Chapter 6

Analysis of a Class of Fractional Delay Integro-Differential Equations with Riesz-Caputo Derivative

6.1 Introduction

In Chapters 3 and 4, we have studied the existence and uniqueness of mild solutions and the controllability of FDEs with delay employing the Caputo fractional derivative in Banach spaces. Further, in Chapter 5, we generalized our previous study on the existence and uniqueness of solutions for FDEs with the Caputo fractional derivative to include the Riesz-Caputo fractional derivative. We discussed the existence and uniqueness of results for FDEs along with the Riesz-Caputo fractional derivative. Now, we pay our attention to the integro-differential equations of fractional order. In particular, this chapter studies the existence and uniqueness of results for fractional order integro-differential equations with delay, incorporating the Riesz-Caputo fractional derivative by utilizing FC techniques and fixed-point theorems.

In 2019, Gu et al. [123], discussed a class of FDEs involving the Riesz-Caputo derivative, and introduced physical meaning. They provided the existence results for positive solutions by relying on Krasnoselskii's and Leray–Schauder's fixed-point theorems. In the work presented in [134], the authors successfully demonstrated the existence, uniqueness, and stability of solutions for a specific class of implicit functional FDEs characterized by finite delays and instantaneous impulses. The proof for these results was established by employing the Banach contraction mapping principle in conjunction with Schaefer's and Darbo's fixed-point theorems. The study in [135] focused on addressing the existence, uniqueness, and Ulam-Hyers-Rassias stability for a specific set of coupled systems in the context of implicit FDEs involving Riesz-Caputo fractional derivatives. These results were established through the application of the Banach contraction mapping principle and Schauder's fixed-point theorem. To explore more such research involving Riesz-Caputo derivatives, one can go through [14, 87, 88, 136] and cite theirs in.

Here, we examine the following family of fractional delay integro-differential equations:

$$\begin{cases} {}_0^{RC}D_t^{\eta}\mathsf{x}(t) = \mathsf{f}\big(t, x(\mu_1(t)), \mathsf{x}(\mu_2(t)), \dots, \mathsf{x}(\mu_n(t)), Q\mathsf{x}(\mu_{n+1}(t))\big), \;\; t \in J, \\ \mathsf{x}(0) = \mathsf{x}_0, \quad \mathsf{x}(T) = \mathsf{x}_T, \end{cases} \tag{6.1}$$

where ${}_0^{RC}D_t^{\eta}$ is the Riesz-Caputo fractional derivative of order $\eta \in (0,1)$, $J = [0,T]$, and X is a Banach space. The continuous functions $\mu_i : [0,T] \to [0,T]$ are delay terms satisfying the conditions $0 \leq \mu_i(t) \leq t$ for $i = 1, 2, \ldots, n, n+1$, where $n \in \mathbb{N}$. Moreover, $\mathsf{f} : [0, T] \times X^{n+1} \to X$ is a nonlinear function and Q is an integral operator defined as:

$$Q\mathsf{x}(t) = \int_0^t q\big(t, p, \mathsf{x}(\mu_{n+1}(p))\big) dp,$$

along with the function $q : \Delta \times X \to X$ is nonlinear and continuous such that $\Delta = \{(t, \mathsf{x}) : 0 \leq \mathsf{x} \leq t \leq T\}$.

Our goal is to delve deeper into this field and establish results beyond what has already been achieved. First, we investigate the existence and uniqueness of solutions of fractional delay integro-differential equations (6.1) in Banach space along with the Riesz-Caputo fractional derivative. We utilize FC techniques and fixed-point theorems to establish the above results. Further, by introducing a partial order in a Banach space of all continuous functions, we look into the existence of extremal solutions. In the end, a few examples are showcased to evince the proficiency of the offered results.

This chapter is organized in the following manner: Section 6.2 covers the fundamental definitions, theorems, and lemmas essential for establishing our study's main results. The existence and uniqueness of the solution of (6.1) are examined in Section 6.3. The extremal solutions of (6.1) are discussed in Section 6.4. Relevant examples associated with our findings have been discussed in Section 6.5. In the end, we give the conclusion in 6.6.

6.2 Preliminary Results

This section introduces some definitions and basic results that will be used throughout this chapter.

Lemma 6.1. *[137] **(Nonlinear contraction mapping)** In a non-empty complete metric space (X, d), a mapping $\psi : X \to X$ is characterized as a nonlinear contraction if there exists a continuous, non-decreasing function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the conditions: (i) $\omega(0) = 0$, and (ii) $\omega(r) < r$ for $r > 0$ such that $d\big(\psi(\mathsf{x}), \psi(\bar{\mathsf{x}})\big) \leq \omega\big(d(\mathsf{x}, \bar{\mathsf{x}})\big)$ for all $\mathsf{x}, \bar{\mathsf{x}} \in X$.*

Lemma 6.2. *[138]**(Nonlinear fixed point theorem)** Suppose ψ is a nonlinear contraction mapping from a non-empty complete metric space (X, d) to itself. Then, ψ has a unique fixed point.*

Lemma 6.3. *[89] Suppose ψ is a completely continuous operator from a Banach space X into itself, and the set $\Delta = \{\mathsf{x} \in X \mid \mathsf{x} = \varrho\psi(\mathsf{x}),\ 0 < \varrho < 1\}$ is bounded. Then, ψ admits at least one fixed point in X.*

Definition 6.4. [137] Let " $\preceq$ " be a partial order. A Solution x_M to the BVP (6.1) is said to be maximal if for any other solution x, one has $\mathsf{x} \preceq \mathsf{x}_M$. Again a solution x_m is called a minimal solution of the BVP (6.1) if $\mathsf{x}_m \preceq \mathsf{x}$.

Lemma 6.5. *[137] Suppose $[\mathsf{x}_*, \bar{\mathsf{x}}_*] \subseteq X$ is an order interval with partial order " $\preceq$ ", where X is a Banach space and let Φ from $[\mathsf{x}_*, \bar{\mathsf{x}}_*]$ to itself be a nondecreasing function. If each sequence $\{\Phi\mathsf{x}_n\} \subseteq \Phi\big([\mathsf{x}_*, \bar{\mathsf{x}}_*]\big)$ converges, whenever $\{\mathsf{x}_n\}$ is a monotone sequence in $[\mathsf{x}_*, \bar{\mathsf{x}}_*]$, then the sequence of Φ-iteration of x_* converges to the least fixed-point ξ_* of Φ and the sequence of Φ-iteration of $\bar{\mathsf{x}}_*$ converges to the greatest fixed-point ξ^* of Φ. Moreover,*

$$\xi_* = \min\{\bar{y} \in [\mathsf{x}_*, \bar{\mathsf{x}}_*] : \bar{y} \succeq \Phi\bar{y}\}, \text{ and } \ \xi^* = \max\{\bar{y} \in [\mathsf{x}_*, \bar{\mathsf{x}}_*] : \bar{y} \preceq \Phi\bar{y}\}. \tag{6.2}$$

6.3 Existence and Uniqueness Results

Lemma 6.6. *The BVP* (6.1) *can be expressed as the subsequent integral equation:*

$$\begin{aligned}\mathsf{x}(t) &= \frac{1}{2}(\mathsf{x}_0 + \mathsf{x}_T)\\ &+ \frac{1}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), Q\mathsf{x}(\mu_{n+1}(p))\big)dp\\ &+ \frac{1}{\Gamma(\eta)}\int_t^T (p-t)^{\eta-1}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), Q\mathsf{x}(\mu_{n+1}(p))\big)dp.\end{aligned}$$

Proof. By Definition 1.11, we have

$${}_0I_T^{\eta}\, {}^{RC}_{0}D_T^{\eta}\mathsf{x}(t) = \frac{1}{2}\big({}_0I_t^{\eta}\, {}^{C}_{0}D_t^{\eta} + (-1)^n\, {}_tI_T^{\eta}\, {}^{C}_{t}D_T^{\eta}\big)\mathsf{x}(t),$$

where ${}_0I_t^{\eta}$ and ${}_tI_T^{\eta}$ are the left, right Rieamann-Liouville integrals respectively.

Now, applying Lemma 5.10 for $0 < \eta < 1$, we have

$${}_0I_T^{\eta}\, {}^{RC}_{0}D_T^{\eta}\mathsf{x}(t) = \mathsf{x}(t) - \frac{1}{2}\Big[\mathsf{x}(0) + \mathsf{x}(T)\Big].$$

This gives

$$\mathsf{x}(t) = \frac{1}{2}\Big[\mathsf{x}(0) + \mathsf{x}(T)\Big] +_0 I_T^{\eta}\, {}^{RC}_{0}D_T^{\eta}\mathsf{x}(t). \tag{6.3}$$

Now, by equation (6.1) and (6.3), we have

$$\begin{aligned}\mathsf{x}(t) &= \frac{1}{2}(\mathsf{x}_0 + \mathsf{x}_T)\\ &+ \frac{1}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), Q\mathsf{x}(\mu_{n+1}(p))\big)dp\\ &+ \frac{1}{\Gamma(\eta)}\int_t^T (p-t)^{\eta-1}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), Q\mathsf{x}(\mu_{n+1}(p))\big)dp.\end{aligned}$$

It completes the proof. □

Theorem 6.7. *Assuming that* f *is a continuous mapping, and* $\exists$ *a* $M \in \mathbb{R}^+$ *such that for* $\mathsf{x}(\mu_i(t)) \in X, t \in J, (i = 1, 2, \ldots, n, n+1)$*:*

$$\left\|\mathsf{f}\big(t, \mathsf{x}(\mu_1(t)), \mathsf{x}(\mu_2(t)), \ldots, \mathsf{x}(\mu_n(t)), Q\mathsf{x}(\mu_{n+1}(t))\big)\right\| < M.$$

Then, for any $\mathsf{x}_0, \mathsf{x}_T \in X$*, the system* (6.1) *has a solution* $\mathsf{x}^* \in C(J, X)$*.*

Proof. Let us define an operator Φ from $C(J, X)$ to itself given by

$$\begin{aligned}
\Phi\mathsf{x}(t) = {} & \frac{1}{2}(\mathsf{x}_0 + \mathsf{x}_T) \\
& + \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{\eta-1} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), Q\mathsf{x}(\mu_{n+1}(p))\big) dp \\
& + \frac{1}{\Gamma(\eta)} \int_t^T (p-t)^{\eta-1} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), Q\mathsf{x}(\mu_{n+1}(p))\big) dp.
\end{aligned}$$

Since f is continuous in $J \times X^n$, then we get Φ is continuous on $C(J, X)$. Next, we show that $\Phi(B_r)$ is bounded and equicontinuous, where

$$B_r = \{\mathsf{x} \in C(J, X) : \|\mathsf{x}\| \le r\}.$$

For every x in the closed ball B_r and each $t \in J$, we obtain

$$\begin{aligned}
& \|\Phi\mathsf{x}(t)\| \\
& \le \frac{1}{2}\|\mathsf{x}_0 + \mathsf{x}_T\| \\
& \quad + \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{\eta-1} \left\|\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), Q\mathsf{x}(\mu_{n+1}(p))\big)\right\| dp \\
& \quad + \frac{1}{\Gamma(\eta)} \int_t^T (p-t)^{\eta-1} \left\|\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), Q\mathsf{x}(\mu_{n+1}(p))\big)\right\| dp \\
& \le \frac{1}{2}\|\mathsf{x}_0 + \mathsf{x}_T\| + \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{\eta-1} M + \frac{1}{\Gamma(\eta)} \int_t^T (p-t)^{\eta-1} M
\end{aligned}$$

$$\leq \frac{1}{2}\|\mathsf{x}_0 + \mathsf{x}_T\| + \frac{2T^\eta M}{\Gamma(\eta+1)} < \infty.$$

This illustrates that the set $\Phi(B_r)$ is bounded.

Next, we show that $\Phi(B_r)$ is equicontinuous for $\mathsf{x} \in B_r$ and $t_1, t_2 \in J$ such that $t_1 < t_2$.

$$\begin{aligned}
&\|\Phi\mathsf{x}(t_2) - \Phi\mathsf{x}(t_1)\| \\
&= \Bigg\|\Bigg[-\frac{1}{\Gamma(\eta)}\int_0^{t_1}(t_1-p)^{\eta-1}\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}(\mu_n(p)),Q\mathsf{x}(\mu_{n+1}(p))\big)dp \\
&\quad - \frac{1}{\Gamma(\eta)}\int_{t_1}^{T}(p-t_1)^{\eta-1}\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}(\mu_n(p)),Q\mathsf{x}(\mu_{n+1}(p))\big)dp\Bigg] \\
&\quad + \Bigg[\frac{1}{\Gamma(\eta)}\int_0^{t_2}(t_2-p)^{\eta-1}\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}u(\mu_n(p)),Q\mathsf{x}(\mu_{n+1}(p))\big)dp \\
&\quad + \frac{1}{\Gamma(\eta)}\int_{t_2}^{T}(p-t_2)^{\eta-1}\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}(\mu_n(p)),Q\mathsf{x}(\mu_{n+1}(p))\big)dp\Bigg]\Bigg\| \\
&\leq \Bigg\|\Bigg[\frac{1}{\Gamma(\eta)}\int_0^{t_2}(t_2-p)^{\eta-1}\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}(\mu_n(p)),Q\mathsf{x}(\mu_{n+1}(p))\big)dp \\
&\quad - \frac{1}{\Gamma(\eta)}\int_0^{t_1}(t_1-p)^{\eta-1}\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}(\mu_n(p)),Q\mathsf{x}(\mu_{n+1}(p))\big)dp\Bigg]\Bigg\| \\
&\quad + \Bigg\|\Bigg[\frac{1}{\Gamma(\eta)}\int_{t_2}^{T}(p-t_2)^{\eta-1}\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}(\mu_n(p)),Q\mathsf{x}(\mu_{n+1}(p))\big)dp \\
&\quad - \frac{1}{\Gamma(\eta)}\int_{t_1}^{T}(p-t_1)^{\eta-1}\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}(\mu_n(p)),Q\mathsf{x}(\mu_{n+1}(p))\big)dp\Bigg]\Bigg\| \\
&= I_1 + I_2.
\end{aligned}$$

We have

$$\begin{aligned}
I_1 = &\Bigg\|\frac{1}{\Gamma(\eta)}\int_0^{t_1}(t_2-p)^{\eta-1}\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}(\mu_n(p)),Q\mathsf{x}(\mu_{n+1}(p))\big)dp \\
&- \frac{1}{\Gamma(\eta)}\int_0^{t_1}(t_1-p)^{\eta-1}\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}(\mu_n(p)),Q\mathsf{x}(\mu_{n+1}(p))\big)dp \\
&+ \frac{1}{\Gamma(\eta)}\int_{t_1}^{t_2}(t_2-p)^{\eta-1}\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}(\mu_n(p)),Q\mathsf{x}(\mu_{n+1}(p))\big)dp\Bigg\|
\end{aligned}$$

$$\leq \frac{1}{\Gamma(\eta)}\int_0^{t_1}\left[(t_2-p)^{\eta-1}-(t_1-p)^{\eta-1}\right]Mds+\frac{1}{\Gamma(\eta)}\int_{t_1}^{t_2}(t_2-p)^{\eta-1}Mdp$$
$$\to 0 \ \text{ as } t_1\to t_2.$$

Likewise, we can assert that $I_2 \to 0$ as $t_1 \to t_2$, leading to $\|\Phi\mathsf{x}(t_2)-\Phi\mathsf{x}(t_1)\| \to 0$ as $t_1 \to t_2$, consequently, $\Phi(B_r)$ is equicontinuous. Applying the Arzelà-Ascoli theorem, we conclude that Φ is a completely continuous operator.

Finally, consider the set $\Delta_s = \{\mathsf{x} \in C(J,X) : \ \mathsf{x} = \varrho\Phi(\mathsf{x}) \ , 0 < \varrho < 1\}$. We show that this set is bounded. For each $\mathsf{x} \in \Delta_s$, some $\varrho \in (0,1)$, and for each $t \in J$, we have

$$\begin{aligned}
&\frac{1}{\varrho}\|\mathsf{x}(t)\| \\
&\leq \frac{1}{2}\|\mathsf{x}_0+\mathsf{x}_T\| \\
&\quad+\frac{1}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1}\big\|\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}(\mu_n(p)),Q\mathsf{x}(\mu_{n+1}(p))\big)\big\|dp \\
&\quad+\frac{1}{\Gamma(\eta)}\int_t^T (p-t)^{\eta-1}\big\|\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\ldots,\mathsf{x}(\mu_n(p)),Q\mathsf{x}(\mu_{n+1}(p))\big)\big\|dp \\
&\leq \frac{1}{2}\|\mathsf{x}_0+\mathsf{x}_T\|+\frac{M}{\Gamma(\eta)}\left[\int_0^t (t-p)^{\eta-1}dp+\int_t^T (p-t)^{\eta-1}dp\right] \\
&\leq \frac{1}{2}\|\mathsf{x}_0+\mathsf{x}_T\|+\frac{2T^\eta M}{\Gamma(\eta+1)}.
\end{aligned}$$

Therefore,

$$\|\mathsf{x}(t)\| \leq \varrho\left[\frac{1}{2}\|\mathsf{x}_0+\mathsf{x}_T\|+\frac{2T^\eta M}{\Gamma(\eta+1)}\right] < \infty.$$

This shows that Δ_s is bounded. Therefore, according to Lemma 6.3, Φ has at least one fixed-point. With this, the proof is concluded. □

Theorem 6.8. *Let us assume the following*

(R_1) *There is a positive constant L such that for any $\mathsf{x}, \bar{\mathsf{x}} \in X$*

$$\big|q(t,p,\mathsf{x}) - q(t,p,\bar{\mathsf{x}})\big| \le L|\mathsf{x} - \bar{\mathsf{x}}|, \quad \textit{and,} \quad \frac{2T^{\eta}}{\Gamma(\eta+1)} \le 1.$$

(R_2) *The function f from $J \times X^{n+1}$ to X is continuous and satisfies*

$$\begin{aligned}
&\big|\mathsf{f}\big(t, \mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n, \mathsf{x}_{n+1}\big) - \mathsf{f}\big(t, \bar{\mathsf{x}}_1, \bar{\mathsf{x}}_2, \ldots, \bar{\mathsf{x}}_n, \bar{\mathsf{x}}_{n+1}\big)\big| \\
&\qquad \le \max\left\{\frac{|\mathsf{x}_1 - \bar{\mathsf{x}}_1|}{a^2 + |\mathsf{x}_1 - \bar{\mathsf{x}}_1|}, \frac{|\mathsf{x}_2 - \bar{\mathsf{x}}_2|}{a^2 + |\mathsf{x}_2 - \bar{\mathsf{x}}_2|}, \frac{|\mathsf{x}_n - \bar{\mathsf{x}}_n|}{a^2 + |\mathsf{x}_n - \bar{\mathsf{x}}_n|}, \right. \\
&\qquad\quad \left. \ldots, \frac{|\mathsf{x}_{n+1} - \bar{\mathsf{x}}_{n+1}|}{a^2 + |\mathsf{x}_{n+1} - \bar{\mathsf{x}}_{n+1}|}\right\},
\end{aligned}$$

for all $\mathsf{x}, \bar{\mathsf{x}} \in X$, $t \in J$, where $a \in \mathbb{R}$ such that $|a| > \max\{1, LT\}$.

Then, for any $\mathsf{x}_0, \mathsf{x}_T \in X$, the system (6.1) has a unique solution $\mathsf{x}^* \in C(J, X)$.

Proof. Let us define an operator Φ from $C(J, X)$ to itself given by

$$\begin{aligned}
\Phi\mathsf{x}(t) = {} & \frac{1}{2}(\mathsf{x}_0 + \mathsf{x}_T) \\
& + \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{\eta-1} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), Q\mathsf{x}(\mu_{n+1}(p))\big)\, dp \\
& + \frac{1}{\Gamma(\eta)} \int_t^T (p-t)^{\eta-1} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), Q\mathsf{x}(\mu_{n+1}(p))\big)\, dp.
\end{aligned}$$

Now, we show that Φ is a nonlinear contraction mapping on $C(J, X)$. Let $t \in J$, $\mathsf{x}\big(\mu_i(t)\big), \bar{\mathsf{x}}\big(\mu_i(t)\big) \in X$. Then, by assumption (R_2), we have

$$\begin{aligned}
&\big|\Phi\mathsf{x}(t) - \Phi\bar{\mathsf{x}}(t)\big| \\
&\quad \le \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{\eta-1} \big|\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), Q\mathsf{x}(\mu_{n+1}(p))\big) \\
&\qquad - \mathsf{f}\big(p, \bar{\mathsf{x}}(\mu_1(p)), \bar{\mathsf{x}}(\mu_2(p)), \ldots, \bar{\mathsf{x}}(\mu_n(p)), Q\bar{\mathsf{x}}(\mu_{n+1}(p))\big)\big|\, dp
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\eta)}\int_t^T (p-t)^{\eta-1}\Big|\mathsf{f}\big(p,\mathsf{x}(\mu_1(p)),\mathsf{x}(\mu_2(p)),\dots,\mathsf{x}(\mu_n(p)),Q\mathsf{x}(\mu_{n+1}(p))\big)\\
&-\mathsf{f}\big(p,\bar{\mathsf{x}}(\mu_1(p)),\bar{\mathsf{x}}(\mu_2(p)),\dots,\bar{\mathsf{x}}(\mu_n(p)),Q\bar{\mathsf{x}}(\mu_{n+1}(p))\big)\Big|dp\\
\leq{}& \frac{1}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1}\max\Bigg\{\frac{\big|\mathsf{x}(\mu_1(p))-\bar{\mathsf{x}}(\mu_1(p))\big|}{a^2+\big|\mathsf{x}(\mu_1(p))-\bar{\mathsf{x}}(\mu_1(p))\big|},\\
&\frac{\big|\mathsf{x}(\mu_2(p))-\bar{\mathsf{x}}(\mu_2(p))\big|}{a^2+\big|\mathsf{x}(\mu_2(p))-\bar{\mathsf{x}}(\mu_2(p))\big|},\dots,\frac{\big|\mathsf{x}(\mu_n(p))-\bar{\mathsf{x}}(\mu_n(p))\big|}{a^2+\big|\mathsf{x}(\mu_n(p))-\bar{\mathsf{x}}(\mu_n(p))\big|},\\
&\frac{\big|Q\mathsf{x}(\mu_{n+1}(p))-Q\bar{\mathsf{x}}(\mu_{n+1}(p))\big|}{a^2+\big|Q\mathsf{x}(\mu_{n+1}(p))-Q\bar{\mathsf{x}}(\mu_{n+1}(p))\big|}\Bigg\}dp\\
+{}& \frac{1}{\Gamma(\eta)}\int_t^T (p-t)^{\eta-1}\max\Bigg\{\frac{\big|\mathsf{x}(\mu_1(p))-\bar{\mathsf{x}}(\mu_1(p))\big|}{a^2+\big|\mathsf{x}(\mu_1(p))-\bar{\mathsf{x}}(\mu_1(p))\big|},\\
&\frac{\big|\mathsf{x}(\mu_2(p))-\bar{\mathsf{x}}(\mu_2(p))\big|}{a^2+\big|\mathsf{x}(\mu_2(p))-\bar{\mathsf{x}}(\mu_2(p))\big|},\dots,\frac{\big|\mathsf{x}(\mu_n(p))-\bar{\mathsf{x}}(\mu_n(p))\big|}{a^2+\big|\mathsf{x}(\mu_n(p))-\bar{\mathsf{x}}(\mu_n(p))\big|},\\
&\frac{\big|Q\mathsf{x}(\mu_{n+1}(p))-Q\bar{\mathsf{x}}(\mu_{n+1}(p))\big|}{a^2+\big|Q\mathsf{x}(\mu_{n+1}(p))-Q\bar{\mathsf{x}}(\mu_{n+1}(p))\big|}\Bigg\}dp.
\end{aligned}
$$

Again, by assumption (R_1), we obtain

$$
\begin{aligned}
&\big|Q\mathsf{x}(\mu_{n+1}(t))-Q\bar{\mathsf{x}}(\mu_{n+1}(t))\big|\\
&\qquad=\left|\int_0^t q\big(t,p,\mathsf{x}(\mu_{n+1}(p))\big)dp-\int_0^t q\big(t,p,\bar{\mathsf{x}}(\mu_{n+1}(p))\big)dp\right|\\
&\qquad\leq\int_0^t L\big|\mathsf{x}(\mu_{n+1}(p))-\bar{\mathsf{x}}(\mu_{n+1}(p))\big|dp.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
|\Phi\mathsf{x}(t)-\Phi\bar{\mathsf{x}}(t)|\leq{}&\frac{1}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1}\max\Bigg\{\frac{\big|\mathsf{x}(\mu_1(p))-\bar{\mathsf{x}}(\mu_1(p))\big|}{a^2+\big|\mathsf{x}(\mu_1(p))-\bar{\mathsf{x}}(\mu_1(p))\big|},\\
&\frac{\big|\mathsf{x}(\mu_2(p))-\bar{\mathsf{x}}(\mu_2(p))\big|}{a^2+\big|\mathsf{x}(\mu_2(p))-\bar{\mathsf{x}}(\mu_2(p))\big|},\dots,\frac{\big|\mathsf{x}(\mu_n(p))-\bar{\mathsf{x}}(\mu_n(p))\big|}{a^2+\big|\mathsf{x}(\mu_n(p))-\bar{\mathsf{x}}(\mu_n(p))\big|},\\
&\frac{\displaystyle\int_0^t L\big|\mathsf{x}(\mu_{n+1}(p))-\bar{\mathsf{x}}(\mu_{n+1}(p))\big|dp}{a^2+\displaystyle\int_0^t L\big|\mathsf{x}(\mu_{n+1}(p))-\bar{\mathsf{x}}(\mu_{n+1}(p))\big|dp}\Bigg\}dp
\end{aligned}
$$

$$+\frac{1}{\Gamma(\eta)}\int_t^T (p-t)^{\eta-1}\max\Bigg\{\frac{\big|\mathsf{x}(\mu_1(p))-\bar{\mathsf{x}}(\mu_1(p))\big|}{a^2+\big|\mathsf{x}(\mu_1(p))-\bar{\mathsf{x}}(\mu_1(p))\big|},$$
$$\frac{\big|\mathsf{x}(\mu_2(p))-\bar{\mathsf{x}}(\mu_2(p))\big|}{a^2+\big|\mathsf{x}(\mu_2(p))-\bar{\mathsf{x}}(\mu_2(p))\big|},\dots,\frac{\big|\mathsf{x}(\mu_n(p))-\bar{\mathsf{x}}(\mu_n(p))\big|}{a^2+\big|\mathsf{x}(\mu_n(p))-\bar{\mathsf{x}}(\mu_n(p))\big|},$$
$$\frac{\int_0^t L\big|\mathsf{x}(\mu_{n+1}(p))-\bar{\mathsf{x}}(\mu_{n+1}(p))\big|dp}{a^2+\int_0^t L\big|\mathsf{x}(\mu_{n+1}(p))-\bar{\mathsf{x}}(\mu_{n+1}(p))\big|dp}\Bigg\}dp.$$

Now, here $0 \le p \le t \le T$ and $0 \le \mu_i(t) \le t$ for all $i = 1, 2, \dots, n, n+1$. So, by taking supremum over t, $\forall\ t \in J$, we obtain

$$\begin{aligned}
\|\Phi\mathsf{x}(t)-\Phi\bar{\mathsf{x}}(t)\| &\le \frac{1}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1}\max\left\{\frac{\|\mathsf{x}-\bar{\mathsf{x}}\|}{a^2+\|\mathsf{x}-\bar{\mathsf{x}}\|},\frac{LT\|\mathsf{x}-\bar{\mathsf{x}}\|}{a^2+LT\|\mathsf{x}-\bar{\mathsf{x}}\|}\right\}dp\\
&\quad+\frac{1}{\Gamma(\eta)}\int_t^T (p-t)^{\eta-1}\max\left\{\frac{\|\mathsf{x}-\bar{\mathsf{x}}\|}{a^2+\|\mathsf{x}-\bar{\mathsf{x}}\|},\frac{LT\|\mathsf{x}-\bar{\mathsf{x}}\|}{a^2+LT\|\mathsf{x}-\bar{\mathsf{x}}\|}\right\}dp\\
&\le \max\frac{2T^\eta}{\Gamma(\eta+1)}\left\{\frac{\|\mathsf{x}-\bar{\mathsf{x}}\|}{a^2+\|\mathsf{x}-\bar{\mathsf{x}}\|},\frac{LT\|\mathsf{x}-\bar{\mathsf{x}}\|}{a^2+LT\|\mathsf{x}-\bar{\mathsf{x}}\|}\right\}\\
&\le \max\left\{\frac{\|\mathsf{x}-\bar{\mathsf{x}}\|}{a^2+\|\mathsf{x}-\bar{\mathsf{x}}\|},\frac{LT\|\mathsf{x}-\bar{\mathsf{x}}\|}{a^2+LT\|\mathsf{x}-\bar{\mathsf{x}}\|}\right\}\\
&\le \omega(\|\mathsf{x}-\bar{\mathsf{x}}\|),\ \forall\ \mathsf{x}\big(\mu_i(p)\big),\bar{\mathsf{x}}\big(\mu_i(p)\big)\in X,
\end{aligned}$$

where

$$\omega(r)=\max\left\{\frac{r}{a^2+r},\frac{LTr}{a^2+LTr}\right\}<r,\quad \text{for } r>0,$$

which shows that Φ is a nonlinear contraction on $C(J, X)$. Therefore, by Lemma 6.2, the system (6.1) possesses a unique solution. This concludes the proof. □

In the following Theorem 6.9, we establish the uniqueness of the solution to the system (6.1) by imposing a set of somewhat weaker constraints on Φ.

Theorem 6.9. *Let us assume the following:*

(H_1) *The function* $\mathsf{f} : J \times X^{n+1} \to X$ *is continuous, and there exist non-negative constants* M_i *such that* $\forall\ \mathsf{x}_i, \bar{\mathsf{x}}_i \in X\ (i = 1, 2, \ldots, n, n+1), t \in J$,

$$\big\|\mathsf{f}\big(t, \mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n, \mathsf{x}_{n+1}\big) - \mathsf{f}\big(t, \bar{\mathsf{x}}_1, \bar{\mathsf{x}}_2, \ldots, \bar{\mathsf{x}}_n, \bar{\mathsf{x}}_{n+1}\big)\big\| \leq \sum_{i=1}^{n+1} M_i \big\|\mathsf{x}_i - \bar{\mathsf{x}}_i\big\|,$$

and

$$M = \max_{t \in J} \|\mathsf{f}(t, 0, \ldots, 0)\|.$$

(H_2) *For any* $\mathsf{x}, \bar{\mathsf{x}} \in X$ *there is a positive constant* L_1 *such that*

$$\|q(t, p, \mathsf{x}) - q(t, p, \bar{\mathsf{x}})\| \leq L_1 \|\mathsf{x} - \bar{\mathsf{x}}\|.$$

(H_3) *Let* $\gamma = \dfrac{2T^\eta}{\Gamma(\eta+1)}$ *and there exists* $q_1 = \gamma \sum_{i=1}^{n} M_i + M_{n+1} L_1 T$ *such that* $0 \leq q_1 < 1$.

Then, for any $\mathsf{x}_0, \mathsf{x}_T \in X$, *the system* (6.1) *has a unique solution* $\mathsf{x}^* \in C\,(J, X)sy$.

Proof. Define the operator $\Phi : C\,(J, X) \to C\,(J, X)$ as follows:

$$\begin{aligned}\Phi\mathsf{x}(t) &= \frac{1}{2}(\mathsf{x}_0 + \mathsf{x}_T) \\ &\quad + \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{\eta-1} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), Q\mathsf{x}(\mu_{n+1}(p))\big)\,dp \\ &\quad + \frac{1}{\Gamma(\eta)} \int_t^T (p-t)^{\eta-1} \mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), Q\mathsf{x}(\mu_{n+1}(p))\big)\,dp.\end{aligned}$$

Again, for $\mathsf{x}, \bar{\mathsf{x}} \in C\,(J, X)$ and for $t \in J$, using conditions (H_1) and (H_2), we have

$$\begin{aligned}&\|\Phi\mathsf{x}(t) - \Phi\bar{\mathsf{x}}(t)\| \\ &\qquad \leq \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{\eta-1} \big\|\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), Q\mathsf{x}(\mu_{n+1}(p))\big)\end{aligned}$$

$$
\begin{aligned}
&- \mathsf{f}\big(p, \bar{\mathsf{x}}(\mu_1(p)), \bar{\mathsf{x}}(\mu_2(p)), \ldots, \bar{\mathsf{x}}(\mu_n(p)), Q\bar{\mathsf{x}}(\mu_{n+1}(p))\big)\big\|dp \\
&+ \frac{1}{\Gamma(\eta)} \int_t^T (p-t)^{\eta-1} \big\|\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), Q\mathsf{x}(\mu_{n+1}(p))\big) \\
&- \mathsf{f}\big(p, \bar{\mathsf{x}}(\mu_1(p)), \bar{\mathsf{x}}(\mu_2(p)), \ldots, \bar{\mathsf{x}}(\mu_n(p)), Q\bar{\mathsf{x}}(\mu_{n+1}(p))\big)\big\|dp \\
\leq & \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{\eta-1} \sum_{i=1}^{n} M_i \big\|\mathsf{x}(\mu_i(p)) - \bar{\mathsf{x}}(\mu_i(p))\big\| \\
&+ M_{n+1} \|Q\mathsf{x}(\mu_{n+1}(p)) - Q\bar{\mathsf{x}}(\mu_{n+1}(p))\|dp \\
&+ \frac{1}{\Gamma(\eta)} \int_t^T (p-t)^{\eta-1} \sum_{i=1}^{n} M_i \big\|\mathsf{x}(\mu_i(p)) - \bar{\mathsf{x}}(\mu_i(p))\big\| \\
&+ M_{n+1} \big\|Q\mathsf{x}(\mu_{n+1}(p)) - Q\bar{\mathsf{x}}(\mu_{n+1}(p))\big\|dp \\
\leq & \frac{T^\eta}{\Gamma(\eta+1)} \left[\sum_{i=1}^{n} M_i \ \|\mathsf{x} - \bar{\mathsf{x}}\| + M_{n+1} T L_1 \|\mathsf{x} - \bar{\mathsf{x}}\| \right] \\
&+ \frac{(T-t)^\eta}{\Gamma(\eta+1)} \left[\sum_{i=1}^{n} M_i \|\mathsf{x} - \bar{\mathsf{x}}\| + M_{n+1} T L_1 \|\mathsf{x} - \bar{\mathsf{x}}\| \right] \\
\leq & \gamma \left[\sum_{i=1}^{n} M_i + M_{n+1} L_1 T \right] \|\mathsf{x} - \bar{\mathsf{x}}\| \\
\leq & q_1 \|\mathsf{x} - \bar{\mathsf{x}}\|,
\end{aligned}
$$

which implies that

$$\|\Phi\mathsf{x}(t) - \Phi\bar{\mathsf{x}}(t)\| \leq q_1 \|\mathsf{x} - \bar{\mathsf{x}}\|.$$

Therefore, the operator Φ acts as a contraction mapping on $C(J, X)$. Using the generalized Banach contraction mapping principle, we deduce that the system (6.1) has a unique solution. This concludes the proof. □

6.4 Existence of Extremal Solutions

In this section, we investigate the existence of extremal solutions for the system (6.1). To facilitate this, we consider $X = \mathbb{R}^n$, and introduce a partial ordering denoted by " $\preceq$ " in $C(J, \mathbb{R}^n)$, defined as $\mathsf{x}_1 \preceq \mathsf{x}_2$ if and only if $\mathsf{x}_1(t) \leq \mathsf{x}_2(t) \forall t \in J$, where the order relation in $\mathbb{R}^n$ is defined coordinate-wise, that is for $a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n,\ a \leq b$ iff $a_i \leq b_i, (i = 1, 2, \ldots, n)$. Furthermore, we define the order interval $[\mathsf{x}_*, \bar{\mathsf{x}}_*]$ as follows:

$$[\mathsf{x}_*, \bar{\mathsf{x}}_*] = \{\mathsf{x} \in C(J, X) : \mathsf{x}_* \preceq \mathsf{x} \preceq \bar{\mathsf{x}}_*\}.$$

Before proceeding further, we define the following definitions:

Definition 6.10. [137] **(Lower Solution)** A function $\mathsf{x} \in C(J, \mathbb{R}^n)$ is said to be a lower solution to the BVP (6.1) on J if

$$\begin{cases} {}^{RC}_{0}D^{\eta}_{t}\mathsf{x}(t) \leq \mathsf{f}\big(t, \mathsf{x}(\mu_1(t)), \mathsf{x}(\mu_2(t)), \ldots, \mathsf{x}(\mu_n(t)), Q\mathsf{x}(\mu_{n+1}(t))\big),\ \forall\ t \in J, \\ \mathsf{x}(0) = \mathsf{x}_0, \quad \mathsf{x}(T) = \mathsf{x}_T. \end{cases}$$

Definition 6.11. [137] **(Upper Solution)** A function $\mathsf{x} \in C(J, \mathbb{R}^n)$ is said to be an upper solution to the BVP (6.1) on J if

$$\begin{cases} {}^{RC}_{0}D^{\eta}_{t}\mathsf{x}(t) \geq \mathsf{f}\big(t, \mathsf{x}(\mu_1(t)), \mathsf{x}(\mu_2(t)), \ldots, \mathsf{x}(\mu_n(t)), Q\mathsf{x}(\mu_{n+1}(t))\big),\ \forall\ t \in J, \\ \mathsf{x}(0) = \mathsf{x}_0, \quad \mathsf{x}(T) = \mathsf{x}_T. \end{cases}$$

Theorem 6.12. *Let us assume the following:*

(Q_1) $t \to \mathsf{f}(t, \mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n, \mathsf{x}_{n+1})$ *is measurable* $\forall\ \mathsf{x}_i \in \mathbb{R}^n$, $(i = 1, 2, \ldots, n, n+1)$.

(Q_2) $\mathsf{f}(t, \mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n, \mathsf{x}_{n+1})$ *is non-decreasing in* x_i $(i = 1, 2, \ldots, n, n+1)$, $t \in J$.

(Q_3) *For* $0 \leq p \leq t \leq T$, $q(t, p, \mathsf{x})$ *is non-decreasing in* x.

(Q_4) *For any* $r \in \mathbb{R}^+$, $\exists\ g_r \in L^1(J, \mathbb{R})$ *such that for* $B_r = \{\mathsf{x} \in \mathbb{R}^n : \|\mathsf{x}\| \leq r\}$,

$$\sup_{\mathsf{x}_i \in B_r} |\mathsf{f}(t, \mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n, \mathsf{x}_{n+1})| \leq g_r(t), \quad t \in J.$$

(Q_5) *The BVP* (6.1) *possesses an upper solution* $\bar{\mathsf{x}}_*$ *and a lower solution* x_* *on* J *with* $\bar{\mathsf{x}}_* \geq \mathsf{x}_*$.

Then, the equation (6.1) *has both a minimal and a maximal solution in the interval* J.

Proof. By Lemma 6.6, the BVP (6.1) can be expressed equivalently as the following integral equation:

$$\begin{aligned}\Phi\mathsf{x}(t) = &\frac{1}{2}(\mathsf{x}_0 + \mathsf{x}_T)\\ &+ \frac{1}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), Q\mathsf{x}(\mu_{n+1}(p))\big)dp\\ &+ \frac{1}{\Gamma(\eta)}\int_t^T (p-t)^{\eta-1}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), Q\mathsf{x}(\mu_{n+1}(p))\big)dp.\end{aligned}$$

The proof consists of the following steps:

Step 1: Using the assumptions $(Q_2), (Q_3)$, we conclude that Φ is non-decreasing in $[\mathsf{x}_*, \bar{\mathsf{x}}_*]$. Let $\mathsf{x}, \bar{\mathsf{x}} \in [\mathsf{x}_*, \bar{\mathsf{x}}_*]$ be such that $\mathsf{x} \preceq \bar{\mathsf{x}}$. Then, we have

$$\begin{aligned}\Phi\mathsf{x}(t) = &\frac{1}{2}(\mathsf{x}_0 + \mathsf{x}_T)\\ &+ \frac{1}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), Q\mathsf{x}(\mu_{n+1}(p))\big)dp\\ &+ \frac{1}{\Gamma(\eta)}\int_t^T (p-t)^{\eta-1}\mathsf{f}\big(p, \mathsf{x}(\mu_1(p)), \mathsf{x}(\mu_2(p)), \ldots, \mathsf{x}(\mu_n(p)), Q\mathsf{x}(\mu_{n+1}(p))\big)dp\\ \leq &\frac{1}{2}(\bar{\mathsf{x}}_0 + \bar{\mathsf{x}}_T)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\eta)}\int_0^t (t-p)^{\eta-1}\mathsf{f}\big(p,\bar{\mathsf{x}}(\mu_1(p)),\bar{\mathsf{x}}(\mu_2(p)),\ldots,\bar{\mathsf{x}}(\mu_n(p)),Q\bar{\mathsf{x}}(\mu_{n+1}(p))\big)dp\\
&+\frac{1}{\Gamma(\eta)}\int_t^T (p-t)^{\eta-1}\mathsf{f}\big(p,\bar{\mathsf{x}}(\mu_1(p)),\bar{\mathsf{x}}(\mu_2(p)),\ldots,\bar{\mathsf{x}}(\mu_n(p)),Q\bar{\mathsf{x}}(\mu_{n+1}(p))\big)dp.\\
&=\Phi\bar{\mathsf{x}}(t) \quad \text{for all} \quad t\in J.
\end{aligned}
$$

Step 2: Next, we show that Φ is a mapping from $[\mathsf{x}_*,\bar{\mathsf{x}}_*]$ to itself. Since Φ is a nondecreasing operator in $[\mathsf{x}_*,\bar{\mathsf{x}}_*]$, then for any $\mathsf{x}\in[\mathsf{x}_*,\bar{\mathsf{x}}_*]$,

$$
\mathsf{x}_*(t)=\Phi\mathsf{x}_*(t)\le\Phi\mathsf{x}(t)\le\Phi\bar{\mathsf{x}}_*(t)=\bar{\mathsf{x}}_*(t) \ \text{ for all } t\in J.
$$

Hence, Φ is a mapping from $[\mathsf{x}_*,\bar{\mathsf{x}}_*]$ to itself.

Step 3: Consider a monotone sequence $\{\mathsf{x}_m\}$ in the interval $[\mathsf{x}_*,\bar{\mathsf{x}}_*]$. Our objective is to demonstrate that the sequence $\{\Phi\mathsf{x}_m\}$ converges in $\Phi\big([\mathsf{x}_*,\bar{\mathsf{x}}_*]\big)$. By the monotonicity of $\{\mathsf{x}_m\}$ it follows that the sequence $\{\Phi\mathsf{x}_m\}$ is monotone in $\Phi\big([\mathsf{x}_*,\bar{\mathsf{x}}_*]\big)$. Moreover, if the conditions $(Q_1)-(Q_5)$ are satisfied, then there exists a Lebesgue integrable function $g:J\to\mathbb{R}^+$, as mentioned in [137], defined by:

$$
\begin{aligned}
g(t)=&\Big|\mathsf{f}\big(t,\mathsf{x}_*(\mu_1(t)),\mathsf{x}_*(\mu_2(t)),\ldots,\mathsf{x}_*(\mu_n(t)),Q\mathsf{x}_*(\mu_{n+1}(t))\big)\Big|\\
&+\Big|\mathsf{f}\big(t,\bar{\mathsf{x}}_*(\mu_1(t)),\bar{\mathsf{x}}_*(\mu_2(t)),\ldots,\bar{\mathsf{x}}_*(\mu_n(t)),Q\bar{\mathsf{x}}_*(\mu_{n+1}(t))\big)\Big|,
\end{aligned}
$$

such that $\forall\ \mathsf{x}(t)\in[\mathsf{x}_*,\bar{\mathsf{x}}_*],\ t\in J$,

$$
\Big|\mathsf{f}\big(t,\mathsf{x}(\mu_1(t)),\mathsf{x}(\mu_2(t)),\ldots,\mathsf{x}(\mu_n(t)),Q\mathsf{x}(\mu_{n+1}(t))\big)\Big|\le g(t).
$$

Further, we shall establish that $\forall t \in J,$ and $\mathsf{x}_m \in [\mathsf{x}_*, \bar{\mathsf{x}}_*]$, Φ is uniformly bounded and equicontinuous.

$$
\begin{aligned}
&\big|\Phi\mathsf{x}_m(t)\big| \\
&= \frac{1}{2}|\mathsf{x}_0 + \mathsf{x}_T| \\
&\quad + \frac{1}{\Gamma(\eta)} \int_0^t (t-p)^{\eta-1} \big|\mathsf{f}\big(p, \mathsf{x}_m(\mu_1(p)), \mathsf{x}_m(\mu_2(p)), \ldots, \mathsf{x}_m(\mu_n(p)), Q\mathsf{x}_m(\mu_{n+1}(p))\big)\big| dp \\
&\quad + \frac{1}{\Gamma(\eta)} \int_t^T (p-t)^{\eta-1} \big|\mathsf{f}\big(p, \mathsf{x}_m(\mu_1(p)), \mathsf{x}_m(\mu_2(p)), \ldots, \mathsf{x}_m(\mu_n(p)), Q\mathsf{x}_m(\mu_{n+1}(p))\big)\big| dp \\
&\le \frac{1}{2}|\mathsf{x}_0 + \mathsf{x}_T| + \frac{1}{\Gamma(\eta)} \left[\int_0^t (t-p)^{\eta-1} g(p) dp + \int_t^T (p-t)^{\eta-1} g(p) dp \right] \\
&= \overline{M}(\text{say}), \ \forall \ m.
\end{aligned}
$$

By taking supremum over $t, \ \forall \ t \in J$, we get $\big\|\Phi\mathsf{x}_m\big\| \le \overline{M}$, which shows that $\{\Phi\mathsf{x}_m\}$ is uniformly bounded.

Again, for $\mathsf{x}_m \in [\mathsf{x}_*, \bar{\mathsf{x}}_*], \ t_1, t_2 \in J$ such that $t_1 < t_2$, we get

$$
\begin{aligned}
&\big|\Phi\mathsf{x}_m(t_2) - \Phi\mathsf{x}_m(t_1)\big| \\
&\qquad = \Bigg| \Bigg[-\frac{1}{\Gamma(\eta)} \int_0^{t_1} (t_1 - p)^{(\eta-1)} \\
&\qquad \times \big\{\mathsf{f}\big(p, \mathsf{x}_m(\mu_1(p)), \mathsf{x}_m(\mu_2(p)), \ldots, \mathsf{x}_m(\mu_n(p)), Q\mathsf{x}_m(\mu_{n+1}(p))\big)\big\} dp \\
&\qquad - \frac{1}{\Gamma(\eta)} \int_{t_1}^{T} (p - t_1)^{(\eta-1)} \\
&\qquad \times \big\{\mathsf{f}\big(p, \mathsf{x}_m(\mu_1(p)), \mathsf{x}_m(\mu_2(p)), \ldots, \mathsf{x}_m(\mu_n(p)), Q\mathsf{x}_m(\mu_{n+1}(p))\big)\big\} dp \Bigg] \\
&\qquad + \Bigg[\frac{1}{\Gamma(\eta)} \int_0^{t_2} (t_2 - p)^{(\eta-1)} \\
&\qquad \times \big\{\mathsf{f}\big(p, \mathsf{x}_m(\mu_1(p)), \mathsf{x}_m(\mu_2(p)), \ldots, \mathsf{x}_m(\mu_n(p)), Q\mathsf{x}_m(\mu_{n+1}(p))\big)\big\} dp \\
&\qquad + \frac{1}{\Gamma(\eta)} \int_{t_2}^{T} (p - t_2)^{(\eta-1)} \\
&\qquad \times \big\{\mathsf{f}\big(p, \mathsf{x}_m(\mu_1(p)), \mathsf{x}_m(\mu_2(p)), \ldots, \mathsf{x}_m(\mu_n(p)), Q\mathsf{x}_m(\mu_{n+1}(p))\big)\big\} dp \Bigg] \Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \bigg|\bigg[\frac{1}{\Gamma(\eta)}\int_0^{t_2}(t_2-p)^{(\eta-1)} \\
&\quad\times \Big\{\mathsf{f}\big(p,\mathsf{x}_m(\mu_1(p)),\mathsf{x}_m(\mu_2(p)),\ldots,\mathsf{x}_m(\mu_n(p)),Q\mathsf{x}_m(\mu_{n+1}(p))\big)\Big\}dp \\
&\quad - \frac{1}{\Gamma(\eta)}\int_0^{t_1}(t_1-p)^{(\eta-1)} \\
&\quad\times \Big\{\mathsf{f}\big(p,\mathsf{x}_m(\mu_1(p)),\mathsf{x}_m(\mu_2(p)),\ldots,\mathsf{x}_m(\mu_n(p)),Q\mathsf{x}_m(\mu_{n+1}(p))\big)\Big\}dp\bigg]\bigg| \\
&\quad + \bigg|\bigg[\frac{1}{\Gamma(\eta)}\int_{t_2}^{T}(p-t_2)^{(\eta-1)} \\
&\quad\times \Big\{\mathsf{f}\big(p,\mathsf{x}_m(\mu_1(p)),\mathsf{x}_m(\mu_2(p)),\ldots,\mathsf{x}_m(\mu_n(p)),Q\mathsf{x}_m(\mu_{n+1}(p))\big)\Big\}dp \\
&\quad - \frac{1}{\Gamma(\eta)}\int_{t_1}^{T}(p-t_1)^{(\eta-1)} \\
&\quad\times \Big\{\mathsf{f}\big(p,\mathsf{x}_m(\mu_1(p)),\mathsf{x}_m(\mu_2(p)),\ldots,\mathsf{x}_m(\mu_n(p)),Q\mathsf{x}_m(\mu_{n+1}(p))\big)\Big\}dp\bigg]\bigg| \\
&= I_1 + I_2.
\end{aligned}
$$

We have

$$
\begin{aligned}
I_1 &= \bigg|\frac{1}{\Gamma(\eta)}\int_0^{t_1}(t_2-p)^{(\eta-1)} \\
&\quad\times \Big[\mathsf{f}\big(p,\mathsf{x}_m(\mu_1(p)),\mathsf{x}_m(\mu_2(p)),\ldots,\mathsf{x}_m(\mu_n(p)),Q\mathsf{x}_m(\mu_{n+1}(p))\big)\Big]dp \\
&\quad - \frac{1}{\Gamma(\eta)}\int_0^{t_1}(t_1-p)^{(\eta-1)} \\
&\quad\times \Big[\mathsf{f}\big(p,\mathsf{x}_m(\mu_1(p)),\mathsf{x}_m(\mu_2(p)),\ldots,\mathsf{x}_m(\mu_n(p)),Q\mathsf{x}_m(\mu_{n+1}(p))\big)\Big]dp \\
&\quad + \frac{1}{\Gamma(\eta)}\int_{t_1}^{t_2}(t_2-p)^{(\eta-1)} \\
&\quad\times \Big[\mathsf{f}\big(p,\mathsf{x}_m(\mu_1(p)),\mathsf{x}_m(\mu_2(p)),\ldots,\mathsf{x}_m(\mu_n(p)),Q\mathsf{x}_m(\mu_{n+1}(p))\big)\Big]dp\bigg| \\
&\leq \frac{1}{\Gamma(\eta)}\bigg[\int_0^{t_1}\big[(t_2-p)^{\eta-1}-(t_1-p)^{\eta-1}\big]g(p)dp + \int_{t_1}^{t_2}(t_2-p)^{\eta-1}g(p)dp\bigg] \\
&\quad \to 0 \;\text{ as } t_1 \to t_2.
\end{aligned}
$$

Similarly, we can say that $I_2 \to 0$ as $t_1 \to t_2$, and hence $\big|\Phi\mathsf{x}_m(t_2) - \Phi\mathsf{x}_m(t_1)\big| \to 0$ as

$t_1 \to t_2$. By taking supremum over t, $\forall\ t \in J$, we get $\big\|\Phi \mathsf{x}_m(t_2) - \Phi \mathsf{x}_m(t_1)\big\| \to 0$ as $t_1 \to t_2$. Therefore, $\{\Phi \mathsf{x}_m\}$ is equicontinuous.

Thus, by applying the Arzelá-Ascoli theorem, the sequence $\{\Phi \mathsf{x}_m\}$ converges in $\Phi\big([\mathsf{x}_*, \bar{\mathsf{x}}_*]\big)$. Consequently, by Lemma 6.5, the BVP (6.1) has a minimal and a maximal solution over the interval J. □

6.5 Applications

In this section, we provide two illustrative examples demonstrating the efficacy of our proposed outcomes.

Example 6.1. *Consider the following FDDEs:*

$$\begin{cases} {}_0^{RC}D_t^{1/2}\mathsf{x}(t) = \dfrac{e^{-t}\cos t}{55}\mathsf{x}(\mu_1(t)) + \dfrac{1}{55}\displaystyle\int_0^t \sin t\ p\ \mathsf{x}\big(\mu_{n+1}(p)\big)dp,\ \ t \in [0,1], \\ \mathsf{x}(0) = 0, \quad \mathsf{x}(1) = 1. \end{cases} \tag{6.4}$$

Clearly, f is continuous. Now,

$$\begin{aligned} \big\|\mathsf{f}\big(t, \mathsf{x}(\mu_1(t)), &\mathsf{x}(\mu_2(t)), \ldots, \mathsf{x}(\mu_n(t)), \mathsf{x}(\mu_{n+1}(t))\big) \\ &- \mathsf{f}\big(t, \bar{\mathsf{x}}(\mu_1(t)), \bar{\mathsf{x}}(\mu_2(t)), \ldots, \bar{\mathsf{x}}(\mu_n(t)), \bar{\mathsf{x}}(\mu_{n+1}(t))\big)\big\| \\ &\leq \frac{1}{55}\big\|\mathsf{x}(\mu_1(t)) - \bar{\mathsf{x}}(\mu_1(t))\big\| + \frac{1}{55}\big\|Q\mathsf{x}(\mu_{n+1}(t)) - Q\bar{\mathsf{x}}(\mu_{n+1}(t))\big\|, \end{aligned}$$

which gives: $M_1 = M_{n+1} = \dfrac{1}{55}$, and $M_i = 0$, for, $i = 2, 3, \ldots, n$.
Further, $M = \max\limits_{t \in [0,1]} ||\mathsf{f}(t, 0, \ldots, 0)|| = 0$.

Again

$$\big\|q(t, p, \mathsf{x}) - q(t, p, \bar{\mathsf{x}})\big\| \leq \big\|\mathsf{x}\big(\mu_{n+1}(t)\big) - \bar{\mathsf{x}}\big(\mu_{n+1}(t)\big)\big\| \implies L_1 = 1$$

Since $\dfrac{2T^{\eta}}{\Gamma(\eta+1)}\sum_{i=1}^{n} M_i + M_{n+1}L_1T = 0.082063939421 < 1$. Thus, by Theorem (6.9) the differential equation (6.4) has a unique solution.

Example 6.2. *Let us examine the following nonlinear FDDEs:*

$$\begin{cases} {}_0^{RC}D_t^{\eta}\mathsf{x}(t) \;=\dfrac{e^t}{1+e^{-\mathsf{x}(\mu_1(t))}} + e^{2t}\tanh\big(\mathsf{x}(\mu_2(t))\big) \\ \qquad\qquad\quad + \dfrac{1}{5}\displaystyle\int_0^t e^{3t}p \;sgn\big(e^{\mathsf{x}(\mu_{n+1}(p))}\big)dp, \quad t\in[0,T], \quad 0<\eta\le 1, \\ \mathsf{x}(0)=\mathsf{x}_0, \quad \mathsf{x}(T)=\mathsf{x}_T. \end{cases} \tag{6.5}$$

Suppose $X=\mathbb{R}^n$. Clearly, the assumptions $(Q_1)-(Q_3)$ be satisfied.

Now, there exists $g(t)=3e^{3t}\in L^1[0,T]$ such that for all $\mathsf{x}\in B_r$, $t\in[0,T]$, where $B_r=\{\mathsf{x}\in X : \|\mathsf{x}\|\le r\}, r\in\mathbb{R}^+$, we have

$$-3e^{3t} \le \frac{e^t}{1+e^{-\mathsf{x}(\mu_1(t))}} + e^{2t}\tanh\big(\mathsf{x}(\mu_2(t))\big) + \frac{1}{5}\int_0^t e^{3t}p\ \mathrm{sgn}\big(e^{\mathsf{x}(\mu_{n+1}(p))}\big)dp \le 3e^{3t}.$$

Therefore, we define the functions θ and ϕ such that,

$$\begin{cases} {}_0^{RC}D_t^{\eta}\theta(t) = -3e^{3t} \\ t\in\big[0,T\big], \quad 0<\eta\le 1, \\ \theta(0)=\theta_0, \quad \theta(T)=\theta_T, \end{cases} \tag{6.6}$$

and

$$\begin{cases} {}_0^{RC}D_t^{\eta}\phi(t) = 3e^{3t} \\ t\in\big[0,T\big], \quad 0<\eta\le 1, \\ \phi(0)=\phi_0, \quad \phi(T)=\phi_T. \end{cases} \tag{6.7}$$

Both differential equations, (6.6) and (6.7), admit a solution as per the Existence Theorem (6.7), namely θ and ϕ. These are the upper and lower solutions, respectively, for the boundary value problem (6.5). So, by employing Theorem 6.12 equation (6.5) possesses both a minimal and a maximal solution over the interval J.

6.6 Conclusion

In this chapter, we explored fractional integro-differential equations with delay, incorporating the Riesz-Caputo fractional derivative. Initially, we examined the existence and uniqueness of solutions for these equations in Banach space using FC techniques and fixed-point theorems. We then introduced a partial order in a Banach space of continuous functions to investigate the existence of extremal solutions. Finally, we presented a few examples to illustrate the effectiveness of our results.

****