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Introduction to singularly perturbed problems

Singularly perturbed problems have many applications in fields of applied mathematics and engineering, such as fluid mechanics, quantum mechanics, elasticity, plasticity, control theory, and diffraction theory. Because of great importance in the above fields, there is a growing interest in finding the solution to them. A classic example of a singularly perturbed problem is the flow of fluid at low viscosity past a body [1]. In viscous flow, the tangential velocity must be zero at the boundary of the body, while in non-viscous flow the fluid can slide along the boundary. Thus, if the viscosity is neglected, the solution of this reduced problem will not approximate the actual viscous flow near the boundary. However, for low viscosity, the actual viscous flow will be closely approximated by the non-viscous flow, except in a narrow region near the boundary. Prandtl [1] was first who developed the concept of singularly perturbed problems during his talk on fluid motion with very small fraction in the Third International Conference of Mathematician in Heidelberg in 1904. Prandtl proved that fluid flow about a body could be treated by dividing it into two regions: a boundary layer and remaining outside region. The term boundary layer was first introduced by Prandtl, but Wasow [2] has generalized this term.

In a physical system, perturbation is defined as a small disturbance. In perturbation theory, we study the effect of a small parameter in models. A perturbation problem can be classified as a regular and singular perturbation problem. These problems can be ordinary differential equations or partial differential equations or simply algebraic equations. Here, we will focus mainly on partial differential equations. The problems in which a small parameter is associated with the highest derivative term are termed as singularly perturbed problems. In singularly perturbed problems, the domain of interest has very narrow regions where the solution undergoes very rapid changes; these regions are known as the boundary layer regions. Although the layer regions are small, these regions significantly influence the overall solution. It is crucial to determine the location and width of all layers for solving such problems.

The principal approaches for finding approximate solutions of singularly perturbed problems are of two types, namely asymptotic expansions and numerical methods. Since the mid 1960s, asymptotic expansions approach has flourished, and its key ideas and methods were given by W.Eckhaus [3]; nevertheless, it may be impossible to construct asymptotic expansions for the given problem, and then there is no alternative other than to use numerical approximations [4–6]. The error bounds of classical numerical methods for singularly perturbed problems have the structure

$$\|\mathrm{u}_\varepsilon - \mathrm{U}_\varepsilon\| \leq C(\varepsilon)N^{-p},$$

where U_ε is an approximation to the exact solution u_ε, N is the discretization parameter, $C(\varepsilon)$ depends on the perturbation parameter ε and approaches to infinity as the perturbation parameter ε tends to zero. It signifies that the discretization parameter N should be proportional to some (negative) power of the perturbation parameter ε, which is impractical. It is therefore of both theoretical and practical interest to develop special numerical method whose convergence does not depend upon the perturbation parameter ε. Such a method is known as parameter-uniform or parameter-robust or uniformly convergent or robust methods in the literature.

Definition 1.0.1. *Let u_ε be the exact solution of a singularly perturbed problem and let U_ε^N be an approximation to u_ε generated by some numerical method. The numerical method is said to be uniformly convergent or robust with respect to ε in a given norm $||.||_*$ if there exists $N_0 > 0$ independent of ε such that $N \geq N_0$*

$$||\mathrm{u}_\varepsilon - \mathrm{U}_\varepsilon^N||_* \leq C\nu(N), \qquad \lim_{N\to\infty} \nu(N) = 0,$$

with a function ν that is independent of ε and constant $C > 0$ that is independent of ε and N.

1.1 Parameter-robust numerical methods

For the development of parameter-robust numerical methods, several non-classical numerical approaches, namely fitted operator, fitted mesh, and domain decomposition have been studied in the literature. Fitted operator approach involves substituting standard finite difference/ finite element operator by a specially designed finite difference/ finite element operator on a uniform mesh. These operators are designed to reflect the solution behavior in the boundary layer regions. In this approach, the coefficients of the difference equation are chosen to ensure parameter-robust accuracy of the approximate solution. The construction of this approach does not require a priori knowledge about the location and size of the boundary layer. Fitted operator numerical methods can be found in [5, 7] and the references therein. In the fitted mesh approach, we make use of a specially designed mesh. This approach uses the standard classical difference equations, but the nodes of the mesh are distributed such that parameter-robust convergence can be achieved. The first special graded mesh was introduced in 1969 by Bakhvalov [8]. In 1988 Shishkin [9] proposed a simple piecewise-uniform mesh. Fitted mesh approach is easily extendable to higher dimensional problems and requires a priori knowledge about the location and size of boundary layers. In past few decades, we have considerable progress in the construction of robust methods on special layer resolving meshes of Bakhvalov and Shishkin types [5, 6]. Domain Decomposition is another approach that is described later.

1.2 Literature review on parameter-robust methods

1.2.1 Singularly perturbed parabolic reaction-diffusion problems

On going through the literature, we found that the numerical treatment of these problems has drawn a lot of attention of many researchers. Miller et al. [10] derived a numerical method that uses the central difference approximation on a piecewise uniform Shishkin mesh to discretize spatial derivative and implicit Euler approximation on a uniform mesh to discretize time derivative. This method is shown to be parameter-uniform of order $O(M_t^{-1} + N_x^{-2} \ln^2 N_x)$, where M_t and N_x are the number of mesh intervals in time and space directions, respectively. Using Green's functions based approach, Linß and Madden [11] presented a categorization of fitted mesh methods that avoid the use of barrier functions and solution decompositions. They gave stability and convergence analysis for Shishkin and Bakhvalov meshes in the same framework. Natesan and Deb [12] developed a numerical method which involves the classical finite difference and the cubic spline scheme to discretize in space, together with backward difference scheme to discretize in time. They derived stability analysis and established parameter-robust error estimates of order $O(M_t^{-1} + N_x^{-2} \ln^2 N_x)$. Kumar and Rao [13] constructed a numerical method which combines HOC (High order Compact) finite difference scheme on a generalized Shishkin mesh in space, together with the Crank–Nicolson scheme on a uniform mesh in time. This method is shown to be parameter-robust, having convergence of order two in time and almost four in space. Clavero and Gracia [14] constructed a uniformly convergent numerical method which involves the backward Euler scheme on a uniform mesh in time and the classical central difference scheme on a special mesh, and obtained convergence of order almost two in space and one in time. To get higher order accuracy in space, Clavero and Gracia [15] developed and analyzed a new numerical method which uses a compact fourth order difference scheme (HODIE) on a generalized Shishkin mesh to discretize in space and the implicit Euler method on a uniform mesh to discretize in time. This method is shown to be uniformly convergent, having order almost four in space and one in time. Later, Clavero and Gracia [16] constructed another high order uniformly convergent scheme to improve computational accuracy in time, which combines the idea of [15] with the use of the Richardson extrapolation approach to discretize in time and obtained improved convergence of order two in time.

1.2.2 Singularly perturbed parabolic reaction-diffusion problems with time delay

Delay differential equations arise as mathematical models describing various natural phenomena in Biosciences (see [17, 18]). Among these, a considerable class includes singularly perturbed delay differential equations, that is, equations with a small parameter multiplying the highest derivative (see, for example, [19–22]). Further, a set of examples illustrating the wide range of existing delay PDE models can be found in [18]. For solving these problems, Ansari et al. [23] developed a numerical scheme by considering central difference approximation in space and Euler implicit approximation in time and proved it to be uniform convergent of $O(M_t^{-1} + N_x^{-2} \ln^2 N_x)$. To get high order parameter-uniform approximations, Kumar and Kumar [24] discretized the problem by a hybrid finite difference scheme on a generalized Shishkin mesh in space and the implicit Euler difference scheme on a uniform mesh in time. Further, they designed a Richardson extrapolation approach for increasing the accuracy in time. Finally, they proved uniform convergence of $O(M_t^{-2} + N_x^{-4} \ln^4 N_x)$. Gowrisankar and Natesan [25] solved the problem by an upwind numerical scheme on layer-adapted grids generated using equidistribution of a monitor function, and proved parameter-uniform convergence of order $O(M_t^{-1} + N_x^{-2})$. Bashier and Patidar [26] developed the fitted operator numerical method in which Crank–Nicolson's discretization based fitted operator was used to replace the classical differential operator. They analyzed the stability and convergence of the method and proved it to be unconditionally stable and parameter-robust with order $O(M_t^{-2} + N_x^{-2})$. Wang [27] developed a reliable analytical technique wherein the series representation of outer and inner exact solutions in reproducing kernel Hilbert space is considered as the part of the exact solution. Using a similar approach, Rao and Chakravarthy [28] constructed an unconditionally stable fitted Numerov method on a uniform mesh.

1.2.3 Coupled systems of singularly perturbed parabolic reaction -diffusion problems

The numerical solution of this class of problems has been investigated in both the cases of two equations [29–32] and an arbitrary number of equations [33–35]. For a system of two equations, the following three cases have been studied in the literature:

$$(i)\ \varepsilon_1 = \varepsilon_2 = \varepsilon \quad (ii)\ \varepsilon_1 = \varepsilon,\ \varepsilon_2 = 1 \quad (iii)\ \varepsilon_1,\ \varepsilon_2 \ \text{ are arbitrary.}$$

Of the three cases, case (iii) is the most interesting and challenging to solve as both components of the solution have boundary layers, and the first component also has a sublayer for $\varepsilon_1 << \varepsilon_2$. For the case of two equations with distinct parameters, Gracia and Lisbona [29] gave the asymptotic analysis for the behavior of the solution and its partial derivatives. They used the implicit Euler method to discretize time derivative on a uniform mesh and the central difference scheme to discretize spatial derivative on a piecewise uniform Shishkin mesh, and proved parameter-robust convergence of $O(M_t^{-1} + N_x^{-2+s} \ln^2 N_x)$ under the assumption $N_x^{-s} \leq CM_t^{-1},\ 0 < s < 1$. To get higher order of convergence for time variable, Clavero et al. [30] constructed a numerical method considering the Crank-Nicolson scheme for time discretization on a uniform mesh and the central finite difference scheme for spatial discretization on a piecewise uniform Shishkin mesh. The resulting method is shown to be uniformly convergent of $O(M_t^{-2} + N_x^{-2+s} \ln^2 N)$ where $N_x^{-s} \leq CM_t^{-1},\ 0 < s < 1$. Clavero and Gracia [31] considered two additive schemes for time derivative discretization on a uniform mesh and the classical central difference approximation for spatial derivative discretization on a piecewise uniform Shishkin mesh, and established the uniform convergence for the semidiscretized and the fully discretized problems. Later, the same authors gave asymptotic analysis for the behavior of the exact solution and its partial derivatives up to six orders in [32]. They constructed a numerical method with backward Euler scheme for time derivative discretization on a uniform mesh and a hybrid finite difference scheme for spatial derivative discretization on a mesh of Vulanovic–Shishkin type and proved that it has uniformly convergent behavior of third and first order in space and time, respectively. Kumar and Rao [33] proposed a high order uniformly convergent method for an arbitrary sized system with one perturbation parameter. They used the Crank-Nicolson method on a uniform mesh for time derivative discretization in conjunction with a hybrid finite difference scheme to discretize spatial derivative on a generalized Shishkin mesh, to get higher order approximations to the exact solution. For solving systems of an arbitrary number of equations, each with a distinct small parameter, Gracia et al. [34] constructed a numerical method which involves the backward Euler scheme for time derivative discretization on a uniform mesh and the central differences scheme for spatial derivative discretization on an appropriate layer-adapted piecewise-uniform mesh. The authors gave the asymptotic analysis of the solution behavior using a new decomposition of the solution and analyzed the uniform convergence of the approximate solution computed from this method. Later, Clavero and Jorge [35] introduced a numerical method for this type of systems combining the central difference scheme for spatial derivative and the fractional implicit Euler method, together with a splitting by components, to discretize time derivative. This method is proven to be uniformly convergent of $O(M_t^{-1} + N_x^{-2} \ln^2 N_x)$.

1.2.4 Singularly perturbed parabolic semilinear reaction- diffusion problems

There are only a few papers in the literature that consider the finite difference approximations to singularly perturbed parabolic semilinear reaction-diffusion problems [36–39]. Kopteva and Savescu [36] established the uniqueness of both continuous and discrete solutions and investigated the accuracy of the conventional and stabilized methods on Shishkin and Bakhvalov types layer-adapted meshes. They established that the considered methods enjoy the uniform convergence of $O(M_t^{-1} + N_x^{-2})$ under restriction $\varepsilon \leq C(N_x^{-1} + M_t^{-1/2})$. Kopteva and Linss [37] gave a posteriori error estimates and considered semi and full discretizations of the backward Euler and the Crank-Nicolson methods. Boglaev [38] constructed a monotone iterative method of lower and upper solutions, which solves only linear discrete systems at each iterative step. The author established its uniform convergence properties and proved it coverges monotonically to the solution of the nonlinear difference scheme. Boglaev [39] analyzed a domain decomposition method wherein nonoverlapping subdomains are used for domain partitioning, and small interfacial subdomains are introduced near the interface of nonoverlapping subdomains.

1.3 Domain decomposition methods

During the last two decades, a great deal of research work on domain decomposition methods has been reported in the field of mathematics and computer science. Methods based on domain decomposition approaches have received more popularity due to their remarkable speed and parallel computing. The mathematics involved in this method is inspired by the work of Schwarz [40] in the nineteenth century. The principle of this approach is that it reduces the problem domain to smaller subdomains and the solution on each subdomain is computed by separate processes where the exchange of solution values between neighboring subdomains is carried out by some appropriate interface conditions. This approach deals with simplified subproblems on simple subdomains rather than solving a complicated problem on the original domain. The final complete solution on the original domain is obtained by combining the solutions of subdomain problems. The benefit of this approach is that the discretizations for each subdomain can be independent and different, which makes the method inherently suitable for parallelization. Domain decomposition methods may be applied either to discrete approximations of differential equations–the discrete approach, or directly to differential equations–the continuous approach. Domain decomposition allows different treatment of each subdomain, that is, users are free to choose different numerical schemes; if there is any singularity in the solution in a particular subdomain it can be treated locally; a non-uniform mesh can be avoided.

The classical way to solve time dependent problems using domain decomposition approach is to first discretize in the time direction by an implicit scheme and then to apply domain decomposition at each time level to solve steady state problems obtained after the time discretization. One drawback of this approach is that it requires to exchange information between subdomains at each time level. This may increase the communication time in parallel environments where starting time to communicate is more. The other way to solve time dependent problems is to discretize in the spatial direction and then apply domain decomposition on resulting system of initial value problems. But the convergence of such methods is proven to be slow. For a comprehensive overview related to domain decomposition methods, we refer the reader to the following references [41,42]. To overcome disadvantages of the classical approach, a new class of domain decomposition methods called Schwarz waveform relaxation (SWR) is introduced in the literature [43–45]. This method is based on decomposition in space and an iteration, where only subproblems in space-time need to be solved. However, for faster convergence, it is essential to use valid transmission conditions between the space-time subdomains. The SWR framework allows use of subdomains with or without overlap, and thus time and space discretization in each subdomain can be taken independently. This method allows solving the model problem over the whole time interval before exchanging information across neighboring subdomains and modifying the boundary conditions at interfaces of the overlapped subdomains.

We next provide the definition of a robust numerical method in domain decomposition framework.

Definition 1.3.1. *Let* u_ε *be the exact solution of a singularly perturbed problem and let* $\mathrm{U}_\varepsilon^{N,k}$ *be an approximation to* u_ε *generated by* k*th step of some iterative numerical method. The numerical method is said to be uniformly convergent with respect to* ε *in a given norm* $||.||_*$ *if there exists* $N_0 > 0$ *independent of* ε *such that* $N \geq N_0$

$$||\mathrm{u}_\varepsilon - \mathrm{U}_\varepsilon^{N,k}||_* \leq C\rho^k + C\nu(N), \qquad 0 \leqslant \rho < 1, \;\; \lim_{N\to\infty} \nu(N) = 0,$$

with a function ν *that is independent of* ε *and constant* $C > 0$ *that is independent of* ε, N *and* k.

1.4 Literature review on domain decomposition methods

During the last two decades, researchers have shown an increased interest in the development of robust domain decomposition methods. H. MacMullen et al. [46] showed that discrete Schwarz domain decomposition method based on central difference approximation with a uniform mesh on each subdomain is parameter-uniform for stationary singularly perturbed reaction-diffusion problems. In [47, 48], the method is extended to a coupled system of stationary singularly perturbed reaction-diffusion problems with two small perturbation parameters. Stephens and Madden [47] proved that the method provides almost second order accurate approximations. Kumar and Kumar [48] gave the convergence analysis of domain decomposition method that is based on introducing some auxiliary problems and separately estimating the contribution to the global error from the iteration and the discretization errors. Moreover, the numerical approximations generated by the method are proved to be almost second order uniformly convergent. For an arbitrarily size coupled system of stationary problems with one perturbation parameter, Stephens and Madden [49] constructed overlapping Schwarz domain decomposition method and established second order uniform convergence which was later improved to almost fourth order convergence by Rao and Kumar [50]. Further in [49], the authors proved that when the perturbation parameter is sufficiently small only one iteration is required to get the desired accuracy. Kopteva et al. [51] applied an overlapping Schwarz domain decomposition method to a stationary semilinear reaction-diffusion problem and proved that second order uniform convergence on Bakhvalov and Shishkin type layer-adapted meshes could be obtained in one iteration when $\varepsilon \leq CN_x^{-1}$. Rao and Kumar [52] established almost fourth order convergence for a stationary singularly perturbed semilinear reaction-diffusion problem, which was later extended to a system of stationary singularly perturbed semilinear reaction-diffusion equations [53]. For coupled nonlinear parabolic equations, Boglaev [54] proposed a domain decomposition method, which is a combination of the monotone approach and Schwarz alternating domain decomposition method.

1.5 Objectives

Over the last few years a new class of decomposition methods, namely Schwarz waveform relaxation methods have been developed for the parallel solution of evolution problems (see [43–45]). A search of the literature shows that Schwarz waveform relaxation methods are very popular for solving parabolic problems of regular type (by regular we mean problems that are not having singular perturbation nature). However, best to our knowledge, they are not yet explored for solving singularly perturbed time dependent problems. The only work we are aware of is due to Kumar and Rao [55], where a discrete Schwarz waveform relaxation method is developed for solving singularly perturbed parabolic problems, respectively. So, the main objectives are to develop domain decomposition methods of discrete Schwarz waveform relaxation types for the following classes of singularly perturbed problems.

- Singularly perturbed parabolic reaction-diffusion problems with time delay.
- Singularly perturbed semilinear parabolic reaction-diffusion problems.
- Coupled systems of singularly perturbed parabolic reaction-diffusion problems with different parameter in each equation.
- Coupled systems of singularly perturbed parabolic reaction-diffusion problems with time delay.

Further, we provide theoretical results for robust convergence analysis of numerical approximations generated from the methods. To validate our theoretical convergence results, we provide numerical results for some test examples.

1.6 Organization of the thesis

The subject of the thesis is to design and analyze domain decomposition methods of SWR types for some classes of singularly perturbed parabolic problems.

The work embodied in the thesis is divided into six chapters.

Chapter 1 firstly introduces singularly perturbed problems, parameter-robust convergence, domain decomposition methods, and after that literature survey and organization of the thesis is given.

In Chapter 2, we design and analyze a domain decomposition method of SWR type for solving singularly perturbed parabolic reaction-diffusion problems with time delay. Using the asymptotic behavior of the solution, we decompose the original domain of the problem into three overlapping subdomains, two of which are boundary layer subdomains, and one is a regular subdomain. On each subdomain, we discretize the problem by the backward Euler scheme in the time direction and the central difference scheme in the spatial direction. The proposed method is shown to be uniformly convergent, having almost second order in space and first order in time. In addition, we prove that the proposed method converges much faster for small values of the perturbation parameter ε.

In Chapter 3, we develop a domain decomposition method of SWR type and analyze its robust convergence for singularly perturbed parabolic semilinear reaction-diffusion problem. We decompose the spatial domain into three overlapping subdomains and obtain the numerical solution by iteratively solving the considered problem on each subdomain. Within each subdomain, we apply a combination of the backward Euler difference approximation for the time variable and the central difference approximation for the space variable. We analyze the independence of the convergence with respect to the size of the perturbation parameter. More notably, we show that the iterative process converges in one iteration when parameter is small.

In Chapter 4, we develop and analyze a domain decomposition method of SWR type for solving a coupled system of singularly perturbed parabolic reaction-diffusion problems with distinct small positive parameters. The proposed method is based on decomposing the original computational domain into five overlapping subdomains and employing the central difference scheme in spatial direction and the backward

Euler scheme in time direction to solve subdomain problems iteratively. Further, we use appropriate interface conditions between space-time subdomain and present the convergence analysis of the method. In particular, it is proved that the proposed method converges uniformly of almost second order in spatial direction and first order in time direction.

In Chapter 5, a coupled system of singularly perturbed reaction-diffusion problems having distinct parameters with time delay is considered. We provide asymptotic behavior of the solution. A domain decomposition method is constructed to obtain parameter-uniform approximations for this system. We show how to partition the computational domain of the problem into overlapping subdomains. Further, the central difference scheme in spatial direction and the backward Euler scheme in time direction are used for the discretization of the model problem. The method can be easily implemented, provides parameter-uniform approximations to the solution, and requires less number of iterations. Moreover, the method is shown to be almost second order convergent in space and first order in time.

In Chapter 6, we design and analyze an improved domain decomposition method to solve singularly perturbed coupled system of parabolic reaction-diffusion problems. We divide the problem domain into three overlapping subdomains: two finely meshed layer subdomains and a coarsely meshed regular subdomain. To solve subdomain problems, we employ a combination of the central difference approximation in space and the backward Euler approximation in time. The uniform convergence analysis is done by introducing some auxiliary problems and splitting the iteration and discretization errors. The method is proved to be first order accurate in time and second order in space. More precisely, in the iterative process when perturbation parameters are of small magnitude, just one iteration is needed to achieve the desired accuracy which is shown numerically.

Numerical experiments are conducted for some test problems to demonstrate the robustness and validity of the theoretical error bounds established for the proposed methods.