43)$ so $T_m(\theta) = cos(m\theta)$ and hence $Eq.(3.43)$ as follows:

$$C_{n,m,n'm'} = \frac{2^{(-k+1)/2}}{\sqrt{\pi}} \int_0^1 \omega(x)\Phi^c_{n',m'}(x) \int_0^\pi \frac{1}{\sqrt{1-cos^2\theta}} \frac{\partial^\Upsilon u(\frac{cos\theta+2n-1}{2^k},x)}{\partial x^\Upsilon} cos(m\theta) sin\theta d\theta dx,$$

or,

$$C_{n,m,n'm'} = \frac{2^{(-k+1)/2}}{\sqrt{\pi}} \int_0^1 \omega(x)\Phi^c_{n',m'}(x) \int_0^\pi \frac{\partial^\Upsilon u(\frac{cos\theta+2n-1}{2^k},x)}{\partial x^\Upsilon} cos(m\theta) d\theta dx,$$

or,

$$C_{n,m,n'm'} = -\frac{2^{(-k+1)/2}}{\sqrt{\pi}} \int_0^1 \omega(x)\Phi^c_{n',m'}(x) \int_0^\pi \frac{\partial^{\Upsilon+1} u(\frac{cos\theta+2n-1}{2^k},x)}{\partial\theta\partial x^\Upsilon} \frac{sin(m\theta)}{m} d\theta dx,$$

or,

$$C_{n,m,n'm'} = -\frac{2^{(-k+1)/2}}{m\sqrt{\pi}} \int_0^\pi sin(m\theta) \int_0^1 \omega(x)\Phi^c_{n',m'}(x) \frac{\partial^{\Upsilon+1} u(\frac{cos\theta+2n-1}{2^k},x)}{\partial\theta\partial x^\Upsilon} dx d\theta,$$

or,

$$C_{n,m,n'm'} = -\frac{2^{(-k+1)/2}}{m\sqrt{\pi}} \int_0^\pi sin(m\theta) \int_{\frac{n'-1}{2^{k'-1}}}^{\frac{n}{2^{k'-1}}} \omega_{n'}(2^{k'}x-2n'+1) \frac{2^{(k'+1)/2}}{\sqrt{\pi}} T_{m'}(2^{k'}-2n'+1) \frac{\partial^{\Upsilon+1} u(\frac{cos\theta+2n-1}{2^k},x)}{\partial\theta\partial x^\Upsilon} dx d\theta. \tag{3.44}$$

Using $Eq.(1.13)$, putting $2^{k'}x-2n'+1=\eta$ in $Eq.(3.44)$ and processing the same as done above, we get

$$C_{n,m,n'm'} = -\frac{2^{(-k+1)/2}}{m\sqrt{\pi}}\frac{2^{(-k'+1)/2}}{\sqrt{\pi}}\int_0^{\pi} sin(m\theta)\int_{-1}^{1}\omega_{n'}(\eta)$$
$$T_{m'}(\eta)\frac{\partial^{\Upsilon+1}u(\frac{cos\theta+2n-1}{2^k},\frac{\eta+2n'-1}{2^{k'}})}{\partial\theta\partial\eta^{\Upsilon}}d\eta d\theta,$$

or,

$$C_{n,m,n'm'} = -\frac{2^{(-k+1)/2}}{m\sqrt{\pi}}\frac{2^{(-k'+1)/2}}{\sqrt{\pi}}\int_0^{\pi} sin(m\theta)\int_{-1}^{1}\frac{T_{m'}(\eta)}{\sqrt{1-\eta^2}}$$
$$\frac{\partial^{\Upsilon+1}u(\frac{cos\theta+2n-1}{2^k},\frac{\eta+2n'-1}{2^{k'}})}{\partial\theta\partial\eta^{\Upsilon}}d\eta d\theta. \tag{3.45}$$

Putting $\eta = cos\theta'$ in $Eq.(3.45)$ so $T_{m'} = cos(m'\theta')$ and hence $Eq.(3.45)$ as follows:

$$C_{n,m,n'm'} = -\frac{2^{(-k+1)/2}}{m\sqrt{\pi}}\frac{2^{(-k'+1)/2}}{\sqrt{\pi}}\int_0^{\pi} sin(m\theta)\int_0^{\pi} cos(m'\theta')$$
$$\frac{\partial^{\Upsilon+1}u(\frac{cos\theta+2n-1}{2^k},\frac{cos\theta'+2n'-1}{2^{k'}})}{\partial\theta\partial\theta'^{\Upsilon}}d\theta d\theta',$$

or,

$$C_{n,m,n'm'} = \frac{2^{(-k+1)/2}}{m\sqrt{\pi}}\frac{2^{(-k'+1)/2}}{m'\sqrt{\pi}}\int_0^{\pi}\int_0^{\pi}\frac{\partial^{\Upsilon+2}u(\frac{cos\theta+2n-1}{2^k},\frac{cos\theta'+2n'-1}{2^{k'}})}{\partial\theta\partial\theta'^{\Upsilon+1}}$$
$$sin(m\theta)sin(m'\theta')d\theta' d\theta,$$

or,

$$C_{n,m,n'm'} = \frac{2^{-(k+k'-1)/2}}{mm'\pi}\int_0^{\pi}\int_0^{\pi}\frac{\partial^{\Upsilon+2}u(\frac{cos\theta+2n-1}{2^k},\frac{cos\theta'+2n'-1}{2^{k'}})}{\partial\theta\partial\theta'^{\Upsilon+1}}$$
$$sin(m\theta)sin(m'\theta')d\theta d\theta'. \tag{3.46}$$

Now, $Eq.(3.46)$ as follows:

$$|C_{n,m,n'm'}|^2 \leq \frac{2^{-(k+k'-1)}}{(mm'\pi)^2}\int_0^{\pi}\int_0^{\pi}\left|\frac{\partial^{\Upsilon+2}u(\frac{cos\theta+2n-1}{2^k},\frac{cos\theta'+2n'-1}{2^{k'}})}{\partial\theta\partial\theta^{\Upsilon+1}}\right|^2$$
$$|sin(m\theta)|^2\,|sin(m'\theta')|^2\,d\theta d\theta',$$

or,

$$|C_{n,m,n'm'}|^2 \leq \frac{2^{-(k+k'-1)}}{(mm'\pi)^2}\int_0^{\pi}\int_0^{\pi} K_2^2 d\theta d\theta',$$

or,

$$|C_{n,m,n'm'}|^2 \leq \frac{2^{-(k+k'-1)}}{(mm'\pi)^2}K_2^2\int_0^{\pi}\int_0^{\pi} d\theta d\theta'.$$

or,

$$|C_{n,m,n'm'}|^2 \leq \frac{2^{-(k+k'-1)}}{(mm')^2}K_2^2.$$

So,

$$\begin{aligned}\sum_{n=1}^{2^{k-1}}\sum_{m=N}^{\infty}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=N}^{\infty}|C_{n,m,n'm'}|^2 &\leq \sum_{n=1}^{2^{k-1}}\sum_{m=N}^{\infty}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=N}^{\infty}\frac{2^{-(k+k'-1)}}{(mm')^2}K_2^2,\\ &= \frac{K_2^2}{2}(F[1,N])^2,\end{aligned}$$

where, $F[1,N]$ is polygamma function.

Hence,

$$\left\|\left(\frac{\partial^{\Upsilon}u}{\partial x^{\Upsilon}}\right)-\left(\frac{\partial^{\Upsilon}u}{\partial x^{\Upsilon}}\right)_N\right\|^2 \leq \frac{K_2^2}{2}(F[1,N])^2.$$

Lemma 3.5.5 *Let $u(t,x)$ be the sufficiently smooth function in Ω and $\left(\frac{\partial^2 u(t,x)}{\partial x^2}\right)_N$ be the CWA of $\left(\frac{\partial^2 u(t,x)}{\partial x^2}\right)$. Assume that the mixed derivative of $\left(\frac{\partial^2 u(t,x)}{\partial x^2}\right)$ is bounded by a constant S_1.i.e.*

$$\left|\frac{\partial^4 u(t,x)}{\partial t\partial x^3}\right| < S_1,$$

then we have the following upper bound of error:

$$\left\|\frac{\partial^2 u(t,x)}{\partial x^2}-\left(\frac{\partial^2 u(t,x)}{\partial x^2}\right)_N\right\|_{L^2}^2 < \frac{S_1^2}{2}(F[1,N])^2,$$

where, $F[1, N]$ is polygamma function.

Proof Proof is similar as *theorem 3.5.4*, if $\Upsilon = 2$.

Lemma 3.5.6 *Let $u(t,x)$ be the sufficiently smooth function in Ω and $\left(\frac{\partial^2 u(t,x)}{\partial t^2}\right)_N$ be the CWA of $\left(\frac{\partial^2 u(t,x)}{\partial t^2}\right)$. Assume that the mixed derivative of $\left(\frac{\partial^2 u(t,x)}{\partial t^2}\right)$ is bounded by a constant S_2.i.e.*

$$\left|\frac{\partial^4 u(t,x)}{\partial t^3 \partial x}\right| < S_2,$$

then we have the following upper bound of error:

$$\left\|\frac{\partial^2 u(t,x)}{\partial t^2} - \left(\frac{\partial^2 u(t,x)}{\partial t^2}\right)_N\right\|_{L^2}^2 < \frac{S_2^2}{2}(F[1,N])^2,$$

where, $F[1, N]$ is polygamma function.

Proof Proof is similar as *theorem 3.5.4*.

3.6 Error analysis

Since our problem $Eq.(3.1)$ reduced in $Eq.(3.15)$ as follows:

$$\frac{1}{\Gamma(1-\{\alpha\})}(\frac{\partial}{\partial t})^{[\alpha]+1}\int_0^t \frac{u(s,x)}{(t-s)^{\{\alpha\}}}ds - \lambda_1 \frac{1}{\Gamma(1-\{\beta\})}(\frac{\partial}{\partial t})^{[\beta]+1}\int_0^t \frac{u(s,x)}{(t-s)^{\{\beta\}}}ds$$
$$-\lambda_2(u_{tt}(t,x) + u_{xx}(t,x)) = f(t,x), \tag{3.47}$$

and approximate solution of $Eq.(3.45)$ as follows:

$$\frac{1}{\Gamma(1-\{\alpha\})}(\frac{\partial}{\partial t})^{[\alpha]+1}\int_0^t \frac{u_N(s,x)}{(t-s)^{\{\alpha\}}}ds - \lambda_1 \frac{1}{\Gamma(1-\{\beta\})}(\frac{\partial}{\partial t})^{[\beta]+1}\int_0^t \frac{u_N(s,x)}{(t-s)^{\{\beta\}}}ds$$
$$-\lambda_2\left((u_{tt})_N(t,x) + (u_{xx})_N(t,x)\right) = f_N(t,x). \tag{3.48}$$

Let $e_n(t,x) = u(t,x) - u_N(t,x)$ be the bounded error function of approximation solution $u_N(t,x)$ to exact solution.

Subtracting $Eq.(3.48)$ from $Eq.(3.47)$ as follows:

$$\frac{1}{\Gamma(1-\{\alpha\})}(\frac{\partial}{\partial t})^{[\alpha]+1}\int_0^t \frac{u(s,x)-u_N(s,x)}{(t-s)^{\{\alpha\}}}ds - \lambda_1 \frac{1}{\Gamma(1-\{\beta\})}(\frac{\partial}{\partial t})^{[\beta]+1} \int_0^t \frac{u(s,x)-u_N(s,x)}{(t-s)^{\{\beta\}}}ds = \lambda_2(u_{tt}(t,x)-(u_{tt})_N(t,x) + u_{xx}(t,x)-(u_{xx})_N(t,x)) + f(t,x) - f_N(t,x)\ . \quad (3.49)$$

Taking L^2-norm both side in $Eq.(3.49)$ as follows:

$$\left\| \frac{1}{\Gamma(1-\{\alpha\})}(\frac{\partial}{\partial t})^{[\alpha]+1}\int_0^t \frac{u(s,x)-u_N(s,x)}{(t-s)^{\{\alpha\}}}ds - \lambda_1 \frac{1}{\Gamma(1-\{\beta\})}(\frac{\partial}{\partial t})^{[\beta]+1} \int_0^t \frac{u(s,x)-u_N(s,x)}{(t-s)^{\{\beta\}}}ds \right\| = \left\| \lambda_2(u_{tt}(t,x)-(u_{tt})_N(t,x) + u_{xx}(t,x)-(u_{xx})_N(t,x)) + f(t,x) - f_N(t,x) \right\|,$$

or,

$$|\frac{1}{\Gamma(1-\{\alpha\})}||(\frac{\partial}{\partial t})^{[\alpha]+1}\int_0^t \frac{\|u(s,x)-u_N(s,x)\|}{(t-s)^{\{\alpha\}}}ds - |\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}||(\frac{\partial}{\partial t})^{[\beta]+1} \int_0^t \frac{\|u(s,x)-u_N(s,x)\|}{(t-s)^{\{\beta\}}}ds \leq |\lambda_2|(\|u_{tt}(t,x)-(u_{tt})_N(t,x)\| + \|u_{xx}(t,x)-(u_{xx})_N(t,x)\|) + \|f(t,x) - f_N(t,x)\|,$$

or,

$$\|u(t,x)-u_N(t,x)\|(|\frac{1}{\Gamma(1-\{\alpha\})}||(\frac{\partial}{\partial t})^{[\alpha]+1}\int_0^t \frac{1}{(t-s)^{\{\alpha\}}}ds - |\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}| |(\frac{\partial}{\partial t})^{[\beta]+1} \int_0^t \frac{1}{(t-s)^{\{\beta\}}}ds) \leq |\lambda_2|(\|u_{tt}(t,x)-(u_{tt})_N(t,x)\| + \|u_{xx}(t,x)-(u_{xx})_N(t,x)\|) + \|f(t,x) - f_N(t,x)\|,$$

or,

$$\|u(t,x)-u_N(t,x)\|(|\frac{1}{\Gamma(1-\{\alpha\})}||(\frac{\partial}{\partial t})^{[\alpha]+1}(\frac{t^{1-\{\alpha\}}}{\{\alpha\}-1}) - |\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}| |(\frac{\partial}{\partial t})^{[\beta]+1}(\frac{t^{1-\{\beta\}}}{\{\beta\}-1}) \leq |\lambda_2|(\|u_{tt}(t,x)-(u_{tt})_N(t,x)\| + \|u_{xx}(t,x)-(u_{xx})_N(t,x)\|) + \|f(t,x) - f_N(t,x)\|, \quad (3.50)$$

since, $1 \leq \beta < \alpha < 3$ then

$$[\alpha] = \begin{cases} 1 & \text{if } 1 \leq \alpha < 2; \\ 2 & \text{if } 2 \leq \alpha < 3. \end{cases} \tag{3.51}$$

and also

$$[\beta] = \begin{cases} 1 & \text{if } 1 \leq \beta < 2; \\ 2 & \text{if } 2 \leq \beta < 3. \end{cases} \tag{3.52}$$

thats why $Eq.$(3.50) arieses three cases as follows:

Case 1. Error bound if $1 < \alpha < 2$ and $1 \leq \beta < 2$ then $Eq.$(3.50) as follows:

$$\begin{aligned} \|u(t,x) - u_N(t,x)\|(|\frac{1}{\Gamma(1-\{\alpha\})}|(\frac{\partial}{\partial t})^2(\frac{t^{1-\{\alpha\}}}{\{\alpha\}-1}) - |\lambda_1||\frac{1}{\Gamma(1-\{\beta\})} \\ |(\frac{\partial}{\partial t})^2(\frac{t^{1-\{\beta\}}}{\{\beta\}-1}) \leq |\lambda_2|(\|u_{tt}(t,x) - (u_{tt})_N(t,x)\| \\ +\|u_{xx}(t,x) - (u_{xx})_N(t,x)\|) + \|f(t,x) - f_N(t,x)\|, \end{aligned}$$

or,

$$\begin{aligned} \|u(t,x) - u_N(t,x)\|(|\frac{1}{\Gamma(1-\{\alpha\})}|(-\{\alpha\})t^{-1-\{\alpha\}} - |\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|) \\ (-\{\beta\})t^{-1-\{\beta\}}) \leq |\lambda_2|\|u_{tt}(t,x) - (u_{tt})_N(t,x)\| \\ +|\lambda_2|\|u_{xx}(t,x) - (u_{xx})_N(t,x)\| + \|f(t,x) - f_N(t,x)\|, \end{aligned}$$

or,

$$\begin{aligned} &\|u(t,x) - u_N(t,x)\| \\ &\quad \leq \frac{\|f(t,x) - f_N(t,x)\| + |\lambda_2|\|u_{tt}(t,x) - (u_{tt})_N(t,x)\|}{(|\frac{1}{\Gamma(1-\{\alpha\})}|(-\{\alpha\})t^{-1-\{\alpha\}} + |\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|)(\{\beta\})t^{-1-\{\beta\}})} \\ &\quad + \frac{|\lambda_2|\|u_{xx}(t,x) - (u_{xx})_N(t,x)\|}{(|\frac{1}{\Gamma(1-\{\alpha\})}|(-\{\alpha\})t^{-1-\{\alpha\}} + |\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|)(\{\beta\})t^{-1-\{\beta\}})}. \end{aligned} \tag{3.53}$$

Case 2. Error bound if $2 \leq \alpha < 3$ and $2 \leq \beta < 3$ then $Eq.(3.50)$ as follows:

$$\begin{aligned}
&\|u(t,x)-u_N(t,x)\| \\
&\leq \frac{\|f(t,x)-f_N(t,x)\|+|\lambda_2|\|u_{tt}(t,x)-(u_{tt})_N(t,x)\|}{(|\frac{1}{\Gamma(1-\{\alpha\})}|\{\alpha\}(1+\{\alpha\})t^{-2-\{\alpha\}}-|\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|)\{\beta\}(1+\{\beta\})t^{-2-\{\beta\}})} \\
&+\frac{|\lambda_2|\|u_{xx}(t,x)-(u_{xx})_N(t,x)\|}{(|\frac{1}{\Gamma(1-\{\alpha\})}|\{\alpha\}(1+\{\alpha\})t^{-2-\{\alpha\}}-|\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|)\{\beta\}(1+\{\beta\})t^{-2-\{\beta\}})}.
\end{aligned} \tag{3.54}$$

Case 3. Error bound if $2 \leq \alpha < 3$ and $1 \leq \beta < 2$ then $Eq.(3.50)$ as follows:

$$\begin{aligned}
&\|u(t,x)-u_N(t,x)\| \\
&\leq \frac{\|f(t,x)-f_N(t,x)\|+|\lambda_2|\|u_{tt}(t,x)-(u_{tt})_N(t,x)\|}{(|\frac{1}{\Gamma(1-\{\alpha\})}|(\{\alpha\}(1+\{\alpha\})t^{-2-\{\alpha\}}+|\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|(\{\beta\})t^{-1-\{\beta\}})} \\
&+\frac{|\lambda_2|\|u_{xx}(t,x)-(u_{xx})_N(t,x)\|}{(|\frac{1}{\Gamma(1-\{\alpha\})}|(\{\alpha\}(1+\{\alpha\})t^{-2-\{\alpha\}}+|\lambda_1||\frac{1}{\Gamma(1-\{\beta\})}|(\{\beta\})t^{-1-\{\beta\}})}.
\end{aligned} \tag{3.55}$$

3.7 Numerical examples

In this section, six examples are given to demonstrate the efficiency and accuracy of our proposed method using Legendre wavelet approximations (LWA) as well as Chebshev wavelet approximation (CWA) for $1 \leq \beta < \alpha < 3$ with basis function at $k = k' = 1, M = M^{'} = 3$ (see graph 3.1-3.6). Also, we are given error tables (see table 3.2,3.4,3.6,3.8,3.10,3.12), $l_2 - error$ and $l_\infty - error$ (see table 3.3,3.5,3.7,3.9,3.11,3.13) for each example.

Example 3.1 Consider the following FPDE for $\alpha = \frac{4}{3}$ and $\beta = \frac{5}{4}$:

$$\begin{cases} ({}_0D_t^{\alpha}u)(t,x) - \frac{1}{4}({}_0D_t^{\beta}u)(t,x) - \frac{1}{5}\nabla^2 u(t,x) = f(t,x); \\ u(t,0) = 0; \\ f(t,x) = \frac{x}{t^{\frac{1}{3}}\Gamma\frac{2}{3}} - \frac{x}{4t^{\frac{1}{4}}\Gamma\frac{3}{4}}. \end{cases}$$

For these conditions there is known analytical solution

$$u(t,x) = tx.$$

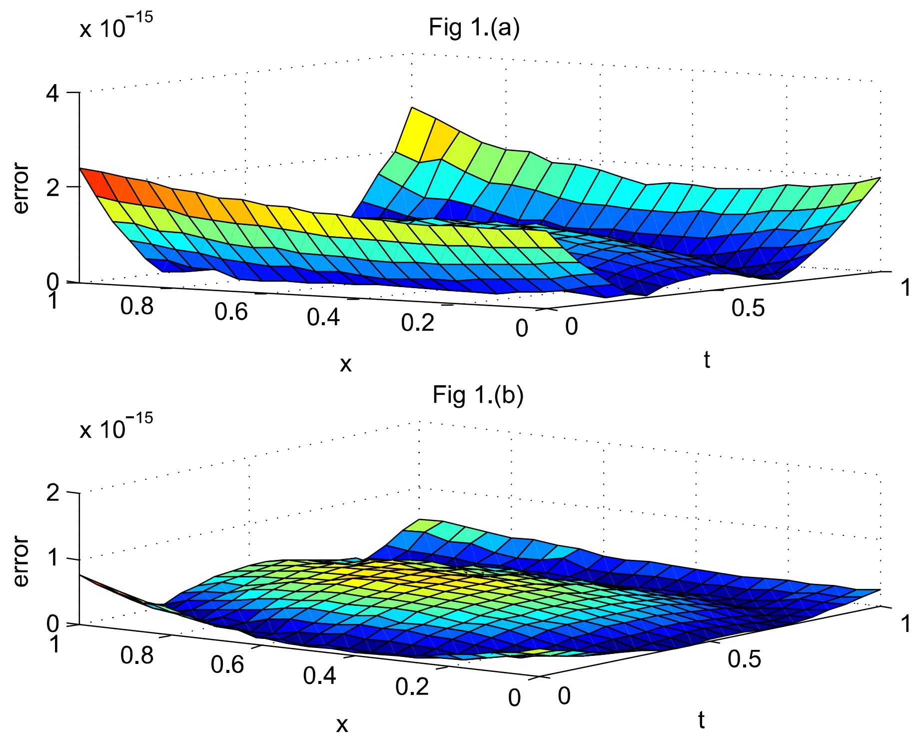

Figure 3.1: Fig 1.(a), represent the errors between the exact solution of *example* 3.1 and its numerical solution to utilizing the LWA for $k = k' = 1, M = M' = 3$ and Fig 1.(b), represent the errors between the exact solution of *example* 3.1 and its numerical solution to utilizing the CWA for $k = k' = 1, M = M' = 3$.

Example 3.2 Consider the following FPDE for $\alpha = \frac{7}{3}$ and $\beta = \frac{9}{4}$:

$$\begin{cases} ({}_0D_t^{\alpha}u)(t,x) - \frac{1}{4}({}_0D_t^{\beta}u)(t,x) - \frac{1}{5}\nabla^2 u(t,x) = f(t,x); \\ u(t,0) = 0; \\ f(t,x) = \frac{-x}{3t^{\frac{4}{3}}\Gamma\frac{2}{3}} - \frac{x}{16t^{\frac{1}{4}}\Gamma\frac{3}{4}}. \end{cases}$$

For these conditions there is known analytical solution

$$u(t,x) = tx.$$

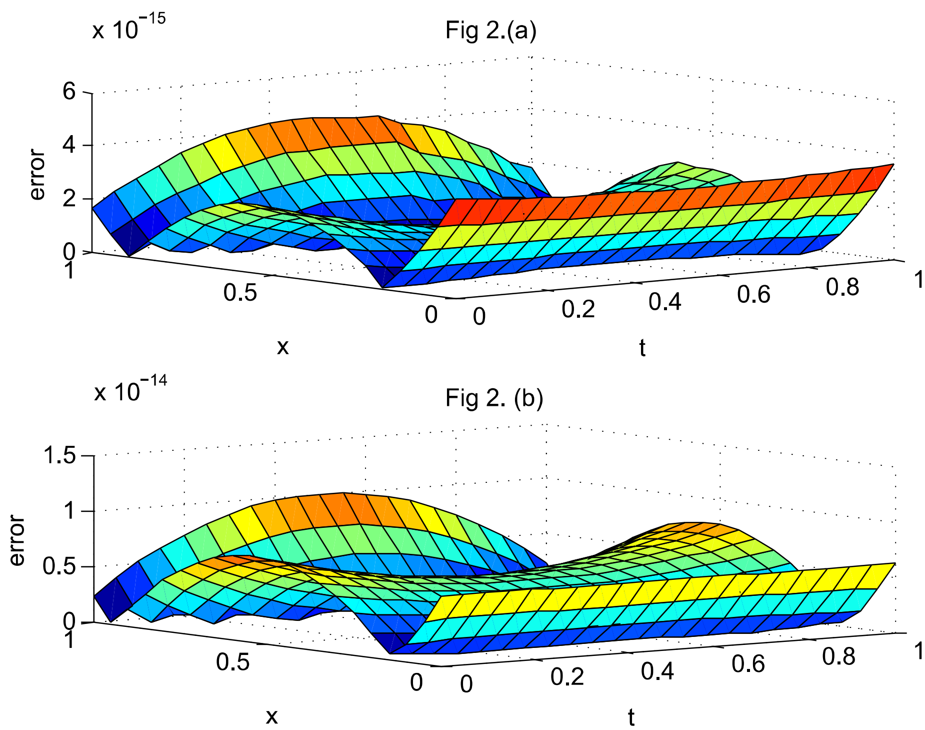

Figure 3.2: Fig 2.(a), represent the errors between the exact solution of *example* 3.2 and its numerical solution to utilizing the LWA for $k = k' = 1, M = M' = 3$ and Fig 2.(b), represent the errors between the exact solution of *example* 3.2 and its numerical solution to utilizing the CWA for $k = k' = 1, M = M' = 3$.

Example 3.3 Consider the following FPDE for $\alpha = \frac{7}{3}$ and $\beta = \frac{5}{4}$:

$$\begin{cases} ({}_0D_t^{\alpha}u)(t,x) - \frac{1}{4}({}_0D_t^{\beta}u)(t,x) - \frac{1}{5}\nabla^2 u(t,x) = f(t,x); \\ u(t,0) = 0; \\ f(t,x) = \frac{-x}{3t^{\frac{4}{3}}\Gamma\frac{2}{3}} - \frac{x}{4t^{\frac{1}{4}}\Gamma\frac{3}{4}}. \end{cases}$$

For these conditions there is known analytical solution

$$u(t,x) = tx.$$

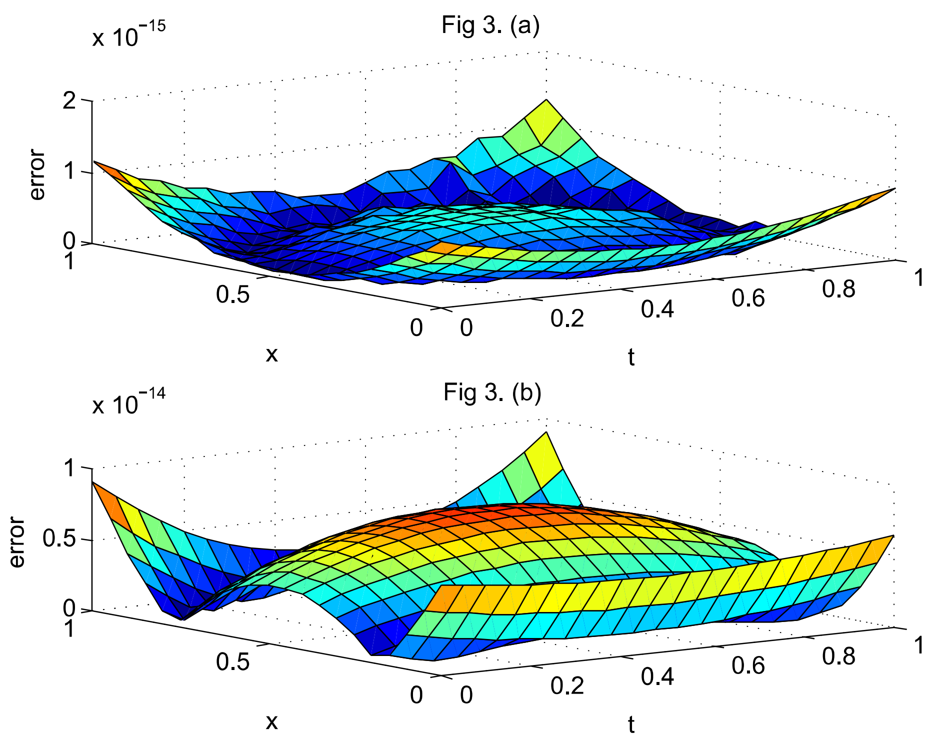

Figure 3.3: Fig 3.(a), represent the errors between the exact solution of *example* 3.3 and its numerical solution to utilizing the LWA for $k = k' = 1, M = M' = 3$ and Fig 3.(b), represent the errors between the exact solution of *example* 3.3 and its numerical solution to utilizing the CWA for $k = k' = 1, M = M' = 3$.

Example 3.4 Consider the following FPDE for $\alpha = \frac{4}{3}$ and $\beta = \frac{5}{4}$:

$$\begin{cases} ({}_0D_t^{\alpha}u)(t,x) - \frac{1}{4}({}_0D_t^{\beta}u)(t,x) - \frac{1}{5}\nabla^2 u(t,x) = f(t,x); \\ u(t,0) = 0; \\ f(t,x) = \frac{1}{5}[2(1-t)t + 2x(1-x)] + \frac{1}{4}[\frac{16\sqrt{t}x(1-x)}{5\sqrt{\pi}} + \frac{(4t-1)(1-x)x}{5\sqrt{\pi}\sqrt{t}}] \\ + \frac{9x(x-1)}{4t^{\frac{1}{3}}\Gamma\frac{2}{3}} + \frac{(3t-4)x(x-1)}{12t^{\frac{4}{3}}\Gamma\frac{2}{3}}. \end{cases}$$

For these conditions there is known analytical solution

$$u(t,x) = tx(t-1)(x-1)).$$

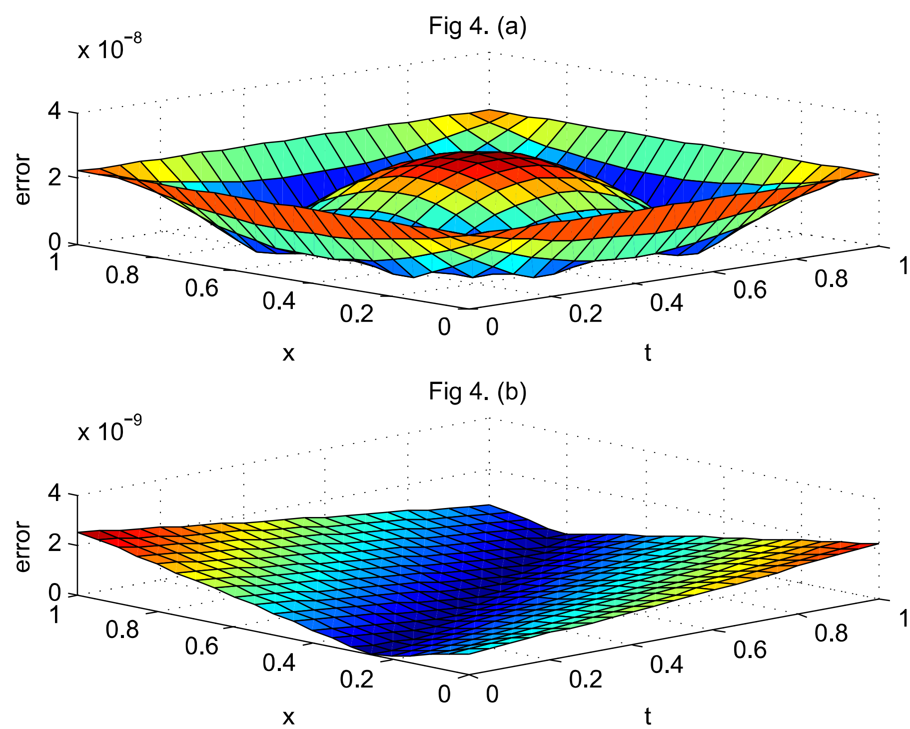

Figure 3.4: Fig 4.(a), represent the errors between the exact solution of *example* 3.4 and its numerical solution to utilizing the LWA for $k = k' = 1, M = M' = 3$ and Fig 4.(b), represent the errors between the exact solution of *example* 3.4 and its numerical solution to utilizing the CWA for $k = k' = 1, M = M' = 3$.

Example 3.5 Consider the following FPDE for $\alpha = \frac{7}{3}$ and $\beta = \frac{9}{4}$:

$$\begin{cases} ({}_0D_t^{\alpha}u)(t,x) - \frac{1}{4}({}_0D_t^{\beta}u)(t,x) - \frac{1}{5}\nabla^2 u(t,x) = f(t,x); \\ u(t,0) = 0; \\ f(t,x) = \frac{1}{5}[2(1-t)t + 2x(1-x)] + \frac{1}{4}[\frac{-64t^{\frac{3}{4}}x(1-x)}{33\Gamma\frac{3}{4}} - \frac{(8t-11)(1-x)x}{11t^{\frac{1}{4}}\Gamma\frac{3}{4}}] \\ + \frac{9t^{\frac{2}{3}}x(x-1)}{4\Gamma\frac{2}{3}} + \frac{(3t-4)x(x-1)}{4t^{\frac{1}{3}}\Gamma\frac{2}{3}}. \end{cases}$$

For these conditions there is known analytical solution

$$u(t,x) = tx(t-1)(x-1)).$$

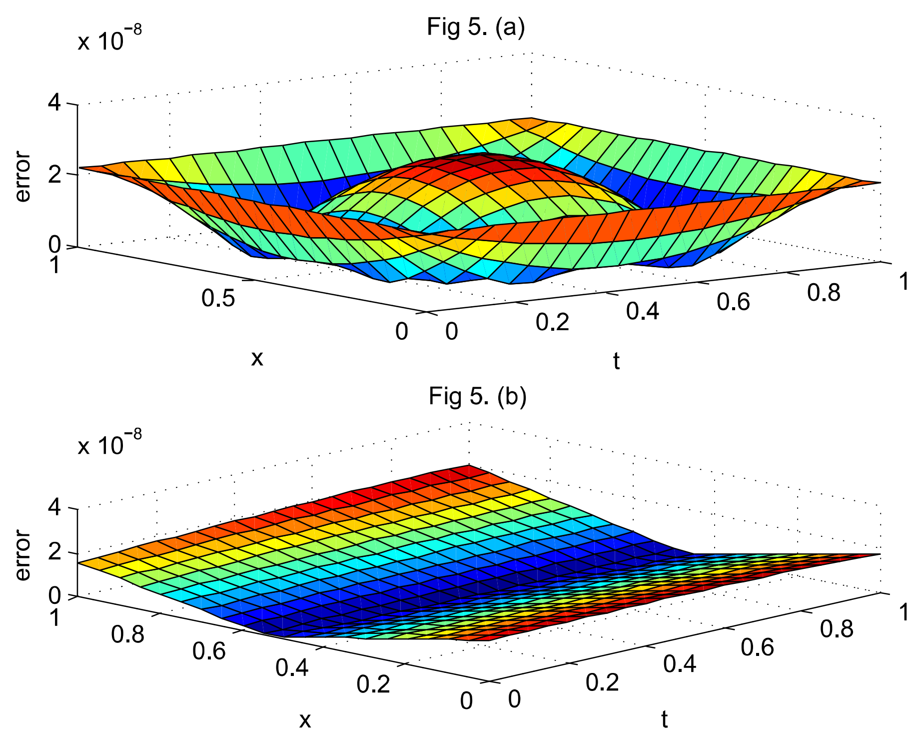

Figure 3.5: Fig 5.(a), represent the errors between the exact solution of *example* 3.5 and its numerical solution to utilizing the LWA for $k = k' = 1, M = M' = 3$ and Fig 5.(b), represent the errors between the exact solution of *example* 3.5 and its numerical solution to utilizing the CWA for $k = k' = 1, M = M' = 3$.

Example 3.6 Consider the following FPDE for $\alpha = \frac{7}{3}$ and $\beta = \frac{5}{4}$:

$$\begin{cases} ({}_0D_t^{\alpha}u)(t,x) - \frac{1}{4}({}_0D_t^{\beta}u)(t,x) - \frac{1}{5}\nabla^2 u(t,x) = f(t,x); \\ u(t,0) = 0; \\ f(t,x) = \frac{1}{5}[2(1-t)t + 2x(1-x)] + \frac{1}{4}[\frac{-64t^{\frac{3}{4}}x(1-x)}{33\Gamma\frac{3}{4}} - \frac{(8t-11)(1-x)x}{11t^{\frac{1}{4}}}] \\ +\frac{9x(x-1)}{4t^{\frac{1}{3}}\Gamma\frac{2}{3}} + \frac{(3t-4)x(x-1)}{12t^{\frac{4}{3}}\Gamma\frac{2}{3}}. \end{cases}$$

For these conditions there is known analytical solution

$$u(t,x) = tx(t-1)(x-1)).$$

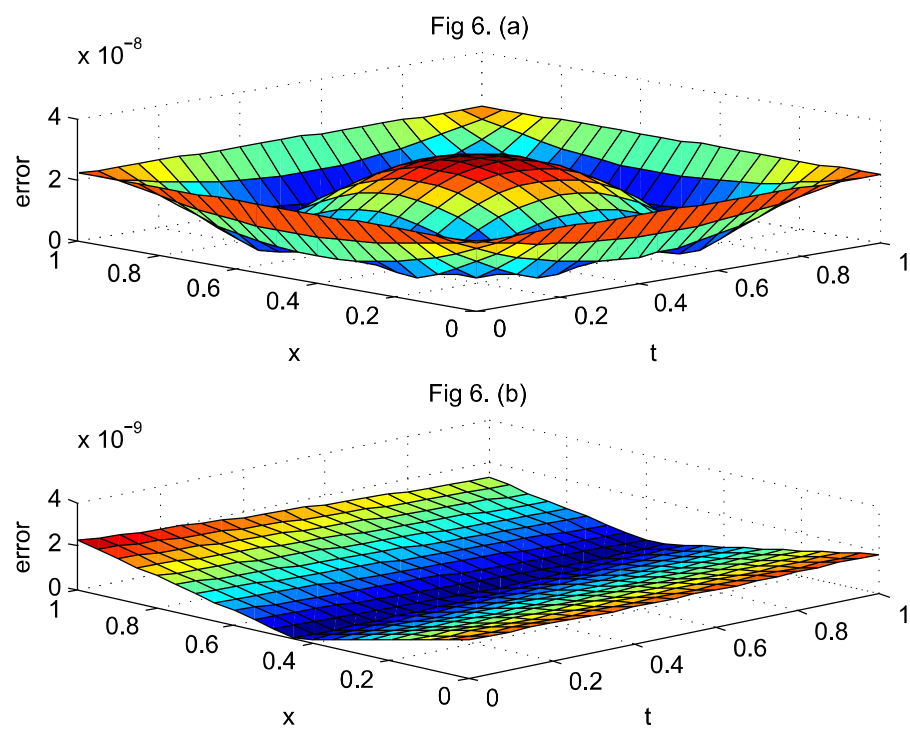

Figure 3.6: Fig 6.(a), represent the errors between the exact solution of *example* 3.6 and its numerical solution to utilizing the LWA for $k = k' = 1, M = M' = 3$ and Fig 6.(b), represent the errors between the exact solution of *example* 3.6 and its numerical solution to utilizing the CWA for $k = k' = 1, M = M' = 3$.

Table 3.2: Error by using LWA and CWA for *example* 3.1 at $\alpha = \frac{4}{3}$, $\beta = \frac{7}{3}$,$k = k' = 1$ and $M = M' = 3$.

(t, x)	**LWA**	**CWA**
$(0.1, 0.1)$	1.01e-14	1.35e-15
$(0.2, 0.2)$	1.10e-14	1.24e-15
$(0.3, 0.3)$	1.06e-14	3.11e-15
$(0.4, 0.4)$	1.08e-14	1.17e-15
$(0.5, 0.5)$	1.06e-14	4.85e-15
$(0.6, 0.6)$	1.02e-14	4.43e-15
$(0.7, 0.7)$	1.07e-14	3.57e-15
$(0.8, 0.8)$	1.02e-14	1.06e-15
$(0.9, 0.9)$	9.44e-15	1.12e-15

Table 3.3: $l_2 - error$ and $l_\infty - error$ by using LWA and CWA for *example* 3.1 at $\alpha = \frac{4}{3},\ \beta = \frac{5}{4}$,$k = k' = 1$ and $M = M' = 3$.

Norm	LWA	CWA
l_2	3.36e-14	1.71e-15
l_∞	1.10e-14	5.25e-16

Table 3.4: Error by using LWA and CWA for *example* 3.2 at $\alpha = \frac{7}{3},\ \beta = \frac{9}{4}$, $k = k' = 1$ and $M = M' = 3$.

(t, x)	LWA	CWA
$(0.1, 0.1)$	4.74e-16	1.69e-15
$(0.2, 0.2)$	4.20e-17	1.61e-16
$(0.3, 0.3)$	2.64e-16	3.74e-16
$(0.4, 0.4)$	5.27e-16	4.76e-15
$(0.5, 0.5)$	5.56e-16	4.79e-15
$(0.6, 0.6)$	4.44e-16	4.05e-15
$(0.7, 0.7)$	2.78e-16	2.70e-16
$(0.8, 0.8)$	0.00e-16	1.02e-16
$(0.9, 0.9)$	5.56e-16	2.60e-16

Table 3.5: $l_2 - error$ and $l_\infty - error$ by using LWA and CWA for *example* 3.2 at $\alpha = \frac{7}{3},\ \beta = \frac{5}{4}$,$k = k' = 1$ and $M = M' = 3$.

Norm	LWA	CWA
l_2	2.03e-15	1.17e-14
l_∞	1.30e-15	2.44e-15

Table 3.6: Error by using LWA and CWA for *example* 3.3 at $\alpha = \frac{7}{3}$, $\beta = \frac{5}{4}$, $k = k' = 1$ and $M = M' = 3$.

(t, x)	**LWA**	**CWA**
$(0.1, 0.1)$	1.60e-15	1.89e-15
$(0.2, 0.2)$	1.32e-16	1.94e-15
$(0.3, 0.3)$	8.19e-16	4.87e-15
$(0.4, 0.4)$	1.22e-15	6.83e-15
$(0.5, 0.5)$	1.33e-15	7.67e-15
$(0.6, 0.6)$	1.05e-15	7.32e-15
$(0.7, 0.7)$	5.56e-16	5.59e-15
$(0.8, 0.8)$	2.22e-16	2.45e-15
$(0.9, 0.9)$	9.98e-16	4.57e-15

Table 3.7: $l_2 - error$ and $l_\infty - error$ by using LWA and CWA for *example* 3.3 at $\alpha = \frac{7}{3}$, $\beta = \frac{9}{4}$,$k = k' = 1$ and $M = M' = 3$.

Norm	**LWA**	**CWA**
l_2	5.07e-15	1.89e-14
l_∞	3.69e-15	9.09e-15

Table 3.8: Error by using LWA and CWA for *example* 3.4 at $\alpha = \frac{4}{3}$, $\beta = \frac{7}{3}$, $k = k' = 1$ and $M = M' = 3$.

(t, x)	**LWA**	**CWA**
$(0.1, 0.1)$	1.57e-8	2.31e-8
$(0.2, 0.2)$	1.74e-9	2.09e-8
$(0.3, 0.3)$	1.31e-8	1.89e-8
$(0.4, 0.4)$	2.38e-8	1.69e-8
$(0.5, 0.5)$	2.78e-8	1.15e-8
$(0.6, 0.6)$	2.39e-8	1.29e-8
$(0.7, 0.7)$	1.31e-8	1.11e-8
$(0.8, 0.8)$	1.74e-9	9.18e-9
$(0.9, 0.9)$	1.57e-8	4.41e-9

Table 3.9: $l_2 - error$ and $l_\infty - error$ by using LWA and CWA for *example* 3.4 at $\alpha = \frac{4}{3}$, $\beta = \frac{5}{4}$,$k = k' = 1$ and $M = M' = 3$.

Norm	**LWA**	**CWA**
l_2	6.11e-8	1.43e-8
l_∞	2.78e-8	5.58e-9

Table 3.10: Error by using LWA and CWA for *example* 3.5 at $\alpha = \frac{7}{3},\ \beta = \frac{9}{4}$, $k = k' = 1$ and $M = M' = 3$.

(t, x)	**LWA**	**CWA**
$(0.1, 0.1)$	5.52e-9	1.26e-8
$(0.2, 0.2)$	8.50e-9	9.67e-9
$(0.3, 0.3)$	7.28e-9	6.59e-9
$(0.4, 0.4)$	1.41e-8	3.35e-9
$(0.5, 0.5)$	1.67e-8	0.00e-9
$(0.6, 0.6)$	1.40e-8	3.43e-9
$(0.7, 0.7)$	7.28e-9	6.90e-9
$(0.8, 0.8)$	8.50e-9	1.04e-8
$(0.9, 0.9)$	5.52e-8	1.38e-8

Table 3.11: $l_2 - error$ and $l_\infty - error$ by using LWA and CWA for *example* 3.5 at $\alpha = \frac{7}{3},\ \beta = \frac{5}{4}$,$k = k' = 1$ and $M = M' = 3$.

Norm	**LWA**	**CWA**
l_2	6.32e-8	3.45e-8
l_∞	5.52e-8	1.71e-8

Table 3.12: Error by using LWA and CWA for *example* 3.6 at $\alpha = \frac{7}{3},\ \beta = \frac{5}{4}$, $k = k' = 1$ and $M = M' = 3$.

(t, x)	**LWA**	**CWA**
$(0.1, 0.1)$	6.48e-9	1.33e-8
$(0.2, 0.2)$	2.05e-8	9.66e-9
$(0.3, 0.3)$	3.52e-8	6.23e-9
$(0.4, 0.4)$	4.61e-8	3.01e-9
$(0.5, 0.5)$	5.00e-8	0.00e-9
$(0.6, 0.6)$	4.68e-8	2.86e-9
$(0.7, 0.7)$	3.53e-8	5.58e-9
$(0.8, 0.8)$	2.05e-8	8.21e-9
$(0.9, 0.9)$	6.45e-9	1.08e-8

Table 3.13: $l_2 - error$ and $l_\infty - error$ by using LWA and CWA for *example* 3.6 at $\alpha = \frac{7}{3},\ \beta = \frac{9}{4}$,$k = k' = 1$ and $M = M' = 3$.

Norm	**LWA**	**CWA**
l_2	1.01e-7	3.18e-8
l_∞	5.00e-8	1.33e-9

3.8 Conclusions

In this chapter, we first transform the original fractional partial differential equations into a equivalent weak singular fractional partial integro-differential equations and then we applied numerical wavelet collocation method based on Legendre and Chebyshev wavelets approximation and utilized its operational matrices to solve the fractional partial integro-differential equations. For this purpose proposed fractional partial integro-differential equations is written in the system of linear algebraic equation by using wavelets operational matrix of integration and differentiation based on Legendre wavelet approximation and Chebyshev wavelet approximation. Moreover, we studied the convergence analysis, and error analysis of the proposed method when $1 \leq \beta < \alpha < 3$. Finally, from the examples considered here, it can be easily seen that numerical wavelet collocation method for fractional partial differential equations arising from electromagnetic waves in dielectric media obtained result as accurate as possible.

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