Chapter 8

Conclusion and future directions

In this chapter, we provide an overall summary and conclusion of all the chapters in the thesis. The chapter also presents some possible directions for future research.

Through this thesis, for interval optimization:

- Various inclusions and inequalities in interval analysis, which are instrumental in developing the theories of IVFs and IOPs have been proven.

- A rigorous analysis of various nonsmooth concepts, such as the Normal and Tangent cones for set of interval variables, gH-Weak subdifferential, gH-Dini-Hadamard subdifferential, gH-Clarke subdifferential, along with corresponding examples, has been conducted. Additionally, criteria for the existence of these subdifferentials for IVFs have been investigated.

- These concepts have been shown to hold several common algebraic preserving rules, such as the sum rule, product rule, partial chain rule, intersection rule, and linear transformation rule. These rules are very useful for studying optimality conditions for non-smooth interval-valued functions IVFs and IOPs.

- Using the proposed concepts, various important optimality conditions have been studied:

  - necessary efficient condition using Lagrange-multiplier rule via Normal cone for smooth IVFs.
  - optimality conditions for IOPs whose objectives are difference of two IVFs.
  - necessary optimality condition using augmented normal cone and gH-weak subdifferential.
  - Fritz-John-type (FJ) and KKT-type optimality conditions for constrained non-smooth IOP using gH-Dini Hadamard subdifferential.

  - sufficient optimality condition, weak and strong duality theory with the help of gH-Clarke subdifferential under generalized convexity for non-smooth IOPs.

- Lastly, based on the proposed supremum relation between gH-Directional derivative and gH-weak subdifferential, the $\mathscr{W}$-gH-weak subdifferential method have been designed. The convergence analysis of this method have been shown separately for constant step-size and diminishing step-size.

For set optimization:

- An efficient trust-region method (TRM) algorithm for solving a special type of set optimization problem has been proposed. The global convergence of TRM has also been proven.

- To verify the effectiveness of TRM, numerical experiments have been presented for several test examples.

- Further, two non-monotone modifications of TRM, Max-NTRM, and Avg-NTRM, have been proposed. They have also been shown to maintain global convergence properties.

- To show the performance improvement of non-monotone schemes over TRM, the three algorithms have been compared using 20 test examples based on the performance metric of the probability of convergence, speed of convergence, computation time, and average step size.

8.1 Future scopes of research

The role of a research community is to focus on developing new theories and algorithms, as well as improving existing ones. Based on the analysis of the work presented in this thesis, several areas for future research have been identified.

Interval optimization:

(i) In this thesis, the concept of normal and tangent cones for set of intervals and three types of gH-subdifferential for IVFs have been developed. In the future, one may attempt to study the simultaneous behavior of normal cone and subdifferential to propose the efficiency conditions for the following IOP:

$$(\mathbf{P}_1)\ \min\ \mathbf{T}(y) \text{ subject to } y \in S,$$

where $\mathbf{T} : \mathbb{R}^n \to I(\mathbb{R})$, S is a subset of $\mathbb{R}^n$. The result to be proven is given by

$$\mathbf{0} \in \partial_{\mathcal{F}} \mathbf{T}(y) \oplus \mathcal{N}_{\mathcal{F}}([y, y]; S),$$

where $\partial_{\mathcal{F}}$ and $\mathcal{N}_{\mathcal{F}}$ are Frećhet subdifferential and the Fréchet normal cone for set of intervals, respectively ([177]). Also, future research may focus on analyzing Fréchet's normal cone characterized by viscosity subgradient.

(ii) One may investigate the necessary and sufficient KKT optimality conditions via the normal and tangent cone ([190]) for the following extended IOPs:

$$\begin{aligned} (\mathbf{P}_2) \qquad & \min \ \mathbf{T}(y) \\ \text{subject to} \qquad & \mathbf{M}_j(y) \preceq \mathbf{0}, \ j = 1, 2, \ldots, k \\ & \mathbf{M}_j(y) = \mathbf{0}, \ j = k+1, k+2, \ldots, m, \end{aligned}$$

where $\mathbf{T} : \mathbb{R}^n \to I(\mathbb{R})$ is an IVF and $\mathbf{M}_j : \mathbb{R}^n \to I(\mathbb{R})$ are an IVFs for $j = 1, 2, \ldots, m$ and the feasible set $\mathbf{C}$ is

$$\mathbf{C} = \{\in \mathbb{R}^n : \ \mathbf{M}_j(y) \preceq \mathbf{0}, \ j = 1, \ldots, k, \ \mathbf{M}_j(y) = \mathbf{0}, \ j = k+1, \ldots, m\}.$$

Then, y^* is an efficient solution to $(\mathbf{P}_2)$ if and only if there exists $\lambda = (\lambda_1, \ldots, \lambda_m)^\top \in \mathbb{R}^m$ such that

(i) (Lagrangian efficiency): $\nabla_y \mathbf{L}(y^*, \lambda) = \nabla \mathbf{T}(y^*) \oplus \sum_{j=1}^{k} \lambda_j \odot \mathbf{M}_j(y^*) \oplus \sum_{j=k+1}^{m} \lambda_j \odot \mathbf{M}_j(y^*) \in \mathcal{N}_{\mathbf{C}}(y^*)$,

(ii) (Complementary slackness): $\lambda_j \odot \mathbf{M}_j(y^*) \preceq \mathbf{0}$ for $j = 1, \ldots, k$ and $\lambda_j \odot \mathbf{M}_j(y^*) = \mathbf{0}$ for $j = k+1, \ldots, m$,

(iii) (Dual feasibility): $\lambda_i \geq 0$ for $j = 1, \ldots, m$,

(iv) (Primal feasibility): $\mathbf{M}_j(y) \preceq \mathbf{0}, \ j = 1, \ldots, k, \mathbf{M}_j(y) = \mathbf{0}, \ j = k+1, \ldots, m$.

(iii) In this thesis, the concept of the gH-weak subdifferential for non-smooth and non-convex IOP has been proposed. One may apply the gH-weak subdifferential to achieve a zero duality gap in IOPs and interval-valued differential equations. The method for eliminating the duality gap can be directly applied in the following areas:

- optimal solutions for control problems involving first-order differential equations [150].
- two-person zero-sum game [149].

- difference of convex programming [58].
- Hamilton-Jacobi field theory [150].

(iv) The augmented normal cone and gH-weak subdifferential suggest the possibility of defining supporting cones for a set of intervals in the future. This concept could potentially be utilized to introduce the conic gap, aimed at capturing the geometric characteristics of a non-convex set of intervals.

(v) In the thesis, the optimality condition for the following constrained IOP has been developed in terms of gH-weak subdifferential as follows:

$$\min_{z \in \mathcal{Z}} \{\boldsymbol{\Upsilon}_1(z) \ominus_{gH} \boldsymbol{\Upsilon}_2(z)\}, \tag{8.1}$$

where $\emptyset \neq \mathcal{Z} \subseteq \mathbb{R}^n$ and $\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_2 : \mathcal{Z} \to I(\mathbb{R})$ are two IVFs. In the future, one may study the optimality condition in terms of gH-Dini Hadamard subdifferential.

(vi) For future research, one may consider establishing the necessary condition for a weak efficient point of a cone-constrained IOP in terms of the gH-Dini Hadamrd subdifferential. To define cone-constrained IOP, let $\emptyset \neq \mathcal{Z} \subseteq \mathbb{R}^n$ and $\boldsymbol{\Upsilon} : \mathcal{Z} \to I(\mathbb{R})$ be an IVF given by $\boldsymbol{\Upsilon} = \boldsymbol{\Phi} \ominus_{gH} \boldsymbol{\Psi}$, where dom $\boldsymbol{\Phi} \subseteq$ dom $\boldsymbol{\Psi}$. Also, let $\mathcal{C} \subseteq \mathcal{Z}$ be a convex and closed set, W be a Banach space, and W^* be its topological dual. Consider the cone-constrained IOP

$$\begin{aligned} \min \quad & \boldsymbol{\Upsilon}(z) \\ \text{subject to} \quad & z \in \mathcal{A} = \{z \in \mathcal{C} : k(z) \in -\mathcal{K}\}, \end{aligned} \tag{8.2}$$

where $\emptyset \neq \mathcal{K} \subseteq W$ is a closed and convex cone, and $k : \mathcal{Z} \to W$ is a $\mathcal{K}$-convex function. For the IOP (8.2), one may try to show that if $\bar{z}$ is a weak efficient solution of $\boldsymbol{\Upsilon}$ on $\mathcal{A}$, then the following is true:

$$\partial^-_{\mathscr{H}} \boldsymbol{\Psi}(\bar{z}) \subseteq \partial^-_{\mathscr{H}} \boldsymbol{\Phi}(\bar{z}) \oplus \bigcup_{\substack{\Im^* \in \mathcal{K}^* \\ (\Im^* k)(\bar{z}) = 0}} \partial^-_{\mathscr{H}}((\Im^* k) \oplus \delta_{\mathcal{C}}))(\bar{z}), \tag{8.3}$$

where $\Im^* k : \mathcal{Z} \to \mathbb{R}$ is defined by $(\Im^* k)(z) = \langle \Im^*, k(z) \rangle$ for $\Im^* \in \mathcal{K}^*$ and $\delta_{\mathcal{C}}$ be the indicator function of $\mathcal{C}$.

Set optimization:

(i) In this thesis, three types of trust region schemes — one monotone and two non-monotone (max-type and average-type) have been developed for solving a special class of set optimization problems. Following this line of work, future research may focus on developing the Cauchy, dogleg, and double dogleg trust-region theories for solving this type of SOP.

(ii) As future research, one may extend these trust-region schemes to more abstract set optimization problems.

(iii) From an application point of view, the conventional trust region method [129] is employed to maximize the log-likelihood of logistic model problems, particularly for two-class classification tasks. In many two-class classification scenarios, imprecise data is common, often introducing uncertainty due to errors in measurement. Therefore, one may aim to develop a set-valued logistic regression model that incorporates uncertain variables and solve it using the proposed trust-region methods.