Chapter 3

Pseudo-Differential Operators of Homogeneous symbol class involving the Weinstein transform

3.1 Introduction

In 1965, Kohn and Nirenberg [32] originated pseudo-differential operators of the homogeneous class of C^{∞}- symbol and proved many interesting results by exploiting the theory of Fourier transform. Using the same integral transform theory, Zaidman [82] discussed the properties of pseudo-differential operators and commutators on the homogeneous class of symbol of order zero. Grafakos [20] studied the boundedness of pseudo-differential operators on homogeneous Lipschitz spaces for the Fourier transform concerns. Motivated from the results of [20], Grafakos and Torres [21] found pseudo-differential operators with homogeneous symbols and studied many properties. Benyi and Bownik [3] examined anisotropic classes of homogeneous pseudo-differential symbols and discussed many important observations. Pathak and Upadhyay [56], introduced pseudo-differential operators associated with a homogeneous class of symbols involving Hankel transform and studied various properties. Later on, with the help of the same transform theory, Upadhyay [78] found the properties of the pseudo-differential operator associated with the homogeneous class of symbol in the L^p- norm sense. In 2018, Upadhyay and Chauhan [79] investigated the characterization of pseudo-differential operators associated with the same type of symbol involving n-dimensional Hankel transform. Motivated by the work of [56, 78, 79, 82], our main intention of the present chapter is to study the pseudo-differential operators $\mathcal{P}(x, D)$ and $\mathcal{Q}(x, D)$ associated with a homogeneous class of symbol and studied many properties by exploiting the theory of the Weinstein transform.

We organize the present chapter by the following way:

Section 3.1 is introductory which gives the brief description about pseudo-differential operators associated with the homogeneous class of symbol. In Section 3.2, boundedness of multiplication operators and other results related to the Sobolev type space are given. Section 3.3, provides various properties of pseudo-differential operators $\mathcal{P}(x, D)$ and $\mathcal{Q}(x, D)$ associated with the homogeneous class of symbol by taking the Weinstein transform. In Section 3.4, product of symbols and commutators of two pseudo-differential operators are defined and some results related to commutators of two pseudo-differential operators on $\mathcal{H}_\alpha^{r,2}$ - type Sobolev space are proven.

3.2 Boundedness of Multiplication Operators

In this section, $L^p_\alpha(\mathbb{R}^{n+1}_+)$- type Sobolev space $\mathcal{H}^{s,p}_\alpha$ is defined and the boundedness of multiplication operators is discussed on the Sobolev type space.

Definition 3.2.1. Let $s \in \mathbb{R}$. We define $\mathcal{H}^{s,p}_\alpha, 1 \leq p < \infty$ is the space of all functions $\phi \in S'_*(\mathbb{R}^{n+1}_+)$ which satisfies

$$\|\phi\|_{\mathcal{H}^{s,p}_\alpha} = \|(1+\|\xi\|^2)^{\frac{s}{2}}\mathcal{F}_\alpha\phi\|_{L^p_\alpha(\mathbb{R}^{n+1}_+)}. \tag{3.2.1}$$

This space $\mathcal{H}^{s,p}_\alpha$ is called $L^p_\alpha(\mathbb{R}^{n+1}_+)$- type Sobolev space of order s.

For $p = 2$, (3.2.1) becomes

$$\|\phi\|_{\mathcal{H}^{s,2}_\alpha} = \|(1+\|\xi\|^2)^{\frac{s}{2}}\mathcal{F}_\alpha\phi\|_{L^2_\alpha(\mathbb{R}^{n+1}_+)}. \tag{3.2.2}$$

Lemma 3.2.2. *Let $u \in S_*(\mathbb{R}^{n+1}_+)$. Then $\mathcal{F}_\alpha u \in S_*(\mathbb{R}^{n+1}_+)$ and the following inequality holds*

$$\big|(\mathcal{F}_\alpha u)(\xi)\big| \leq C_q(1+\|\xi\|^2)^{-q}, \quad \forall\ q \in \mathbb{N}_0 \tag{3.2.3}$$

where C_q is a positive constant depends on q.

Proof. The proof of this lemma can be done from (1.4.6) and [50]. □

From Friedrichs and Lax [16], we define the following operator:

The Friedrichs operator $\varphi(D)$ is defined as

$$\varphi(D)u = \mathcal{F}^{-1}_\alpha\Big(\varphi(\xi)(\mathcal{F}_\alpha u)(\xi)\Big), \quad \forall u \in S_*(\mathbb{R}^{n+1}_+) \tag{3.2.4}$$

where φ is a measurable function on $\mathbb{R}_+^{n+1}$.

Theorem 3.2.3. *Let $\alpha > -\frac{1}{2}$ and $\varphi(\xi)$ be a measurable function such that*

$$|\varphi(\xi)| \leq C(1 + \|\xi\|^2)^r, \quad \forall\ r \in \mathbb{N}. \tag{3.2.5}$$

Then the operator $\varphi(D)$ maps continuously from $\mathcal{H}_\alpha^{s+2r,2}$ to $\mathcal{H}_\alpha^{s,2}$ and satisfies the following norm inequality

$$\|\varphi(D)u\|_{\mathcal{H}_\alpha^{s,2}} \leq C\|u\|_{\mathcal{H}_\alpha^{s+2r,2}}, \quad \forall u \in S_*(\mathbb{R}_+^{n+1}) \tag{3.2.6}$$

where C is a positive constant and the operator $\varphi(D)$ is of order $2r$.

Proof. From (3.2.3), we have $\mathcal{F}_\alpha u \in S_*(\mathbb{R}_+^{n+1})$. Then it satisfies

$$|(\mathcal{F}_\alpha u)(\xi)| \leq C_q(1 + \|\xi\|^2)^{-q}, \qquad \forall\ q \in \mathbb{N}_0. \tag{3.2.7}$$

Therefore,

$$|\varphi(\xi)(\mathcal{F}_\alpha u)(\xi)| \leq C(1 + \|\xi\|^2)^r |(\mathcal{F}_\alpha u)(\xi)|. \tag{3.2.8}$$

From (3.2.7), we get

$$\begin{aligned} |\varphi(\xi)(\mathcal{F}_\alpha u)(\xi)| &\leq C_q'(1 + \|\xi\|^2)^r(1 + \|\xi\|^2)^{-q} \\ &\leq C_q'(1 + \|\xi\|^2)^{r-q}, \end{aligned} \tag{3.2.9}$$

where $C_q' = CC_q$ is a positive constant depends only on q.

By taking the Weinstein transform in (3.2.4), we get

$$\mathcal{F}_\alpha\big(\varphi(D)u\big)(\xi) = \varphi(\xi)(\mathcal{F}_\alpha u)(\xi). \tag{3.2.10}$$

Now, from (3.2.2), we have

$$\|\varphi(D)u\|^2_{\mathcal{H}^{s,2}_\alpha} = \int_{\mathbb{R}^{n+1}_+} (1+\|\xi\|^2)^s \big|\mathcal{F}_\alpha\big(\varphi(D)u\big)(\xi)\big|^2 d\mu_\alpha(\xi).$$

Using (3.2.10), we get

$$\|\varphi(D)u\|^2_{\mathcal{H}^{s,2}_\alpha} = \int_{\mathbb{R}^{n+1}_+} (1+\|\xi\|^2)^s \big|\varphi(\xi)(\mathcal{F}_\alpha u)(\xi)\big|^2 d\mu_\alpha(\xi).$$

In view of (3.2.8), the last expression becomes

$$\begin{aligned}\|\varphi(D)u\|^2_{\mathcal{H}^{s,2}_\alpha} &\leq C\int_{\mathbb{R}^{n+1}_+} (1+\|\xi\|^2)^s (1+\|\xi\|^2)^{2r} |(\mathcal{F}_\alpha u)(\xi)|^2 d\mu_\alpha(\xi) \\ &\leq C\int_{\mathbb{R}^{n+1}_+} (1+\|\xi\|^2)^{s+2r} |(\mathcal{F}_\alpha u)(\xi)|^2 d\mu_\alpha(\xi) \\ &\leq C\|u\|^2_{\mathcal{H}^{s+2r,2}_\alpha}.\end{aligned}$$

□

Theorem 3.2.4. *Let $\phi \in S_*(\mathbb{R}^{n+1}_+)$ and $\psi \in S_*(\mathbb{R}^{n+1}_+)$. Then we have*

$$\mathcal{F}_\alpha(\phi *_w \psi) = \mathcal{F}_\alpha(\phi)\mathcal{F}_\alpha(\psi). \tag{3.2.11}$$

Proof. From (1.4.11) and (1.4.13), we have

$$\begin{aligned}(\phi *_w \psi)(x) &= \int_{\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^{n+1}_+}\int_{\mathbb{R}^{n+1}_+} e^{i\langle x',t'\rangle}\hat{J}_\alpha(x_{n+1}t_{n+1}) e^{-i\langle y',t'\rangle}\hat{J}_\alpha(y_{n+1}t_{n+1}) \\ &\quad\times e^{-i\langle z',t'\rangle}\hat{J}_\alpha(z_{n+1}t_{n+1})\phi(y)\psi(z)d\mu_\alpha(t)d\mu_\alpha(y)d\mu_\alpha(z) \\ &= \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',t'\rangle}\hat{J}_\alpha(x_{n+1}t_{n+1})\Big(\int_{\mathbb{R}^{n+1}_+} e^{-i\langle y',t'\rangle}\hat{J}_\alpha(y_{n+1}t_{n+1})\phi(y)d\mu_\alpha(y)\Big) \\ &\quad\times\Big(\int_{\mathbb{R}^{n+1}_+} e^{-i\langle z',t'\rangle}\hat{J}_\alpha(z_{n+1}t_{n+1})\psi(z)d\mu_\alpha(z)\Big)d\mu_\alpha(t).\end{aligned}$$

By (1.4.1), we get

$$(\phi *_w \psi)(x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',t'\rangle} \hat{J}_\alpha(x_{n+1}t_{n+1})(\mathcal{F}_\alpha\phi)(t)(\mathcal{F}_\alpha\psi)(t)d\mu_\alpha(t).$$

By inversion formula of the Weinstein transform, we have

$$(\phi *_w \psi)(x) = \mathcal{F}_\alpha^{-1}\Big(\mathcal{F}_\alpha(\phi)\mathcal{F}_\alpha(\psi)\Big)(x).$$

Therefore, for $\phi, \psi \in S_*(\mathbb{R}_+^{n+1})$ we find

$$\mathcal{F}_\alpha(\phi *_w \psi) = \mathcal{F}_\alpha(\phi)\mathcal{F}_\alpha(\psi).$$

□

Corollary 3.2.5. *Let $\phi, \psi \in S_*(\mathbb{R}_+^{n+1})$. Then we have*

$$\mathcal{F}_\alpha^{-1}(\phi *_w \psi) = \mathcal{F}_\alpha^{-1}(\phi)\mathcal{F}_\alpha^{-1}(\psi). \tag{3.2.12}$$

Proof. Let $\xi \in \mathbb{R}_+^{n+1}$. Then from (1.4.6), we have

$$\mathcal{F}_\alpha^{-1}(\phi *_w \psi)(-\xi) = \mathcal{F}_\alpha(\phi *_w \psi)(\xi).$$

Using (3.2.11), above expression becomes

$$\mathcal{F}_\alpha^{-1}(\phi *_w \psi)(-\xi) = \mathcal{F}_\alpha(\phi)(\xi)\mathcal{F}_\alpha(\psi)(\xi).$$

Again, from (1.4.6), the last expression becomes

$$\mathcal{F}_\alpha^{-1}(\phi *_w \psi)(-\xi) = \mathcal{F}_\alpha^{-1}(\phi)(-\xi)\mathcal{F}_\alpha^{-1}(\psi)(-\xi).$$

Since the last expression holds for all $\xi \in \mathbb{R}_+^{n+1}$. Hence

$$\mathcal{F}_\alpha^{-1}(\phi *_w \psi) = \mathcal{F}_\alpha^{-1}(\phi)\mathcal{F}_\alpha^{-1}(\psi).$$

□

Corollary 3.2.6. *Let $a, u \in S_*(\mathbb{R}_+^{n+1})$. Then the following relation holds*

$$\mathcal{F}_\alpha(a) *_w \mathcal{F}_\alpha(u) = \mathcal{F}_\alpha(au). \tag{3.2.13}$$

Proof. For $\phi, \psi \in S_*(\mathbb{R}_+^{n+1})$, choose $\phi = \mathcal{F}_\alpha(a)$ and $\psi = \mathcal{F}_\alpha(u)$.

Then from (3.2.12), we can find

$$\begin{aligned}\mathcal{F}_\alpha^{-1}\big(\mathcal{F}_\alpha(a) *_w \mathcal{F}_\alpha(u)\big) &= \mathcal{F}_\alpha^{-1}\big(\mathcal{F}_\alpha(a)\big)\mathcal{F}_\alpha^{-1}\big(\mathcal{F}_\alpha(u)\big)\\ &= au.\end{aligned}$$

Taking the Weinstein transform, the last expression yields

$$\mathcal{F}_\alpha(a) *_w \mathcal{F}_\alpha(u) = \mathcal{F}_\alpha(au).$$

□

Theorem 3.2.7. $\big($***Multiplication Operator***$\big)$

Let $\alpha > -\frac{1}{2}, s \in \mathbb{R}$ and let $p, q, r \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. Then for $a \in S_*(\mathbb{R}_+^{n+1})$ the operator $u \to au$ maps continuously from $\mathcal{H}_\alpha^{s,pp'}$ to $\mathcal{H}_\alpha^{0,r}$ and satisfies the following norm inequality

$$\|au\|_{\mathcal{H}_\alpha^{0,r}} \leq C_{\alpha,q,q',s}\, \|u\|_{\mathcal{H}_\alpha^{s,pp'}}, \quad \forall\ u \in S_*(\mathbb{R}_+^{n+1}) \tag{3.2.14}$$

where $C_{\alpha,q,q',s}$ is a positive constant for q' conjugate to p' and $s > max\big(\frac{2\alpha+n+2}{qq'}, \frac{2\alpha+n+2}{pq'}\big)$.

Proof. From (3.2.13), we have

$$\mathcal{F}_\alpha(au) = \mathcal{F}_\alpha(a) *_w \mathcal{F}_\alpha(u).$$

Therefore,

$$\left(\int_{\mathbb{R}_+^{n+1}} \left|\Big(\mathcal{F}_\alpha(au)\Big)(\xi)\right|^r d\mu_\alpha(\xi)\right)^{\frac{1}{r}} = \left(\int_{\mathbb{R}_+^{n+1}} \left|\Big(\mathcal{F}_\alpha(a) *_w \mathcal{F}_\alpha(u)\Big)(\xi)\right|^r d\mu_\alpha(\xi)\right)^{\frac{1}{r}}.$$

Thus, we have

$$\|\mathcal{F}_\alpha(au)\|_{L^r_\alpha} = \|\mathcal{F}_\alpha(a) *_w \mathcal{F}_\alpha(u)\|_{L^r_\alpha}.$$

Using (1.4.17), the last expression becomes

$$\|\mathcal{F}_\alpha(au)\|_{L^r_\alpha} \leq \|\mathcal{F}_\alpha(a)\|_{L^q_\alpha}\|\mathcal{F}_\alpha(u)\|_{L^p_\alpha}. \tag{3.2.15}$$

Now, we find

$$\begin{aligned}\|\mathcal{F}_\alpha(a)\|^q_{L^q_\alpha} &= \int_{\mathbb{R}_+^{n+1}} \left|\Big(\mathcal{F}_\alpha(a)\Big)(\xi)\right|^q d\mu_\alpha(\xi) \\ &= \int_{\mathbb{R}_+^{n+1}} (1+\|\xi\|^2)^{-\frac{sq}{2}}(1+\|\xi\|^2)^{\frac{sq}{2}}\left|\Big(\mathcal{F}_\alpha(a)\Big)(\xi)\right|^q d\mu_\alpha(\xi). \end{aligned} \tag{3.2.16}$$

Let p', q' are conjugate to each other. Then by Hölders inequality, (3.2.16) becomes

$$\begin{aligned}\|\mathcal{F}_\alpha(a)\|^q_{L^q_\alpha} &= \left(\int_{\mathbb{R}_+^{n+1}} (1+\|\xi\|^2)^{-\frac{sqq'}{2}} d\mu_\alpha(\xi)\right)^{\frac{1}{q'}} \\ &\quad \times \left(\int_{\mathbb{R}_+^{n+1}} (1+\|\xi\|^2)^{\frac{sqp'}{2}}\left|\Big(\mathcal{F}_\alpha(a)\Big)(\xi)\right|^{qp'} d\mu_\alpha(\xi)\right)^{\frac{1}{p'}} \\ &= \|(1+\|\xi\|^2)^{-\frac{s}{2}}\|^q_{L^{qq'}_\alpha}\ \|(1+\|\xi\|^2)^{\frac{s}{2}}(\mathcal{F}_\alpha a)\|^q_{L^{qp'}_\alpha}. \end{aligned}$$

Therefore,

$$\|\mathcal{F}_\alpha(a)\|_{L^q_\alpha} = \|(1+\|\xi\|^2)^{-\frac{s}{2}}\|_{L^{qq'}_\alpha} \ \|(1+\|\xi\|^2)^{\frac{s}{2}}(\mathcal{F}_\alpha a)\|_{L^{qp'}_\alpha}. \tag{3.2.17}$$

Similaly, we can find

$$\|\mathcal{F}_\alpha(u)\|_{L^p_\alpha} = \|(1+\|\xi\|^2)^{-\frac{s}{2}}\|_{L^{pq'}_\alpha} \ \|(1+\|\xi\|^2)^{\frac{s}{2}}(\mathcal{F}_\alpha u)\|_{L^{pp'}_\alpha}. \tag{3.2.18}$$

Using (3.2.17) and (3.2.18) in (3.2.15), we obtain

$$\begin{aligned}\|\mathcal{F}_\alpha(au)\|_{L^r_\alpha} \leq \|(1+\|\xi\|^2)^{-\frac{s}{2}}\|_{L^{qq'}_\alpha} \ \|(1+\|\xi\|^2)^{-\frac{s}{2}}\|_{L^{pq'}_\alpha} \\ \times \|(1+\|\xi\|^2)^{\frac{s}{2}}(\mathcal{F}_\alpha a)\|_{L^{qp'}_\alpha} \ \|(1+\|\xi\|^2)^{\frac{s}{2}}(\mathcal{F}_\alpha u)\|_{L^{pp'}_\alpha}.\end{aligned}$$

From (3.2.1), the last expression yields

$$\begin{aligned}\|au\|_{\mathcal{H}^{0,r}_\alpha} &\leq \|(1+\|\xi\|^2)^{-\frac{s}{2}}\|_{L^{qq'}_\alpha} \ \|(1+\|\xi\|^2)^{-\frac{s}{2}}\|_{L^{pq'}_\alpha} \ \|a\|_{\mathcal{H}^{s,qp'}_\alpha}\|u\|_{\mathcal{H}^{s,pp'}_\alpha} \\ &\leq C_{\alpha,q,q',s} \ \|u\|_{\mathcal{H}^{s,pp'}_\alpha},\end{aligned}$$

where $C_{\alpha,q,q',s} = \|(1+\|\xi\|^2)^{-\frac{s}{2}}\|_{L^{qq'}_\alpha} \ \|(1+\|\xi\|^2)^{-\frac{s}{2}}\|_{L^{pq'}_\alpha} \ \|a\|_{\mathcal{H}^{s,qp'}_\alpha}$ is the positive constant for $s > max\left(\frac{2\alpha+n+2}{qq'}, \frac{2\alpha+n+2}{pq'}\right)$. □

Corollary 3.2.8. *Let $\alpha > -\frac{1}{2}$ and $s \in \mathbb{R}$. Then for $a \in S_*(\mathbb{R}^{n+1}_+)$, the operator $u \to au \in S_*(\mathbb{R}^{n+1}_+)$, $\forall \ u \in S_*(\mathbb{R}^{n+1}_+)$ and following norm inequality holds*

$$\|au\|_{\mathcal{H}^{0,1}_\alpha} \leq C_{\alpha,q',s}\|u\|_{\mathcal{H}^{s,p'}_\alpha}, \quad \forall \ p' \geq 1 \tag{3.2.19}$$

where $C_{\alpha,q',s}$ is a positive constant for q' is conjugate to p' and $s > \frac{2\alpha+n+2}{q'}$.

Proof. From the similar technique used in Theorem 3.2.7, we obtained the required result. □

Corollary 3.2.9. *Let $\alpha > -\frac{1}{2}$ and $s \in \mathbb{R}$. Then for $a \in S_*(\mathbb{R}^{n+1}_+)$, the operator $u \to au$ maps continuously from $\mathcal{H}^{s,2p'}_\alpha$ to $\mathcal{H}^{0,2}_\alpha$. Moreover, following norm inequality holds*

$$\|au\|_{\mathcal{H}^{0,2}_\alpha} \leq C_{\alpha,q',s} \; \|u\|_{\mathcal{H}^{s,2p'}_\alpha}, \quad \forall \; u \in S_*(\mathbb{R}^{n+1}_+) \tag{3.2.20}$$

where $C_{\alpha,q',s}$ is a positive constant for q' conjugate to p' and $s > max\big(\frac{2\alpha+n+2}{q'}, \frac{2\alpha+n+2}{2q'}\big)$.

Proof. Using similar steps of Theorem 3.2.7 for $r = 2$, we get the desired result. □

3.3 Pseudo-differential operators associated with the Weinstein transform

In this section, various properties of the pseudo-differential operators $\mathcal{P}(x, D)$ and $\mathcal{Q}(x, D)$ associated with homogeneous class of symbol $\sigma(x, \xi)$ are investigated by exploiting the theory of the Weinstein transform.

Definition 3.3.1. Let Λ denotes the class of all C^∞- functions $\sigma : \mathbb{R}^{n+1}_+ \times \mathbb{R}^{n+1}_+ \to \mathbb{C}$ which satisfies the following properties:

(i) $\sigma(x, t\xi) = \sigma(x, \xi)$ for $t > 0$.

(ii) $\lim_{\|x\|\to\infty} \sigma(x, \xi) = \sigma(\infty, \xi)$ for $\xi \in \mathbb{R}^{n+1}_+$ and $\sigma(\infty, \xi)$ is a C^∞- function.

(iii) Define $\sigma'(x, \xi) = \sigma(x, \xi) - \sigma(\infty, \xi)$ and assume that the estimates

$$(1 + \|x\|^2)^k \big| D^\beta_\xi D^\gamma_x \sigma'(x, \xi)\big| \leq C_{\beta,\gamma,k}, \tag{3.3.1}$$

for all $x \in \mathbb{R}^{n+1}_+$ and $\xi \in \mathbb{R}^{n+1}_+$ such that $\|\xi\| = 1$, where $k \in \mathbb{N}_0$ and $\beta, \gamma \in \mathbb{N}^{n+1}_0$.

Example: Let $\sigma(x,\xi) = \sum_{j=0}^{n} C_j(1+\|x\|^2)^{-\gamma_j}$, $\gamma_j \geq 0$, where C_j are constants. Then $\sigma(\infty,\xi) = C_0$ and $\sigma \in \mathit{\Lambda}$.

Lemma 3.3.2. *Let $s \in \mathbb{N}_0$ and σ be a C^∞- function. Then*

$$\left|\Delta^s_{\alpha,n,x}\sigma(x,\xi)\right| \leq \sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{\delta_1,\cdots,\delta_n\geq 0}\binom{s}{j}\binom{s-j}{\delta_1,\cdots,\delta_n}E'_{\alpha,r}|x_{n+1}|^{r-s}\left|D_x^{2\delta'+r}\sigma(x,\xi)\right|, \tag{3.3.2}$$

where $2\delta'+r = (2\delta_1,\cdots,2\delta_n,r) \in \mathbb{N}_0^{n+1}$ and $2|\delta'|+r = 2\delta_1+\cdots+2\delta_n+r$.

Proof. From [19, p. 14], we can find a constant $E'_{\alpha,r}$ for $r \in \{0,1,\cdots,s\}$ depending only on α satisfying

$$\Delta^s_{\alpha,n,x}\sigma(x,\xi) = \sum_{j=0}^{s}\sum_{r=1}^{2j}\binom{s}{j}E'_{\alpha,r}x_{n+1}^{r-s}(\Delta_n)_{x'}^{s-j}\frac{\partial^r}{\partial x_{n+1}^r}\sigma(x,\xi), \tag{3.3.3}$$

where $(\Delta_n)_{x'}^{s-j} = (\frac{\partial^2}{\partial x_1^2}+\cdots+\frac{\partial^2}{\partial x_n^2})^{s-j}$.

Using multi-index theorem for $s-j \in \mathbb{N}_0$, we have

$$(\Delta_n)_{x'}^{s-j} = \sum_{\delta_1,\cdots,\delta_n\geq 0}\binom{s-j}{\delta_1,\cdots,\delta_n}\frac{\partial^{2\delta_1}}{\partial x_1^{2\delta_1}}\cdots\frac{\partial^{2\delta_n}}{\partial x_n^{2\delta_n}}. \tag{3.3.4}$$

With the help of (3.3.4), (3.3.3) becomes

$$\Delta^s_{\alpha,n,x}\sigma(x,\xi) = \sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{\delta_1,\cdots,\delta_n\geq 0}\binom{s}{j}\binom{s-j}{\delta_1,\cdots,\delta_n}E'_{\alpha,r}x_{n+1}^{r-s}\left[\frac{\partial^{2|\delta'|+r}}{\partial x_1^{2\delta_1}\cdots\partial x_n^{2\delta_n}\partial x_{n+1}^r}\sigma(x,\xi)\right],$$

where $\delta' = (\delta_1,\cdots,\delta_n) \in \mathbb{N}_0^n$ and $|\delta'| = \delta_1+\cdots+\delta_n$.

Therefore,

$$\Delta^s_{\alpha,n,x}\sigma(x,\xi) = \sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{\delta_1,\cdots,\delta_n\geq 0}\binom{s}{j}\binom{s-j}{\delta_1,\cdots,\delta_n}E'_{\alpha,r}x_{n+1}^{r-s}\left[D_x^{2\delta'+r}\sigma(x,\xi)\right]. \tag{3.3.5}$$

Hence,

$$\Big|\Delta^s_{\alpha,n,x}\sigma(x,\xi)\Big| \leq \sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{\delta_1,\cdots,\delta_n\geq 0}\binom{s}{j}\binom{s-j}{\delta_1,\cdots,\delta_n}E'_{\alpha,r}|x_{n+1}|^{r-s}\Big|D_x^{2\delta'+r}\sigma(x,\xi)\Big|.$$

□

Lemma 3.3.3. *Let $u \in S_*(\mathbb{R}^{n+1}_+)$ and $\sigma(x,\xi) \in \Lambda$. Then $u(x)\sigma(x,\xi) \in S_*(\mathbb{R}^{n+1}_+)$ and $\mathcal{F}_\alpha\big[u(x)\sigma(x,\xi)\big] \in S_*(\mathbb{R}^{n+1}_+)$.*

Proof. Let $u \in S_*(\mathbb{R}^{n+1}_+)$ and $\sigma(x,\xi) = \sigma(\infty,\xi) + \sigma'(x,\xi) \in \Lambda$, then from the defintion of the Schwartz space

$$(1+\|x\|^2)^q\big|D^\delta_x u(x)\big| \leq C_{q,\delta}, \quad \forall q \in \mathbb{N}_0, \delta \in \mathbb{N}^{n+1}_0. \tag{3.3.6}$$

Let $q \in \mathbb{N}_0$ and $\rho \in \mathbb{N}^{n+1}_0$. Then from Binomial theorem, we have

$$\begin{aligned}(1+\|x\|^2)^q\Big|D^\rho_x\big[u(x)\sigma(x,\xi)\big]\Big| &\leq (1+\|x\|^2)^q\sum_{\delta\leq\rho}\binom{\rho}{\delta}\big|D^\delta_x u(x)\big|\big|D^{\rho-\delta}_x\sigma(x,\xi)\big| \\ &\leq (1+\|x\|^2)^q\sum_{\delta\leq\rho}\binom{\rho}{\delta}\big|D^\delta_x u(x)\big|\big|D^{\rho-\delta}_x\sigma'(x,\xi)\big|.\end{aligned}$$

Using (3.3.1) and (3.3.6), above expression becomes

$$\begin{aligned}(1+\|x\|^2)^q\Big|D^\rho_x\big[u(x)\sigma(x,\xi)\big]\Big| &\leq (1+\|x\|^2)^q\sum_{\delta\leq\rho}\binom{\rho}{\delta}C_{q,\delta}(1+\|x\|^2)^{-q} \\ &\quad\times C_{0,\rho-\delta,k}(1+\|x\|^2)^{-k} \\ &\leq (1+\|x\|^2)^{-k}\sum_{\delta\leq\rho}\binom{\rho}{\delta}C_{q,\delta}C_{0,\rho-\delta,k}.\end{aligned}$$

Therefore,

$$\sup_{x\in\mathbb{R}_+^{n+1}} (1+\|x\|^2)^q \Big| D_x^{\rho}\big[u(x)\sigma(x,\xi)\big]\Big| \leq C_{q,\rho}, \quad \forall k \in \mathbb{N}_0$$

where $C_{q,\rho}$ is a constant depends only on q and ρ.

Hence, $u \in S_*(\mathbb{R}_+^{n+1})$. Also, from (1.4.6) we get $\mathcal{F}_\alpha\big[u(x)\sigma(x,\xi)\big] \in S_*(\mathbb{R}_+^{n+1})$. □

Theorem 3.3.4. *For all $\xi, \eta \in \mathbb{R}_+^{n+1}$ we have the following estimates*

(i) Let $\sigma(\infty,\xi)$ be a C^∞- function. Then

$$\big|\sigma(\infty,\xi) - \sigma(\infty,\eta)\big| \leq C\Big(\frac{\|\xi-\eta\|}{\|\xi\|+\|\eta\|}\Big) \leq C(1+\|\xi-\eta\|^2)^{1/2}(1+\|\eta\|^2)^{-1/2}. \quad (3.3.7)$$

(ii) Let $\alpha > -\frac{1}{2}$ and $\lambda \in \mathbb{R}_+^{n+1}$. Then

$$(1+\|\lambda\|^2)^p\big|\big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\xi)\big| \leq C_{\alpha,k,p}, \quad \forall\ p \in \mathbb{N} \quad (3.3.8)$$

where $C_{\alpha,k,p}$ is a positive constant for $k > \alpha + \frac{n}{2} + p + 1$.

(iii) Let $\alpha > -\frac{1}{2}$ and $\lambda \in \mathbb{R}_+^{n+1}$. Then

$$(1+\|\lambda\|^2)^p\big|\big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\xi) - \big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\eta)\big| \leq C_{\alpha,k,p}\Big(\frac{\|\xi-\eta\|}{\|\xi\|+\|\eta\|}\Big), \quad \forall\ p \in \mathbb{N} \quad (3.3.9)$$

where $C_{\alpha,k,p}$ is a positive constant for $k > \alpha + \frac{n}{2} + 1$.

Proof. (i) For proof we can refer [82, p. 347].

(ii) Using Binomial theorem, we have

$$(1+\|\lambda\|^2)^p\big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\xi) = \sum_{s=0}^{p}\binom{p}{s}\|\lambda\|^{2s}\big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\xi).$$

In view of (1.4.3), we get

$$(1+\|\lambda\|^2)^p\big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\xi)=\sum_{s=0}^{p}\binom{p}{s}(-1)^s\mathcal{F}_\alpha\Big(\Delta^s_{\alpha,n,x}\sigma'\Big)(\lambda,\xi). \tag{3.3.10}$$

From (1.4.3), above expression becomes

$$(1+\|\lambda\|^2)^p\big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\xi)=\sum_{s=0}^{p}\binom{p}{s}(-1)^s\int_{\mathbb{R}^{n+1}_+}e^{-i\langle x',\lambda'\rangle}\hat{J}_\alpha(x_{n+1}\lambda_{n+1})$$
$$\times\,\Delta^s_{\alpha,n,x}\sigma'(x,\xi)d\mu_\alpha(x).$$

Therefore,

$$(1+\|\lambda\|^2)^p\big|\big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\xi)\big|\le\sum_{s=0}^{p}\binom{p}{s}\int_{\mathbb{R}^{n+1}_+}\big|\Delta^s_{\alpha,n,x}\sigma'(x,\xi)\big|d\mu_\alpha(x). \tag{3.3.11}$$

Using (3.3.2), we can find

$$\Big|\Delta^s_{\alpha,n,x}\sigma'(x,\xi)\Big|\le\sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{\delta_1,\cdots,\delta_n\ge0}\binom{s}{j}\binom{s-j}{\delta_1,\cdots,\delta_n}E'_{\alpha,r}|x_{n+1}|^{r-s}\Big|D_x^{2\delta'+r}\sigma'(x,\xi)\Big|.$$

In view of (3.3.1), the last inequality becomes

$$\Big|\Delta^s_{\alpha,n,x}\sigma'(x,\xi)\Big|\le\sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{\delta_1,\cdots,\delta_n\ge0}\binom{s}{j}\binom{s-j}{\delta_1,\cdots,\delta_n}E'_{\alpha,r}C_{0,2\delta'+r,k}$$
$$\times\,|x_{n+1}|^{r-s}(1+\|x\|^2)^{-k}. \tag{3.3.12}$$

From (3.3.12), (3.3.11) becomes

$$(1+\|\lambda\|^2)^p\big|\big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\xi)\big|\le\sum_{s=0}^{p}\sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{\delta_1,\cdots,\delta_n\ge0}\binom{p}{s}\binom{s}{j}\binom{s-j}{\delta_1,\cdots,\delta_n}E'_{\alpha,r}C_{0,2\delta'+r,k}$$
$$\times\int_{\mathbb{R}^{n+1}_+}|x_{n+1}|^{r-s}(1+\|x\|^2)^{-k}d\mu_\alpha(x).$$

Therefore,

$$(1+\|\lambda\|^2)^p\big|\big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\xi)\big| \leq \sum_{s=0}^{p}\sum_{j=0}^{s}\sum_{\delta_1,\cdots,\delta_n\geq 0}\binom{p}{s}\binom{s}{j}\binom{s-j}{\delta_1,\cdots,\delta_n}E'_{\alpha,2s}C_{0,2\delta'+2s,k} \times \int_{\mathbb{R}^{n+1}_+}(1+\|x\|^2)^{s-k}d\mu_\alpha(x).$$

Using (1.3.4), the last expression becomes

$$\begin{aligned}(1+\|\lambda\|^2)^p\big|\big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\xi)\big| &\leq \sum_{j=0}^{p}\sum_{\delta_1,\cdots,\delta_n\geq 0}\binom{p}{j}\binom{p-j}{\delta_1,\cdots,\delta_n}E'_{\alpha,2p}C_{0,2\delta'+2p,k}\ A_\alpha \\ &\quad\times \int_{\mathbb{R}^{n+1}_+}(1+\|x\|^2)^{p-k}x_{n+1}^{2\alpha+1}dx \\ &\leq C'_{\alpha,p,k}\int_{\mathbb{R}^{n+1}_+}(1+\|x\|^2)^{\frac{2(p-k)+2\alpha+1}{2}}dx \\ &\leq C_{\alpha,k,p},\end{aligned}$$

where $C_{\alpha,k,p}$ is a positive constant for $k>\alpha+\frac{n}{2}+p+1$.

(iii) From (3.3.10), we can find the following

$$(1+\|\lambda\|^2)^p\Big(\big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\xi)-\big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\eta)\Big)=\sum_{s=0}^{p}\binom{p}{s}(-1)^s\mathcal{F}_\alpha\Big[\Delta^s_{\alpha,n,x}\Big(\sigma'(x,\xi)-\sigma'(x,\eta)\Big)\Big].$$

Using (1.4.1), above expression becomes

$$\begin{aligned}(1+\|\lambda\|^2)^p\Big(\big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\xi)-\big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\eta)\Big) &= \sum_{s=0}^{p}\binom{p}{s}(-1)^s\int_{\mathbb{R}^{n+1}_+}e^{-i\langle x',\lambda'\rangle}\hat{J}_\alpha(x_{n+1}\lambda_{n+1}) \\ &\quad\times\Delta^s_{\alpha,n,x}\Big(\sigma'(x,\xi)-\sigma'(x,\eta)\Big)d\mu_\alpha(x) \\ &= \sum_{s=0}^{p}\binom{p}{s}(-1)^s\int_{\mathbb{R}^{n+1}_+}e^{-i\langle x',\lambda'\rangle}\hat{J}_\alpha(x_{n+1}\lambda_{n+1}) \\ &\quad\times(1+\|x\|^2)^k\Delta^s_{\alpha,n,x}\Big(\sigma'(x,\xi)-\sigma'(x,\eta)\Big) \\ &\quad\times(1+\|x\|^2)^{-k}d\mu_\alpha(x). \qquad (3.3.13)\end{aligned}$$

Now, we take

$$F_{k,s}(x,\xi) = (1+\|x\|^2)^k \Delta^s_{\alpha,n,x}\sigma'(x,\xi). \tag{3.3.14}$$

From (3.3.13) and (3.3.14), we can find the estimate

$$(1+\|\lambda\|^2)^p\Big|\big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\xi) - \big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\eta)\Big| \le \sum_{s=0}^{p}\binom{p}{s}\int_{\mathbb{R}^{n+1}_+}\big|F_{k,s}(x,\xi) - F_{k,s}(x,\eta)\big| \times (1+\|x\|^2)^{-k} d\mu_\alpha(x). \tag{3.3.15}$$

From [82, p. 349], we have

$$\big|F_{k,s}(x,\xi) - F_{k,s}(x,\eta)\big| \le C\Big(\frac{\|\xi-\eta\|}{\|\xi\|+\|\eta\|}\Big). \tag{3.3.16}$$

Therefore, from (3.3.15) and (3.3.16) we get

$$(1+\|\lambda\|^2)^p\Big|\big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\xi) - \big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\eta)\Big| \le \sum_{s=0}^{p}\binom{p}{s}C\Big(\frac{\|\xi-\eta\|}{\|\xi\|+\|\eta\|}\Big) \times \int_{\mathbb{R}^{n+1}_+}(1+\|x\|^2)^{-k} d\mu_\alpha(x).$$

Using (1.3.4), the last expression becomes

$$\begin{aligned}(1+\|\lambda\|^2)^p\Big|\big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\xi) - \big(\mathcal{F}_\alpha\sigma'\big)(\lambda,\eta)\Big| &\le CA_\alpha\sum_{s=0}^{p}\binom{p}{s}\Big(\frac{\|\xi-\eta\|}{\|\xi\|+\|\eta\|}\Big)\\ &\quad\times\int_{\mathbb{R}^{n+1}_+}(1+\|x\|^2)^{-k}|x_{n+1}|^{2\alpha+1}dx\\ &\le CA_\alpha\sum_{s=0}^{p}\binom{p}{s}\Big(\frac{\|\xi-\eta\|}{\|\xi\|+\|\eta\|}\Big)\\ &\quad\times\int_{\mathbb{R}^{n+1}_+}(1+\|x\|^2)^{\frac{-2k+2\alpha+1}{2}}dx\\ &\le C_{\alpha,k,p}\Big(\frac{\|\xi-\eta\|}{\|\xi\|+\|\eta\|}\Big),\end{aligned}$$

where $C_{\alpha,k,p}$ is a positive constant for $k > \alpha + \frac{n}{2} + 1$. □

Definition 3.3.5. $\big($**Pseudo-differential operator** $\mathcal{P}(x, D)\big)$

Let $\sigma : C^\infty(\mathbb{R}_+^{n+1} \times \mathbb{R}_+^{n+1}) \to \mathbb{C}$ be a symbol in Λ. Then for $u \in S_*(\mathbb{R}_+^{n+1})$, the pseudo-differential operator $\mathcal{P}(x, D)$ associated to σ is defined by

$$\big[\mathcal{P}(x, D)u\big](x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) F(\xi) d\mu_\alpha(\xi). \tag{3.3.17}$$

From (1.4.2), the above can be expressed by

$$\mathcal{F}_\alpha[\mathcal{P}(x, D)u](\xi) = \sigma(\infty, \xi)(\mathcal{F}_\alpha u)(\xi) + \int_{\mathbb{R}_+^{n+1}} \sigma'_\xi(\eta)(\mathcal{F}_\alpha u)(\eta) d\mu_\alpha(\eta), \tag{3.3.18}$$

where

$$F(\xi) = \sigma(\infty, \xi)(\mathcal{F}_\alpha u)(\xi) + \int_{\mathbb{R}_+^{n+1}} \sigma'_\xi(\eta)(\mathcal{F}_\alpha u)(\eta) d\mu_\alpha(\eta), \tag{3.3.19}$$

and

$$\sigma'_\xi(\eta) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\eta'\rangle} \hat{J}_\alpha(x_{n+1}\eta_{n+1})\big[e^{-i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma'(x, \xi)\big] d\mu_\alpha(x). \tag{3.3.20}$$

Lemma 3.3.6. *Let $\alpha > -\frac{1}{2}$ and $\sigma' \in \Lambda$ be a symbol. Then we found the following inequality*

$$\big|\sigma'_\xi(\eta)\big| \le E_{\alpha,k,p}(1 + \|\xi\|^2)^{-p}(1 + \|\eta\|^2)^{2p}, \tag{3.3.21}$$

where $E_{\alpha,k,p}$ is a positive constant for p and $k \in \mathbb{N}_0$ such that $k > \alpha + \frac{n}{2} + \frac{p}{2} + 1$.

Proof. From (3.3.20), we have

$$\begin{aligned}\sigma'_\xi(\eta) &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\eta'\rangle} \hat{J}_\alpha(x_{n+1}\eta_{n+1})\big[e^{-i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma'(x, \xi)\big] d\mu_\alpha(x)\\ &= \int_{\mathbb{R}_+^{n+1}} e^{-i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\big[e^{i\langle x',\eta'\rangle} \hat{J}_\alpha(x_{n+1}\eta_{n+1})\sigma'(x, \xi)\big] d\mu_\alpha(x).\end{aligned}$$

With the help of Binomial theorem, we have

$$(1+\|\xi\|^2)^p\sigma'_\xi(\eta)=\sum_{s=0}^{p}\binom{p}{s}\|\xi\|^{2s}\int_{\mathbb{R}_+^{n+1}}e^{-i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})$$
$$\times\left[e^{i\langle x',\eta'\rangle}\hat{J}_\alpha(x_{n+1}\eta_{n+1})\sigma'(x,\xi)\right]d\mu_\alpha(x).$$

From (1.4.3), the last expression becomes

$$(1+\|\xi\|^2)^p\sigma'_\xi(\eta)=\sum_{s=0}^{p}\binom{p}{s}(-1)^s\int_{\mathbb{R}_+^{n+1}}e^{-i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})$$
$$\times\Delta^s_{\alpha,n,x}\left[e^{i\langle x',\eta'\rangle}\hat{J}_\alpha(x_{n+1}\eta_{n+1})\sigma'(x,\xi)\right]d\mu_\alpha(x).$$

Therefore,

$$(1+\|\xi\|^2)^p\left|\sigma'_\xi(\eta)\right|\leq\sum_{s=0}^{p}\binom{p}{s}\int_{\mathbb{R}_+^{n+1}}\left|\Delta^s_{\alpha,n,x}\left[e^{i\langle x',\eta'\rangle}\hat{J}_\alpha(x_{n+1}\eta_{n+1})\sigma'(x,\xi)\right]\right|$$
$$\times\, d\mu_\alpha(x). \tag{3.3.22}$$

From (3.3.22), we estimate $\left|\Delta^s_{\alpha,n,x}\left[e^{i\langle x',\eta'\rangle}\hat{J}_\alpha(x_{n+1}\eta_{n+1})\sigma'(x,\xi)\right]\right|$.
Using (3.3.2), we have

$$\left|\Delta^s_{\alpha,n,x}\left[e^{i\langle x',\eta'\rangle}\hat{J}_\alpha(x_{n+1}\eta_{n+1})\sigma'(x,\xi)\right]\right|\leq\sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{\delta_1,\cdots,\delta_n\geq 0}\binom{s}{j}\binom{s-j}{\delta_1,\cdots,\delta_n}E'_{\alpha,r}|x_{n+1}|^{r-s}$$
$$\times\left|D_x^{2\delta'+r}\left[e^{i\langle x',\eta'\rangle}\hat{J}_\alpha(x_{n+1}\eta_{n+1})\sigma'(x,\xi)\right]\right|,$$

where $2\delta'+r=(2\delta_1,\cdots,2\delta_n,r)\in\mathbb{N}_0^{n+1}$ and $|\delta'|=s-j$.

In view of Leibnitz formula, the last expression becomes

$$\left|\Delta^s_{\alpha,n,x}\left[e^{i\langle x',\eta'\rangle}\hat{J}_\alpha(x_{n+1}\eta_{n+1})\sigma'(x,\xi)\right]\right| \leq \sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{\delta_1,\cdots,\delta_n\geq 0}\sum_{\gamma\leq 2\delta'+r}\binom{s}{j}\binom{s-j}{\delta_1,\cdots,\delta_n}\binom{2\delta'+r}{\gamma}$$
$$\times E'_{\alpha,r}|x_{n+1}|^{r-s}\left|D^\gamma_x e^{i\langle x',\eta'\rangle}\hat{J}_\alpha(x_{n+1}\eta_{n+1})\right|$$
$$\times\left|D^{2\delta'+r-\gamma}_x\sigma'(x,\xi)\right|. \quad (3.3.23)$$

Using (1.3.4) and (3.3.1), (3.3.23) yields

$$\left|\Delta^s_{\alpha,n,x}\left[e^{i\langle x',\eta'\rangle}\hat{J}_\alpha(x_{n+1}\eta_{n+1})\sigma'(x,\xi)\right]\right| \leq \sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{\delta_1,\cdots,\delta_n\geq 0}\sum_{\gamma\leq 2\delta'+r}\binom{s}{j}\binom{s-j}{\delta_1,\cdots,\delta_n}\binom{2\delta'+r}{\gamma}$$
$$\times E'_{\alpha,r}C_{0,2\delta'+r-\gamma,k}|x_{n+1}|^{r-s}\|\eta\|^{|\gamma|}(1+\|x\|^2)^{-k}.$$

Therefore,

$$\left|\Delta^s_{\alpha,n,x}\left[e^{i\langle x',\eta'\rangle}\hat{J}_\alpha(x_{n+1}\eta_{n+1})\sigma'(x,\xi)\right]\right| \leq \sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{\delta_1,\cdots,\delta_n\geq 0}\sum_{\gamma\leq 2\delta'+r}\binom{s}{j}\binom{s-j}{\delta_1,\cdots,\delta_n}\binom{2\delta'+r}{\gamma}$$
$$\times E'_{\alpha,r}C_{0,2\delta'+r-\gamma,k}(1+\|\eta\|^2)^{\frac{|\gamma|}{2}}|x_{n+1}|^{r-s}(1+\|x\|^2)^{-k}$$
$$\leq \sum_{j=0}^{s}\sum_{r=1}^{2j}\sum_{\delta_1,\cdots,\delta_n\geq 0}\binom{s}{j}\binom{s-j}{\delta_1,\cdots,\delta_n}E'_{\alpha,r}C_{0,0,k}$$
$$\times(1+\|\eta\|^2)^{\frac{2|\delta'|+r}{2}}|x_{n+1}|^{r-s}(1+\|x\|^2)^{-k}$$
$$\leq \sum_{j=0}^{s}\sum_{r=1}^{2j}\binom{s}{j}E'_{\alpha,r}C_{0,0,k}(1+\|\eta\|^2)^{\frac{2(s-j)+r}{2}}$$
$$\times|x_{n+1}|^{r-s}(1+\|x\|^2)^{-k}$$
$$\leq \sum_{j=0}^{s}\binom{s}{j}E'_{\alpha,2s}C_{0,0,k}(1+\|\eta\|^2)^{\frac{2(s-j)+2s}{2}}$$
$$\times|x_{n+1}|^{s}(1+\|x\|^2)^{-k}$$
$$\leq E''_{\alpha,k,s}(1+\|\eta\|^2)^{2s}(1+\|x\|^2)^{\frac{s-2k}{2}}, \quad (3.3.24)$$

where $E''_{\alpha,k,s}$ is a positive constant.

Now, from (3.3.22) and (3.3.24) we get

$$\begin{aligned}(1+\|\xi\|^2)^p\left|\sigma'_\xi(\eta)\right| &\le \sum_{s=0}^{p}\binom{p}{s}E''_{\alpha,k,s}A_\alpha(1+\|\eta\|^2)^{2s}\int_{\mathbb{R}^{n+1}_+}(1+\|x\|^2)^{\frac{s-2k}{2}}|x_{n+1}|^{2\alpha+1}dx\\ &\le \sum_{s=0}^{p}\binom{p}{s}E''_{\alpha,k,s}A_\alpha(1+\|\eta\|^2)^{2s}\int_{\mathbb{R}^{n+1}_+}(1+\|x\|^2)^{\frac{s-2k+2\alpha+1}{2}}dx\\ &\le E''_{\alpha,k,p}A_\alpha(1+\|\eta\|^2)^{2p}\int_{\mathbb{R}^{n+1}_+}(1+\|x\|^2)^{\frac{p-2k+2\alpha+1}{2}}dx.\end{aligned}$$

For $k > \alpha + \frac{n}{2} + \frac{p}{2} + 1$, there exists a positive constant $E_{\alpha,k,p}$ such that

$$\left|\sigma'_\xi(\eta)\right| \le E_{\alpha,k,p}(1+\|\xi\|^2)^{-p}(1+\|\eta\|^2)^{2p}.$$

□

Theorem 3.3.7. *The function $F(\xi)$ is defined in (3.3.19). Then $F(\xi)$ is Weinstein transformable.*

Proof. Let $\sigma \in \Lambda$ such that $\sigma(x,\xi) = \sigma(\infty,\xi) + \sigma'(x,\xi)$. To prove that $F(\xi)$ is the Weinstein transformable, it is sufficient to show that $F \in S_*(\mathbb{R}^{n+1}_+)$.

Now, from (1.4.1) we have

$$\sigma(\infty,\xi)(\mathcal{F}_\alpha u)(\xi) = \int_{\mathbb{R}^{n+1}_+} e^{-i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(\infty,\xi)u(x)d\mu_\alpha(x). \tag{3.3.25}$$

Also, from (3.3.20) we can find

$$\begin{aligned}&\int_{\mathbb{R}^{n+1}_+}\sigma'_\xi(\eta)(\mathcal{F}_\alpha u)(\eta)d\mu_\alpha(\eta)\\ &\quad= \int_{\mathbb{R}^{n+1}_+}\left(\int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\eta'\rangle}\hat{J}_\alpha(x_{n+1}\eta_{n+1})\left[e^{-i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma'(x,\xi)\right]d\mu_\alpha(x)\right)\\ &\quad\times(\mathcal{F}_\alpha u)(\eta)d\mu_\alpha(\eta).\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}_+^{n+1}} \sigma'_\xi(\eta)(\mathcal{F}_\alpha u)(\eta)d\mu_\alpha(\eta) \\
&\quad = \int_{\mathbb{R}_+^{n+1}} \left(\int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\eta'\rangle} \hat{J}_\alpha(x_{n+1}\eta_{n+1})(\mathcal{F}_\alpha u)(\eta)d\mu_\alpha(\eta) \right) \\
&\qquad \times e^{-i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma'(x,\xi)d\mu_\alpha(x).
\end{aligned}$$

From (1.4.2), we get

$$\begin{aligned}
&\int_{\mathbb{R}_+^{n+1}} \sigma'_\xi(\eta)(\mathcal{F}_\alpha u)(\eta)d\mu_\alpha(\eta) \\
&\quad = \int_{\mathbb{R}_+^{n+1}} \mathcal{F}_\alpha^{-1}(\mathcal{F}_\alpha u)(x)e^{-i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma'(x,\xi)d\mu_\alpha(x) \\
&\quad = \int_{\mathbb{R}_+^{n+1}} e^{-i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma'(x,\xi)u(x)d\mu_\alpha(x). \qquad (3.3.26)
\end{aligned}$$

Using (3.3.25) and (3.3.26), then (3.3.19) becomes

$$\begin{aligned}
F(\xi) &= \int_{\mathbb{R}_+^{n+1}} e^{-i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(\infty,\xi)u(x)d\mu_\alpha(x) \\
&\quad + \int_{\mathbb{R}_+^{n+1}} e^{-i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma'(x,\xi)u(x)d\mu_\alpha(x) \qquad (3.3.27) \\
&= \int_{\mathbb{R}_+^{n+1}} e^{-i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\big(\sigma(\infty,\xi)+\sigma'(x,\xi)\big)u(x)d\mu_\alpha(x) \\
&= \int_{\mathbb{R}_+^{n+1}} e^{-i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(x,\xi)u(x)d\mu_\alpha(x). \qquad (3.3.28)
\end{aligned}$$

Since σ is a symbol and $u \in S_*(\mathbb{R}_+^{n+1})$, therefore by Lemma 3.3.3, we get $\mathcal{F}_\alpha[\sigma(x,\xi)u(x)] \in S_*(\mathbb{R}_+^{n+1})$. Hence, $F(\xi) \in S_*(\mathbb{R}_+^{n+1})$. □

Theorem 3.3.8. *Let $\alpha > -\frac{1}{2}$ and $\sigma(x,\xi)$ be a symbol in Λ. Then for $u \in S_*(\mathbb{R}^{n+1}_+)$, the pseudo-differential operators $\mathcal{P}(x,D)$ associated to σ can be expressed as*

$$[\mathcal{P}(x,D)u](x) = \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) \times \left(\int_{\mathbb{R}^{n+1}_+} e^{-i\langle y',\xi'\rangle} \hat{J}_\alpha(y_{n+1}\xi_{n+1}) \sigma(y,\xi) u(y) d\mu_\alpha(y) \right) \times d\mu_\alpha(\xi). \tag{3.3.29}$$

Proof. From (3.3.17) and (3.3.28), we get

$$[\mathcal{P}(x,D)u](x) = \int_{\mathbb{R}^{n+1}_+} e^{i\langle x',\xi'\rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) \times \left(\int_{\mathbb{R}^{n+1}_+} e^{-i\langle y',\xi'\rangle} \hat{J}_\alpha(y_{n+1}\xi_{n+1}) \sigma(y,\xi) u(y) d\mu_\alpha(y) \right) \times d\mu_\alpha(\xi).$$

In view of (1.4.2), we find

$$\mathcal{F}_\alpha[\mathcal{P}(x,D)u](\xi) = \int_{\mathbb{R}^{n+1}_+} e^{-i\langle y',\xi'\rangle} \hat{J}_\alpha(y_{n+1}\xi_{n+1}) \sigma(y,\xi) u(y) d\mu_\alpha(y). \tag{3.3.30}$$

□

Lemma 3.3.9. *Let $\sigma \in \Lambda$. Then the pseudo-differential operators have the following representations*

$$[\mathcal{P}(x,D)u](x) = \big[\mathcal{P}(\infty,D)u\big](x) + \big[\mathcal{P}'(x,D)u\big](x), \quad \forall u \in S_*(\mathbb{R}^{n+1}_+) \tag{3.3.31}$$

where

$$[\mathcal{P}(\infty, D)u](x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1}) \times\left(\int_{\mathbb{R}_+^{n+1}} e^{-i\langle y',\xi'\rangle}\hat{J}_\alpha(y_{n+1}\xi_{n+1})\sigma(\infty,\xi)u(y)d\mu_\alpha(y)\right)d\mu_\alpha(\xi), \tag{3.3.32}$$

$$[\mathcal{P}'(x, D)u](x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1}) \times\left(\int_{\mathbb{R}_+^{n+1}} e^{-i\langle y',\xi'\rangle}\hat{J}_\alpha(y_{n+1}\xi_{n+1})\sigma'(y,\xi)u(y)d\mu_\alpha(y)\right)d\mu_\alpha(\xi). \tag{3.3.33}$$

and $\sigma(x,\xi) = \sigma'(x,\xi) + \sigma(\infty,\xi)$.

Proof. From (3.3.29), we have

$$\begin{aligned}
[\mathcal{P}(x, D)u](x) &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\bigg(\int_{\mathbb{R}_+^{n+1}} e^{-i\langle y',\xi'\rangle}\hat{J}_\alpha(y_{n+1}\xi_{n+1}) \\
&\quad\times\big(\sigma(\infty,\xi)+\sigma'(y,\infty)\big)u(y)d\mu_\alpha(y)\bigg)d\mu_\alpha(\xi) \\
&= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1}) \\
&\quad\times\left(\int_{\mathbb{R}_+^{n+1}} e^{-i\langle y',\xi'\rangle}\hat{J}_\alpha(y_{n+1}\xi_{n+1})\sigma(\infty,\xi)u(y)d\mu_\alpha(y)\right)d\mu_\alpha(\xi) \\
&\quad+\int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1}) \\
&\quad\times\left(\int_{\mathbb{R}_+^{n+1}} e^{-i\langle y',\xi'\rangle}\hat{J}_\alpha(y_{n+1}\xi_{n+1})\sigma'(y,\xi)u(y)d\mu_\alpha(y)\right)d\mu_\alpha(\xi).
\end{aligned}$$

Hence,

$$[\mathcal{P}(x,D)u](x) = \big[\mathcal{P}(\infty,D)u\big](x) + \big[\mathcal{P}'(x,D)u\big](x).$$

□

Theorem 3.3.10. *Let $\alpha > -\frac{1}{2}$ and $\sigma \in \Lambda$. Then pseudo-differential operators $\mathcal{P}(x,D)$ have the following norm inequality*

$$\|\mathcal{P}(x,D)u\|_{\mathcal{H}_\alpha^{r,2}} \leq E'_{\alpha,k,p,r,t}\|u\|_{\mathcal{H}_\alpha^{r+t,2}}, \qquad \forall\ u \in S_*(\mathbb{R}_+^{n+1}) \tag{3.3.34}$$

where $E'_{\alpha,k,p,r,t}$ is a positive constant for $k,p \in \mathbb{N}_0, r \in \mathbb{R}$ and $t > 0$ such that $t > \alpha + \frac{n}{2} + 4p - r + 1$.

Proof. From (3.3.31), we have

$$\mathcal{P}(x,D) = \mathcal{P}(\infty,D) + \mathcal{P}'(x,D).$$

Using (3.3.25) and (3.3.32), we have

$$\mathcal{F}_\alpha[\mathcal{P}(\infty,D)u](\xi) = \sigma(\infty,\xi)(\mathcal{F}_\alpha u)(\xi). \tag{3.3.35}$$

Now, from (3.2.2) and (3.3.35), we have

$$\begin{aligned}
\|\mathcal{P}(\infty,D)u\|^2_{\mathcal{H}_\alpha^{r,2}} &= \int_{\mathbb{R}_+^{n+1}} (1+\|\xi\|^2)^r|\mathcal{F}_\alpha[\mathcal{P}(\infty,D)u](\xi)|^2 d\mu_\alpha(\xi)\\
&= \int_{\mathbb{R}_+^{n+1}} (1+\|\xi\|^2)^r|\sigma(\infty,\xi)|^2|(\mathcal{F}_\alpha u)(\xi)|^2 d\mu_\alpha(\xi)\\
&\leq \Big(sup_{\|\xi\|=1}|\sigma(\infty,\xi)|\Big)^2 \int_{\mathbb{R}_+^{n+1}} (1+\|\xi\|^2)^r|(\mathcal{F}_\alpha u)(\xi)|^2 d\mu_\alpha(\xi)\\
&\leq C_1^2\|u\|^2_{\mathcal{H}_\alpha^{r,2}},
\end{aligned}$$

where $C_1 = \Big(sup_{\|\xi\|=1}|\sigma(\infty,\xi)|\Big)$ is the positive constant.

Therefore,

$$\|\mathcal{P}(\infty, D)u\|_{\mathcal{H}_\alpha^{r,2}} \leq C_1 \|u\|_{\mathcal{H}_\alpha^{r,2}}. \tag{3.3.36}$$

From (3.3.26) and (3.3.33), we get

$$\mathcal{F}_\alpha[\mathcal{P}'(x, D)u](\xi) = \int_{\mathbb{R}_+^{n+1}} \sigma'_\xi(\eta)(\mathcal{F}_\alpha u)(\eta) d\mu_\alpha(\eta). \tag{3.3.37}$$

Therefore,

$$\Big|(1+\|\xi\|^2)^{\frac{r}{2}}\ \mathcal{F}_\alpha\big[\mathcal{P}'(x,D)u\big](\xi)\Big| \leq \int_{\mathbb{R}_+^{n+1}} (1+\|\xi\|^2)^{\frac{r}{2}}\ |\sigma'_\xi(\eta)||(\mathcal{F}_\alpha u)(\eta)| d\mu_\alpha(\eta).$$

In view of (3.3.21), the last inequality becomes

$$\begin{aligned}\Big|(1+\|\xi\|^2)^{\frac{r}{2}}\ \mathcal{F}_\alpha\big[\mathcal{P}'(x,D)u\big](\xi)\Big| &\leq E_{\alpha,k,p} \int_{\mathbb{R}_+^{n+1}} (1+\|\xi\|^2)^{\frac{r}{2}} (1+\|\xi\|^2)^{-p} (1+\|\eta\|^2)^{2p} \\ &\quad \times |(\mathcal{F}_\alpha u)(\eta)| d\mu_\alpha(\eta) \\ &\leq E_{\alpha,k,p} (1+\|\xi\|^2)^{\frac{r-2p}{2}} \int_{\mathbb{R}_+^{n+1}} (1+\|\eta\|^2)^{\frac{4p-r-t}{2}} (1+\|\eta\|^2)^{\frac{r+t}{2}} \\ &\quad \times |(\mathcal{F}_\alpha u)(\eta)| d\mu_\alpha(\eta).\end{aligned}$$

For $t > \alpha + \frac{n}{2} + 4p - r + 1$, we have

$$\begin{aligned}\Big|(1+\|\xi\|^2)^{\frac{r}{2}}\ \mathcal{F}_\alpha\big[\mathcal{P}'(x,D)u\big](\xi)\Big| &\leq E_{\alpha,k,p} (1+\|\xi\|^2)^{\frac{r-2p}{2}} \Big\|(1+\|\eta\|^2)^{\frac{4p-r-t}{2}}\Big\|_{L^2_\alpha(\mathbb{R}_+^{n+1})} \\ &\quad \times \Big\|(1+\|\eta\|^2)^{\frac{r+t}{2}} (\mathcal{F}_\alpha u)(\eta)\Big\|_{L^2_\alpha(\mathbb{R}_+^{n+1})}.\end{aligned} \tag{3.3.38}$$

From (3.2.2) and (3.3.38), we get

$$\left|(1+\|\xi\|^2)^{\frac{r}{2}}\ \mathcal{F}_\alpha\big(\mathcal{P}'(x,D)u\big)(\xi)\right| \le E'_{\alpha,k,p,r,t}(1+\|\xi\|^2)^{\frac{r-2p}{2}}\|u\|_{\mathcal{H}^{r+t,2}_\alpha},$$

where $E'_{\alpha,k,p,r,t}$ is a positive constant.

Therefore,

$$\left\|(1+\|\xi\|^2)^{\frac{r}{2}}\ \mathcal{F}_\alpha\big(\mathcal{P}'(x,D)u\big)(\xi)\right\|_{L^2_\alpha(\mathbb{R}^{n+1}_+)} \le E'_{\alpha,k,p,r,t}\left\|(1+\|\xi\|^2)^{\frac{r-2p}{2}}\right\|_{L^2_\alpha(\mathbb{R}^{n+1}_+)}\|u\|_{\mathcal{H}^{r+t,2}_\alpha}. \tag{3.3.39}$$

For $p > \frac{1}{2}(\alpha+\frac{n}{2}+r+1)$, (3.3.39) yields

$$\left\|(1+\|\xi\|^2)^{\frac{r}{2}}\ \mathcal{F}_\alpha\big(\mathcal{P}'(x,D)u\big)(\xi)\right\|_{L^2_\alpha(\mathbb{R}^{n+1}_+)} \le E_{\alpha,k,p,r,t}\|u\|_{\mathcal{H}^{r+t,2}_\alpha},$$

where $E_{\alpha,k,p,r,t}$ is a constant depends only on α, k, p, r and t.

Therefore, from (3.2.2) the last expression gives

$$\big\|\mathcal{P}'(x,D)u\big\|_{\mathcal{H}^{r,2}_\alpha} \le E_{\alpha,k,p,r,t}\|u\|_{\mathcal{H}^{r+t,2}_\alpha}. \tag{3.3.40}$$

From (3.3.36) and (3.3.40), we have

$$\begin{aligned}\|\mathcal{P}(x,D)u\|_{\mathcal{H}^{r,2}_\alpha} &= \|\mathcal{P}(\infty,D)u+\mathcal{P}'(x,D)u\|_{\mathcal{H}^{r,2}_\alpha}\\ &\le \|\mathcal{P}(\infty,D)u\|_{\mathcal{H}^{r,2}_\alpha}+\|\mathcal{P}'(x,D)u\|_{\mathcal{H}^{r,2}_\alpha}\\ &\le C_1\|u\|_{\mathcal{H}^{r,2}_\alpha}+E_{\alpha,k,p,r,t}\|u\|_{\mathcal{H}^{r+t,2}_\alpha}.\end{aligned}$$

For $t > 0$, we have

$$\begin{aligned}\|\mathcal{P}(x,D)u\|_{\mathcal{H}_\alpha^{r,2}} &\leq C_1\|u\|_{\mathcal{H}_\alpha^{r+t,2}} + E_{\alpha,k,p,r,t}\|u\|_{\mathcal{H}_\alpha^{r+t,2}}\\ &\leq E'_{\alpha,k,p,r,t}\|u\|_{\mathcal{H}_\alpha^{r+t,2}},\end{aligned}$$

where $E'_{\alpha,k,p,r,t} = \max(C_1, E_{\alpha,k,p,r,t})$. □

Definition 3.3.11. $\big($**Pseudo-differential operator** $\mathcal{Q}(x,D)\big)$

Let $\sigma : C^\infty(\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}) \to \mathbb{C}$ be a symbol in Λ. Then for $u \in S_*(\mathbb{R}_+^{n+1})$, the pseudo-differential operator $\mathcal{Q}(x,D)$ associated with σ is defined by

$$[\mathcal{Q}(x,D)u](x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})G(\xi)d\mu_\alpha(\xi), \tag{3.3.41}$$

where

$$G(\xi) = \sigma(\infty,\xi)(\mathcal{F}_\alpha u)(\xi) + \int_{\mathbb{R}_+^{n+1}} \sigma'_\eta(\xi)(\mathcal{F}_\alpha u)(\eta)d\mu_\alpha(\eta), \tag{3.3.42}$$

and

$$\sigma'_\eta(\xi) = \int_{\mathbb{R}_+^{n+1}} e^{-i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\big[e^{i\langle x',\eta'\rangle}\hat{J}_\alpha(x_{n+1}\eta_{n+1})\sigma'(x,\eta)\big]d\mu_\alpha(x). \tag{3.3.43}$$

Lemma 3.3.12. *Let $\alpha > -\frac{1}{2}$ and $\sigma' \in \Lambda$. Then the following inequality holds*

$$\big|\sigma'_\eta(\xi)\big| \leq F_{\alpha,k,p}(1+\|\xi\|^2)^{-p}(1+\|\eta\|^2)^{2p}, \tag{3.3.44}$$

where $F_{\alpha,k,p}$ is a positive constant and $p, k \in \mathbb{N}_0$ such that $k > \alpha + \frac{n}{2} + \frac{p}{2} + 1$.

Proof. The proof of this lemma is exactly same as Lemma 3.3.6. □

An alternate form of $\mathcal{Q}(x,D)$ is obtained.

Theorem 3.3.13. *Let $\alpha > -\frac{1}{2}$ and σ be a symbol in Λ. Then for $u \in S_*(\mathbb{R}_+^{n+1})$, the pseudo-differential operator $\mathcal{Q}(x, D)$ can be expressed as*

$$[\mathcal{Q}(x, D)u](x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(x, \xi)(\mathcal{F}_\alpha u)(\xi) d\mu_\alpha(\xi). \tag{3.3.45}$$

Proof. We have $\sigma(x, \xi) = \sigma(\infty, \xi) + \sigma'(x, \xi)$ and $\mathcal{F}_\alpha u \in S_*(\mathbb{R}_+^{n+1})$.
From (3.3.41) and (3.3.42), we have

$$\begin{aligned}
\big[\mathcal{Q}(x, D)u\big](x) &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) G(\xi) d\mu_\alpha(\xi) \\
&= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) \\
&\quad \times \left(\sigma(\infty, \xi)(\mathcal{F}_\alpha u)(\xi) + \int_{\mathbb{R}_+^{n+1}} \sigma'_\eta(\xi)(\mathcal{F}_\alpha u)(\eta) d\mu_\alpha(\eta) \right) d\mu_\alpha(\xi) \\
&= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) \sigma(\infty, \xi)(\mathcal{F}_\alpha u)(\xi) d\mu_\alpha(\xi) \\
&\quad + \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) \left(\int_{\mathbb{R}_+^{n+1}} \sigma'_\eta(\xi)(\mathcal{F}_\alpha u)(\eta) d\mu_\alpha(\eta) \right) d\mu_\alpha(\xi) \\
&= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) \sigma(\infty, \xi)(\mathcal{F}_\alpha u)(\xi) d\mu_\alpha(\xi) \\
&\quad + \int_{\mathbb{R}_+^{n+1}} (\mathcal{F}_\alpha u)(\eta) \left(\int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) \sigma'_\eta(\xi) d\mu_\alpha(\xi) \right) d\mu_\alpha(\eta).
\end{aligned} \tag{3.3.46}$$

Taking inverse Weinstein transform in (3.3.43), we get

$$\int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1}) \sigma'_\eta(\xi) d\mu_\alpha(\xi) = e^{i\langle x', \eta' \rangle} \hat{J}_\alpha(x_{n+1}\eta_{n+1}) \sigma'(x, \eta). \tag{3.3.47}$$

Therefore, (3.3.46) and (3.3.47) gives

$$\begin{aligned}[\mathcal{Q}(x,D)u](x) &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(\infty,\xi)(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi)\\ &\quad+\int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\eta'\rangle}\hat{J}_\alpha(x_{n+1}\eta_{n+1})\sigma'(x,\eta)(\mathcal{F}_\alpha u)(\eta)d\mu_\alpha(\eta)\\ &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(\infty,\xi)(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi)\\ &\quad+\int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma'(x,\xi)(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi). \end{aligned} \tag{3.3.48}$$

Using linear property in (3.3.48), we find

$$\begin{aligned}[\mathcal{Q}(x,D)u](x) &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\big(\sigma(\infty,\xi)+\sigma'(x,\xi)\big)(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi)\\ &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(x,\xi)(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi),\end{aligned}$$

where $\sigma(x,\xi)=\sigma(\infty,\xi)+\sigma'(x,\xi)$. □

Lemma 3.3.14. *Let $\sigma \in \Lambda$ and $\mathcal{Q}(x,D)$ be the pseudo-differential operators. Then for all $u \in S_*(\mathbb{R}_+^{n+1})$, the following relation holds*

$$[\mathcal{Q}(x,D)u](x) = \big[\mathcal{Q}(\infty,D)u\big](x) + \big[\mathcal{Q}'(x,D)u\big](x). \tag{3.3.49}$$

Proof. With the help of (3.3.48), we have

$$\begin{aligned}[\mathcal{Q}(x,D)u](x) &= \int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(\infty,\xi)(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi)\\ &\quad+\int_{\mathbb{R}_+^{n+1}} e^{i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma'(x,\xi)(\mathcal{F}_\alpha u)(\xi)d\mu_\alpha(\xi)\\ &= [\mathcal{Q}(\infty,D)u](x)+[\mathcal{Q}'(x,D)u](x),\end{aligned}$$

where

$$[\mathcal{Q}(\infty, D)u](x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(\infty, \xi)(\mathcal{F}_\alpha u)(\xi) d\mu_\alpha(\xi), \qquad (3.3.50)$$

and

$$[\mathcal{Q}'(x, D)u](x) = \int_{\mathbb{R}_+^{n+1}} e^{i\langle x', \xi' \rangle} \hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma'(x, \xi)(\mathcal{F}_\alpha u)(\xi) d\mu_\alpha(\xi). \qquad (3.3.51)$$

□

Theorem 3.3.15. *Let $\alpha > -\frac{1}{2}$ and $\sigma \in \Lambda$ be a symbol. Then pseudo-differential operators $\mathcal{Q}(x, D)$ have the following norm inequality*

$$\|\mathcal{Q}(x, D)u\|_{\mathcal{H}_\alpha^{r,2}} \leq F'_{\alpha,k,p,r,t}\|u\|_{\mathcal{H}_\alpha^{r+t,2}}, \quad \forall\ u \in S_*(\mathbb{R}_+^{n+1}) \qquad (3.3.52)$$

where $F'_{\alpha,k,p,r,t}$ is a positive constant for $k, p \in \mathbb{N}_0, r \in \mathbb{R}$ and $t > 0$ such that $t > \alpha + \frac{n}{2} + 4p - r + 1$.

Proof. Taking the Weinstein transform in (3.3.41), we get

$$\mathcal{F}_\alpha[\mathcal{Q}(x, D)u](\xi) = G(\xi).$$

Using (3.3.42), above expression becomes

$$\mathcal{F}_\alpha[\mathcal{Q}(x, D)u](\xi) = \sigma(\infty, \xi)(\mathcal{F}_\alpha u)(\xi) + \int_{\mathbb{R}_+^{n+1}} \sigma'_\eta(\xi)(\mathcal{F}_\alpha u)(\eta) d\mu_\alpha(\eta). \qquad (3.3.53)$$

From (3.3.53), we have

$$\mathcal{F}_\alpha[\mathcal{Q}(\infty, D)u](\xi) = \sigma(\infty, \xi)(\mathcal{F}_\alpha u)(\xi).$$

In view of (3.2.2), we have

$$\begin{aligned}\|\mathcal{Q}(\infty, D)u\|^2_{\mathcal{H}^{r,2}_\alpha} &= \int_{\mathbb{R}^{n+1}_+}(1+\|\xi\|^2)^r|\sigma(\infty,\xi)|^2|(\mathcal{F}_\alpha u)(\xi)|^2 d\mu_\alpha(\xi)\\ &\leq \Big(sup_{\|\xi\|=1}|\sigma(\infty,\xi)|\Big)^2\int_{\mathbb{R}^{n+1}_+}(1+\|\xi\|^2)^r|(\mathcal{F}_\alpha u)(\xi)|^2 d\mu_\alpha(\xi)\\ &\leq C_1^2\|u\|^2_{\mathcal{H}^{r,2}_\alpha},\end{aligned}$$

where $C_1 = \Big(sup_{\|\xi\|=1}|\sigma(\infty,\xi)|\Big)$ is the positive constant.

Therefore,

$$\|\mathcal{Q}(\infty, D)u\|_{\mathcal{H}^{r,2}_\alpha} \leq C_1\|u\|_{\mathcal{H}^{r,2}_\alpha}. \tag{3.3.54}$$

From (3.3.53), we have

$$\mathcal{F}_\alpha[\mathcal{Q}'(x,D)u](\xi) = \int_{\mathbb{R}^{n+1}_+}\sigma'_\eta(\xi)(\mathcal{F}_\alpha u)(\eta)d\mu_\alpha(\eta).$$

Therefore,

$$\Big|(1+\|\xi\|^2)^{\frac{r}{2}}\ \mathcal{F}_\alpha\big[\mathcal{Q}'(x,D)u\big](\xi)\Big| \leq \int_{\mathbb{R}^{n+1}_+}\Big|(1+\|\xi\|^2)^{\frac{r}{2}}\ |\sigma'_\eta(\xi)||(\mathcal{F}_\alpha u)(\eta)|d\mu_\alpha(\eta).$$

In view of (3.3.44), the last inequality becomes

$$\begin{aligned}\Big|(1+\|\xi\|^2)^{\frac{r}{2}}\ \mathcal{F}_\alpha\big[\mathcal{Q}'(x,D)u\big](\xi)\Big| &\leq F_{\alpha,k,p}\int_{\mathbb{R}^{n+1}_+}(1+\|\xi\|^2)^{\frac{r}{2}}(1+\|\xi\|^2)^{-p}(1+\|\eta\|^2)^{2p}\\ &\quad\times|(\mathcal{F}_\alpha u)(\eta)|d\mu_\alpha(\eta)\\ &\leq F_{\alpha,k,p}(1+\|\xi\|^2)^{\frac{r-2p}{2}}\int_{\mathbb{R}^{n+1}_+}(1+\|\eta\|^2)^{\frac{4p-r-t}{2}}(1+\|\eta\|^2)^{\frac{r+t}{2}}\\ &\quad\times|(\mathcal{F}_\alpha u)(\eta)|d\mu_\alpha(\eta).\end{aligned}$$

Thus, we have

$$\Big|(1+\|\xi\|^2)^{\frac{r}{2}}\ \mathcal{F}_\alpha\big[\mathcal{Q}'(x,D)u\big](\xi)\Big| \le F_{\alpha,k,p}(1+\|\xi\|^2)^{\frac{r-2p}{2}}\big\|(1+\|\eta\|^2)^{\frac{4p-r-t}{2}}\big\|_{L^2_\alpha(\mathbb{R}^{n+1}_+)} \times \big\|(1+\|\eta\|^2)^{\frac{r+t}{2}}(\mathcal{F}_\alpha u)(\eta)\big\|_{L^2_\alpha(\mathbb{R}^{n+1}_+)}.$$

For $t > \alpha + \frac{n}{2} + 4p - r + 1$, we can find a positive constant $F_{\alpha,p,r,t}$ such that

$$\Big|(1+\|\xi\|^2)^{\frac{r}{2}}\ \mathcal{F}_\alpha\big[\mathcal{Q}'(x,D)u\big](\xi)\Big| \le F_{\alpha,k,p}F_{\alpha,p,r,t}(1+\|\xi\|^2)^{\frac{r-2p}{2}} \times \big\|(1+\|\eta\|^2)^{\frac{r+t}{2}}(\mathcal{F}_\alpha u)(\eta)\big\|_{L^2_\alpha(\mathbb{R}^{n+1}_+)}. \tag{3.3.55}$$

From (3.2.2) and (3.3.55), we get

$$\Big|(1+\|\xi\|^2)^{\frac{r}{2}}\ \mathcal{F}_\alpha\big[\mathcal{Q}'(x,D)u\big](\xi)\Big| \le F'_{\alpha,k,p,r,t}(1+\|\xi\|^2)^{\frac{r-2p}{2}}\|u\|_{\mathcal{H}^{r+t,2}_\alpha},$$

where $F'_{\alpha,k,p,r,t}$ is a positive constant.

Therefore,

$$\Big\|(1+\|\xi\|^2)^{\frac{r}{2}}\ \mathcal{F}_\alpha\big[\mathcal{Q}'(x,D)u\big](\xi)\Big\|_{L^2_\alpha(\mathbb{R}^{n+1}_+)} \le F'_{\alpha,k,p,r,t}\big\|(1+\|\xi\|^2)^{\frac{r-2p}{2}}\big\|_{L^2_\alpha(\mathbb{R}^{n+1}_+)}\ \|u\|_{\mathcal{H}^{r+t,2}_\alpha}. \tag{3.3.56}$$

For $p > \frac{1}{2}(\alpha + \frac{n}{2} + r + 1)$, (3.3.39) yields

$$\Big\|(1+\|\xi\|^2)^{\frac{r}{2}}\ \mathcal{F}_\alpha\big[\mathcal{Q}'(x,D)u\big](\xi)\Big\|_{L^2_\alpha(\mathbb{R}^{n+1}_+)} \le F_{\alpha,k,p,r,t}\|u\|_{\mathcal{H}^{r+t,2}_\alpha},$$

where $F_{\alpha,k,p,r,t}$ is a constant depends only on α, k, p, r and t.

Therefore, from (3.2.2) the last expression gives

$$\big\|\mathcal{Q}'(x,D)u\big\|_{\mathcal{H}_\alpha^{r,2}} \leq F_{\alpha,k,p,r,t}\|u\|_{\mathcal{H}_\alpha^{r+t,2}}. \tag{3.3.57}$$

From (3.3.49), we have

$$[\mathcal{Q}(x,D)u](x) = \big[\mathcal{Q}(\infty,D)u\big](x) + \big[\mathcal{Q}'(x,D)u\big](x).$$

Using (3.3.54) and (3.3.57), we obtain

$$\begin{aligned}
\|\mathcal{Q}(x,D)u\|_{\mathcal{H}_\alpha^{r,2}} &= \|\mathcal{Q}(\infty,D)u + \mathcal{Q}'(x,D)u\|_{\mathcal{H}_\alpha^{r,2}} \\
&\leq \|\mathcal{Q}(\infty,D)u\|_{\mathcal{H}_\alpha^{r,2}} + \|\mathcal{Q}'(x,D)u\|_{\mathcal{H}_\alpha^{r,2}} \\
&\leq \|\mathcal{Q}(\infty,D)u\|_{\mathcal{H}_\alpha^{r,2}} + \|\mathcal{Q}'(x,D)u\|_{\mathcal{H}_\alpha^{r,2}} \\
&\leq C_1\|u\|_{\mathcal{H}_\alpha^{r,2}} + \leq F_{\alpha,k,p,r,t}\|u\|_{\mathcal{H}_\alpha^{r+t,2}}.
\end{aligned}$$

Then we have

$$\|\mathcal{Q}(x,D)\|_{\mathcal{H}_\alpha^{r,2}} \leq F'_{\alpha,k,p,r,t}\|u\|_{\mathcal{H}_\alpha^{r+t,2}},$$

where $F'_{\alpha,k,p,r,t} = \max(C_1, F_{\alpha,k,p,r,t})$. □

Theorem 3.3.16. *Let $\alpha > -\frac{1}{2}$ and $\sigma \in \Lambda$ be a symbol. Then we find the following norm inequality*

$$\|\big(\mathcal{P}(x,D) - \mathcal{Q}(x,D)\big)u\|_{\mathcal{H}_\alpha^{r,2}} \leq C_{\alpha,k,p,r,t}\|u\|_{\mathcal{H}_\alpha^{r+t,2}}, \quad \forall\ u \in S_*(\mathbb{R}_+^{n+1}) \tag{3.3.58}$$

where $C_{\alpha,k,p,r,t}$ is a positive constant for $k,p \in \mathbb{N}_0, r \in \mathbb{R}$ and $t > 0$ such that $t > \alpha + \frac{n}{2} + 4p - r + 1$.

Proof. Invoking Theorem 3.3.10 and Theorem 3.3.15, we obtain

$$\begin{aligned}\|\big(\mathcal{P}(x,D)-\mathcal{Q}(x,D)\big)u\|_{\mathcal{H}^{r,2}_\alpha} &\leq \|\mathcal{P}(x,D)u\|_{\mathcal{H}^{r,2}_\alpha}+\|\mathcal{Q}(x,D)u\|_{\mathcal{H}^{r,2}_\alpha}\\ &\leq E'_{\alpha,k,p,r,t}\|u\|_{\mathcal{H}^{r+t,2}_\alpha}+F'_{\alpha,k,p,r,t}\|u\|_{\mathcal{H}^{r+t,2}_\alpha}\\ &\leq C_{\alpha,k,p,r,t}\|u\|_{\mathcal{H}^{r+t,2}_\alpha},\end{aligned}$$

where $C_{\alpha,k,p,r,t}=\max\big(E'_{\alpha,k,p,r,t}\ ,F'_{\alpha,k,p,r,t}\big)$ is a constant. □

3.4 Product and Commutators

In this section, we discuss some properties of the product and the commutator between pseudo-differential operators on $\mathcal{H}^{r,2}_\alpha$ - type Sobolev space.

Theorem 3.4.1. *Let $\sigma_1(x,\xi),\sigma_2(x,\xi)$ be two symbols in Λ. Then $\sigma(x,\xi)=\sigma_1(x,\xi)\sigma_2(x,\xi)$ is a symbol in Λ.*

Proof. For detailed proof, we refer the papers [82, p. 360] and [56, p. 143]. □

Here, we introduce pseudo-differential operators $\mathcal{R}(x,D),\mathcal{P}(x,D),\mathcal{B}(x,D)$ associated with symbol $\rho(x,\xi),\sigma(x,\xi),\tau(x,\xi)$ in Λ, respectively which are defined by the following way:

$$\mathcal{P}(x,D)=\mathcal{P}(\infty,D)+\mathcal{P}'(x,D),$$

$$\mathcal{B}(x,D)=\mathcal{B}(\infty,D)+\mathcal{B}'(x,D),$$

and

$$\begin{aligned}\mathcal{R}(x,D) &= \mathcal{P}(x,D)\mathcal{B}(x,D)\\ &= \mathcal{P}(\infty,D)\mathcal{B}(\infty,D)+\mathcal{P}'(x,D)\mathcal{B}(\infty,D)+\mathcal{P}(\infty,D)\mathcal{B}'(x,D)\\ &\quad+\mathcal{P}'(x,D)\mathcal{B}'(x,D)\\ &= \gamma(D)+K(x,D)+K_1(x,D)+K_2(x,D), \qquad (3.4.1)\end{aligned}$$

where

$$\sigma(\infty,\xi)\tau(\infty,\xi)=\rho(\infty,\xi)=\gamma(\xi), \qquad (3.4.2)$$

$$\sigma'(x,\xi)\tau'(x,\xi)=k(x,\xi), \qquad (3.4.3)$$

$$\sigma(\infty,\xi)\tau'(x,\xi)=k_1(x,\xi), \qquad (3.4.4)$$

and

$$\tau(\infty,\xi)\sigma'(x,\xi)=k_2(x,\xi). \qquad (3.4.5)$$

Using these results, we have

Lemma 3.4.2. *Let $u\in S_*(\mathbb{R}^{n+1}_+)$. Then*

$$\gamma(D)u=\mathcal{P}(\infty,D)\mathcal{B}(\infty,D)u. \qquad (3.4.6)$$

Proof. In view of (3.2.4), we have

$$\mathcal{F}_\alpha\big(\gamma(D)u\big)(\xi)=\int_{\mathbb{R}^{n+1}_+} e^{-i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\gamma(\xi)u(x)d\mu_\alpha(x).$$

From (3.4.2), we obtain

$$\mathcal{F}_\alpha\big(\gamma(D)u\big)(\xi)=\int_{\mathbb{R}^{n+1}_+} e^{-i\langle x',\xi'\rangle}\hat{J}_\alpha(x_{n+1}\xi_{n+1})\sigma(\infty,\xi)\tau(\infty,\xi)u(x)d\mu_\alpha(x).$$

By (1.4.1), we find

$$\mathcal{F}_\alpha\big(\gamma(D)u\big)(\xi) = \mathcal{F}_\alpha\big(\mathcal{P}(\infty, D)\mathcal{B}(\infty, D)u\big)(\xi).$$

By uniqueness of the Weinstein transform, we get the required result. □

Lemma 3.4.3. *Let $u \in S_*(\mathbb{R}^{n+1}_+)$. Then we have*

$$K_1(x, D)u = \mathcal{P}(\infty, D)\mathcal{B}'(x, D)u. \tag{3.4.7}$$

Proof. The proof of this lemma is same as Lemma 3.4.2. □

Lemma 3.4.4. *Let $u \in S_*(\mathbb{R}^{n+1}_+)$. Then we have*

$$K_2(x, D)u = \mathcal{B}(\infty, D)\mathcal{P}'(x, D)u. \tag{3.4.8}$$

Proof. This proof is similar as done in Lemma 3.4.2. □

Lemma 3.4.5. *Following relation holds*

$$\big[\mathcal{P}'(x, D), \mathcal{B}(\infty, D)\big] = \mathcal{P}'(x, D)\mathcal{B}(\infty, D) - \mathcal{B}(\infty, D)\mathcal{P}'(x, D), \tag{3.4.9}$$

where $[\ ,\ ]$ denotes the commutator between two operators.

Proof. The proof of this lemma is obvious. □

Theorem 3.4.6. *Let $\alpha > -\frac{1}{2}$ and $\sigma, b \in \Lambda$. Then the following norm inequality holds*

$$\big\| \big[\mathcal{P}'(x, D), \mathcal{B}(\infty, D)\big] u \big\|_{\mathcal{H}^{r,2}_\alpha} \leq C_{\alpha,p,r,t} \|u\|_{\mathcal{H}^{r+t,2}_\alpha}, \qquad \forall\ u \in S_*(\mathbb{R}^{n+1}_+) \tag{3.4.10}$$

where $[\ ,\ ]$ *denotes the commutator and* $C_{\alpha,p,r,t}$ *be a constant depending upon* $p \in \mathbb{N}_0$, $r \in \mathbb{R}$ *and* $t > 0$ *such that* $t > \alpha + \frac{n}{2} + 4p - r + 2$.

Proof. From (3.3.18) and (3.3.31), we have

$$\mathcal{F}_\alpha\Big(\mathcal{P}'(x,D)\mathcal{B}(\infty,D)u\Big)(\xi) = \int_{\mathbb{R}_+^{n+1}} \sigma'_\xi(\eta)\mathcal{F}_\alpha\Big(\mathcal{B}(\infty,D)u\Big)(\eta)d\mu_\alpha(\eta).$$

From (3.2.4), above expression becomes

$$\mathcal{F}_\alpha\Big(\mathcal{P}'(x,D)\mathcal{B}(\infty,D)u\Big)(\xi) = \int_{\mathbb{R}_+^{n+1}} \sigma'_\xi(\eta)\tau(\infty,\eta)\big(\mathcal{F}_\alpha u\big)(\eta)d\mu_\alpha(\eta). \tag{3.4.11}$$

Also, from (3.2.4) we have

$$\begin{aligned}\mathcal{F}_\alpha\Big(\mathcal{B}(\infty,D)\mathcal{P}'(x,D)u\Big)(\xi) &= \tau(\infty,\xi)\mathcal{F}_\alpha\Big(\mathcal{P}'(x,D)u\Big)(\xi)\\ &= \tau(\infty,\xi)\int_{\mathbb{R}_+^{n+1}} \sigma'_\xi(\eta)(\mathcal{F}_\alpha u)(\eta)d\mu_\alpha(\eta)\\ &= \int_{\mathbb{R}_+^{n+1}} \sigma'_\xi(\eta)\tau(\infty,\xi)(\mathcal{F}_\alpha u)(\eta)d\mu_\alpha(\eta). \end{aligned} \tag{3.4.12}$$

From (3.4.9), we obtain

$$\begin{aligned}\mathcal{F}_\alpha\Big(\big[\mathcal{P}'(x,D),\mathcal{B}(\infty,D)\big]u\Big)(\xi) &= \mathcal{F}_\alpha\Big(\mathcal{P}'(x,D)\mathcal{B}(\infty,D)u - \mathcal{B}(\infty,D)\mathcal{P}'(x,D)u\Big)(\xi)\\ &= \mathcal{F}_\alpha\Big(\mathcal{P}'(x,D)\mathcal{B}(\infty,D)u\Big)(\xi) - \mathcal{F}_\alpha\Big(\mathcal{B}(\infty,D)\mathcal{P}'(x,D)u\Big)(\xi).\end{aligned}$$

Using (3.4.11) and (3.4.12), the last expression yields

$$\mathcal{F}_\alpha\Big(\big[\mathcal{P}'(x,D),\mathcal{B}(\infty,D)\big]u\Big)(\xi) = \int_{\mathbb{R}_+^{n+1}} \big(\tau(\infty,\eta) - \tau(\infty,\xi)\big)\sigma'_\xi(\eta)(\mathcal{F}_\alpha u)(\eta)d\mu_\alpha(\eta).$$

Therefore,

$$\left|(1+\|\xi\|^2)^{\frac{r}{2}}\mathcal{F}_\alpha\Big(\big[\mathcal{P}'(x,D),\mathcal{B}(\infty,D)\big]u\Big)(\xi)\right| \leq \int_{\mathbb{R}_+^{n+1}}(1+\|\xi\|^2)^{\frac{r}{2}}\big|\tau(\infty,\xi)-\tau(\infty,\eta)\big| \times |\sigma'_\xi(\eta)|\ |(\mathcal{F}_\alpha u)(\eta)|d\mu_\alpha(\eta). \quad (3.4.13)$$

Also, from (3.3.7) we have

$$\big|\tau(\infty,\xi)-\tau(\infty,\eta)\big| \leq C(1+\|\xi-\eta\|^2)^{1/2}(1+\|\eta\|^2)^{-1/2}. \quad (3.4.14)$$

Using (3.3.21), (3.4.13) and (3.4.14) we obtain

$$\begin{aligned}\left|(1+\|\xi\|^2)^{\frac{r}{2}}\mathcal{F}_\alpha\Big(\big[\mathcal{P}'(x,D),\mathcal{B}(\infty,D)\big]u\Big)(\xi)\right| &\leq CE_{\alpha,k,p}\int_{\mathbb{R}_+^{n+1}}(1+\|\xi\|^2)^{\frac{r}{2}}(1+\|\xi-\eta\|^2)^{1/2}\\ &\quad\times(1+\|\eta\|^2)^{-1/2}(1+\|\xi\|^2)^{-p}(1+\|\eta\|^2)^{2p}\\ &\quad\times|(\mathcal{F}_\alpha u)(\eta)|d\mu_\alpha(\eta)\\ &\leq CE_{\alpha,k,p}(1+\|\xi\|^2)^{\frac{r-2p}{2}}\int_{\mathbb{R}_+^{n+1}}(1+\|\xi-\eta\|^2)^{1/2}\\ &\quad\times(1+\|\eta\|^2)^{\frac{4p-r-t-1}{2}}(1+\|\eta\|^2)^{\frac{r+t}{2}}|(\mathcal{F}_\alpha u)(\eta)|\\ &\quad\times d\mu_\alpha(\eta).\end{aligned}$$

In view of Cauchy-Schwarz inequality, the last expression yields

$$\begin{aligned}&\left|(1+\|\xi\|^2)^{\frac{r}{2}}\mathcal{F}_\alpha\Big(\big[\mathcal{P}'(x,D),\mathcal{B}(\infty,D)\big]u\Big)(\xi)\right|\\ &\quad\leq CE_{\alpha,k,p}(1+\|\xi\|^2)^{\frac{r-2p}{2}}\left\|\frac{(1+\|\xi-\eta\|^2)^{\frac{1}{2}}}{(1+\|\eta\|^2)^{\frac{r+t-4p+1}{2}}}\right\|_{L^2_\alpha(\mathbb{R}_+^{n+1})}\left\|(1+\|\eta\|^2)^{\frac{r+t}{2}}(\mathcal{F}_\alpha u)(\eta)\right\|_{L^2_\alpha(\mathbb{R}_+^{n+1})}.\end{aligned}$$

From (3.2.2), the last expression becomes

$$\begin{aligned}
&\left|(1+\|\xi\|^2)^{\frac{r}{2}}\mathcal{F}_\alpha\Big(\big[\mathcal{P}'(x,D),\mathcal{B}(\infty,D)\big]u\Big)(\xi)\right| \\
&\leq CE_{\alpha,k,p}\|u\|_{\mathcal{H}^{r+t,2}_\alpha}(1+\|\xi\|^2)^{\frac{r-2p}{2}}\left\|\frac{(1+\|\xi-\eta\|^2)}{(1+\|\eta\|^2)^{\frac{r+t-4p+1}{2}}}\right\|_{L^2_\alpha(\mathbb{R}^{n+1}_+)} \\
&\leq CE_{\alpha,k,p}\|u\|_{\mathcal{H}^{r+t,2}_\alpha}(1+\|\xi\|^2)^{\frac{r-2p}{2}} \\
&\times\left\|\frac{(1+\|\xi\|^2)+(1+\|\eta\|^2)+(1+\|\xi\|^2)(1+\|\eta\|^2)}{(1+\|\eta\|^2)^{\frac{r+t-4p+1}{2}}}\right\|_{L^2_\alpha(\mathbb{R}^{n+1}_+)} \\
&\leq CE_{\alpha,k,p}\|u\|_{\mathcal{H}^{r+t,2}_\alpha}\Bigg((1+\|\xi\|^2)^{\frac{r-2p+2}{2}}\left\|\frac{1}{(1+\|\eta\|^2)^{\frac{r+t-4p+1}{2}}}\right\|_{L^2_\alpha(\mathbb{R}^{n+1}_+)} \\
&+(1+\|\xi\|^2)^{\frac{r-2p}{2}}\left\|\frac{1}{(1+\|\eta\|^2)^{\frac{r+t-4p-1}{2}}}\right\|_{L^2_\alpha(\mathbb{R}^{n+1}_+)} \\
&+(1+\|\xi\|^2)^{\frac{r-2p+2}{2}}\left\|\frac{1}{(1+\|\eta\|^2)^{\frac{r+t-4p-1}{2}}}\right\|_{L^2_\alpha(\mathbb{R}^{n+1}_+)}\Bigg).
\end{aligned}$$

If we choose $t>\alpha+\frac{n}{2}+4p-r+2$, then we have

$$\begin{aligned}
&\left|(1+\|\xi\|^2)^{\frac{r}{2}}\mathcal{F}_\alpha\Big(\big[\mathcal{P}'(x,D),\mathcal{B}(\infty,D)\big]u\Big)(\xi)\right| \\
&\leq CE_{\alpha,k,p}\|u\|_{\mathcal{H}^{r+t,2}_\alpha}\Bigg(C_{\alpha,p,r,t,1}(1+\|\xi\|^2)^{\frac{r-2p+2}{2}}+C_{\alpha,p,r,t,2} \\
&\times(1+\|\xi\|^2)^{\frac{r-2p}{2}}+C_{\alpha,p,r,t,2}(1+\|\xi\|^2)^{\frac{r-2p+2}{2}}\Bigg).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left\|(1+\|\xi\|^2)^{\frac{r}{2}}\mathcal{F}_\alpha\Big(\big[\mathcal{P}'(x,D),\mathcal{B}(\infty,D)\big]u\Big)(\xi)\right\|_{L^2_\alpha(\mathbb{R}^{n+1}_+)} \\
&\leq CE_{\alpha,k,p}\|u\|_{\mathcal{H}^{r+t,2}_\alpha}\Bigg(C_{\alpha,p,r,t,1}\left\|(1+\|\xi\|^2)^{\frac{r-2p+2}{2}}\right\|_{L^2_\alpha(\mathbb{R}^{n+1}_+)}+C_{\alpha,p,r,t,2} \\
&\times\left\|(1+\|\xi\|^2)^{\frac{r-2p}{2}}\right\|_{L^2_\alpha(\mathbb{R}^{n+1}_+)}+C_{\alpha,p,r,t,2}\left\|(1+\|\xi\|^2)^{\frac{r-2p+2}{2}}\right\|_{L^2_\alpha(\mathbb{R}^{n+1}_+)}\Bigg).
\end{aligned}$$

For $p > \frac{1}{2}(\alpha + \frac{n}{2} + r + 3)$, there is a constant $C_{\alpha,p,r,t}$ such that

$$\big\|\big[\mathcal{P}'(x, D), \mathcal{B}(\infty, D)\big]u\big\|_{\mathcal{H}_\alpha^{r,2}} \leq C_{\alpha,p,r,t}\|u\|_{\mathcal{H}_\alpha^{r+t,2}}.$$

□

Conclusion: On the basis of rich calculus of the Weinstein transform and results of Zaidman [82], Pathak and Upadhyay [56], Upadhyay [78], Upadhyay and Chauhan [79] and others pseudo-differential operators $\mathcal{P}(x, D)$ and $\mathcal{Q}(x, D)$ associated with the homogeneous class of symbol of order zero are introduced and discussed boundedness properties on $\mathcal{H}_\alpha^{r,2}$ - type Sobolev space by considering theory of the Weinstein transform. Product of symbols and commutators of two pseudo-differential operators associated with homogeneous class of symbol are defined and its properties on certain type of Sobolev space are discussed. The aforesaid results are useful to find the weak solution of a pseudo-differential equation of pseudo-differential operators $\mathcal{P}(x, D)$ and $\mathcal{Q}(x, D)$ associated with homogeneous symbol $\sigma(x, \xi)$.

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