Chapter 3

Real-analytic submanifolds of finite type

In this chapter, we will prove that the Weyl transform of a smooth measure supported on a finite type real-analytic submanifold of $\mathbb{R}^{2n}$ is compact. We will also prove that the Weyl transform of a smooth measure supported on a finite type smooth curve in $\mathbb{R}^2$ is compact.

3.1 Introduction

In Theorem 2.1.1, we proved that if μ is a smooth measure supported on a compact hypersurface of positive Gaussian curvature, then $W(\mu)$ is compact. We will now consider a smooth measure supported on a submanifold of arbitrary codimension.

Let M be a smooth m-dimensional submanifold of $\mathbb{R}^{2n}$, $1 \leq m \leq 2n-1$. Recall that M is of finite type if at each point, M has at most a finite order of contact with any affine hyperplane. In particular, if we consider M to be a real-analytic submanifold of $\mathbb{R}^{2n}$, then the condition of being finite type is equivalent to M not lying in any affine hyperplane (see [49, p350]). For a hypersurface, the condition of being finite type is equivalent to the condition that at least one of the principal curvatures does not vanish to infinite order. For a curve in $\mathbb{R}^3$, the condition of being finite type is equivalent to the condition that neither the curvature nor the torsion vanishes to infinite order.

The main result of this chapter is the following theorem, which is an analog of Theorem 1.4.5 for the Weyl transform of a smooth measure supported on a real-analytic submanifold of finite type, and is proved in Section 3.3.

Theorem 3.1.1. *Suppose M is a connected real-analytic submanifold of $\mathbb{R}^{2n}$ which is not contained in an affine hyperplane. Let μ be a smooth measure on M. Then $W(\mu)$ is a compact operator.*

In Section 3.4, we prove that if $n = 1$, then the conclusion of Theorem 3.1.1 holds for a smooth submanifold of finite type without the additional assumption of real analyticity.

3.2 Twisted convolution of measures

In this section, we define the twisted convolution of finite Borel measures on $\mathbb{R}^{2n}$, and prove that the Weyl transform is an algebra homomorphism from the algebra of finite Borel measures on $\mathbb{R}^{2n}$ to $\mathcal{B}(\mathcal{H})$. For a detailed discussion, we refer to [26, 30].

Recall that if $f, g \in L^1(\mathbb{R}^{2n})$, then the twisted convolution of f and g, denoted by $f \natural g$, is given by

$$f \natural g(x,y) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x-x', y-y') g(x', y') e^{\pi i (x\cdot y' - y\cdot x')} \, dx' \, dy', \quad (x,y) \in \mathbb{R}^{2n}.$$

It is well known that twisted convolution turns $L^1(\mathbb{R}^{2n})$ into a non-commutative Banach algebra. Moreover, the Weyl transform is an algebra homomorphism from $L^1(\mathbb{R}^{2n})$ to $\mathcal{B}(\mathcal{H})$, i.e., if $f, g \in L^1(\mathbb{R}^{2n})$ then $W(f \natural g) = W(f)W(g)$.

Definition 3.2.1. Let μ and ν be finite Borel measures on $\mathbb{R}^{2n}$. The *twisted convolution* of μ and ν is the measure on $\mathbb{R}^{2n}$, denoted by $\mu \natural \nu$, given by

$$\mu \natural \nu(E) = \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \chi_E(x + x', y + y') e^{\pi i(x \cdot y' - y \cdot x')} \, d\mu(x, y) \, d\nu(x', y'),$$

where E is a Borel subset of $\mathbb{R}^{2n}$, and χ_E is the characteristic function of E.

It follows from Definition 3.2.1 that if $f \in \mathcal{C}_0(\mathbb{R}^{2n})$, then

$$\int_{\mathbb{R}^{2n}} f(x, y) \, d(\mu \natural \nu)(x, y) = \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} f(x + x', y + y') e^{\pi i(x \cdot y' - y \cdot x')} \, d\mu(x, y) \, d\nu(x', y'). \tag{3.1}$$

Let $f, g \in L^1(\mathbb{R}^{2n})$. Let m denote the Lebesgue measure on $\mathbb{R}^{2n}$. Let $\mu_f = fm$ and $\mu_g = gm$. If E is a Borel subset of $\mathbb{R}^{2n}$, then

$$\begin{aligned}
&\mu_f \natural \mu_g(E) \\
&= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \chi_E(x + x', y + y') e^{\pi i(x \cdot y' - y \cdot x')} \, d\mu(x, y) \, d\nu(x', y') \\
&= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \chi_E(x + x', y + y') e^{\pi i(x \cdot y' - y \cdot x')} f(x, y) g(x', y') \, dm(x, y) \, dm(x', y') \\
&= \int_{\mathbb{R}^{2n}} \chi_E(x, y) \int_{\mathbb{R}^{2n}} e^{\pi i(x \cdot y' - y \cdot x')} f(x - x', y - y') g(x', y') \, dm(x', y') \, dm(x, y) \\
&= \int_{\mathbb{R}^{2n}} \chi_E(x, y) (f \natural g)(x, y) \, dm(x, y) \\
&= (f \natural gm)(E) \\
&= \mu_{f \natural g}(E).
\end{aligned}$$

Therefore, the two definitions of twisted convolution coincide for L^1 functions.

Twisted convolution turns $M(\mathbb{R}^{2n})$, the set of finite Borel measures on $\mathbb{R}^{2n}$, into a non-commutative Banach algebra. The following theorem shows that the Weyl transform is an algebra homomorphism from $M(\mathbb{R}^{2n})$ to $\mathcal{B}(\mathcal{H})$.

Theorem 3.2.2. *Let μ and ν be finite Borel measures on $\mathbb{R}^{2n}$. Then*

$$W(\mu \natural \nu) = W(\mu)W(\nu).$$

Proof. By equations (1.4) and (3.1),

$$\begin{aligned}
W(\mu \natural \nu) &= \int_{\mathbb{R}^{2n}} \rho(x,y,1)\, d(\mu \natural \nu)(x,y) \\
&= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \rho(x+x', y+y', 1) e^{\pi i (x\cdot y' - y \cdot x')}\, d\mu(x,y)\, d\nu(x',y') \\
&= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \rho(x+x', y+y', e^{\pi i (x\cdot y' - y \cdot x')})\, d\mu(x,y)\, d\nu(x',y') \\
&= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \rho((x,y,1)(x',y',1))\, d\mu(x,y)\, d\nu(x',y') \\
&= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \rho(x,y,1)\rho(x',y',1)\, d\nu(x',y')\, d\mu(x,y) \\
&= \int_{\mathbb{R}^{2n}} \rho(x,y,1) W(\nu)\, d\mu(x,y) \\
&= W(\mu)W(\nu).
\end{aligned}$$

□

3.3 Proof of the main result

To prove Theorem 3.1.1, we need a result about the absolute continuity of the twisted convolution of measures supported on real-analytic submanifolds of $\mathbb{R}^{2n}$.

In [40, Theorem 5.1], Ragozin proved the following result.

Theorem 3.3.1. *Let* $M_1, \ldots, M_k$ *be connected real-analytic submanifolds of* $\mathbb{R}^n$ *such that for some choice of points* $p_i \in M_i$, $i = 1, \ldots, k$, $T_{p_1}M_1 + \cdots + T_{p_k}M_k = \mathbb{R}^n$, *where* $T_{p_i}M_i$ *denotes the tangent space at* p_i *of* M_i. *If* μ_i *is a smooth measure on* M_i, $i = 1, \ldots, k$, *then* $\mu_1 * \cdots * \mu_k$ *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}^n$.

In particular, Ragozin proved the absolute continuity of the convolution square of the surface measure on a compact analytic hypersurface of $\mathbb{R}^n$. Later, Thangavelu proved that the twisted convolution of the surface measure on a unit sphere in $\mathbb{R}^{2n}$ with itself is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{2n}$ (see [54, Proposition 4.3]).

The following theorem is an analog of Theorem 3.3.1 for twisted convolution.

Theorem 3.3.2. *Let* $M_1, \ldots, M_k$ *be connected real-analytic submanifolds of* $\mathbb{R}^{2n}$ *such that for some choice of points* $p_i \in M_i$, $i = 1, \ldots, k$, $T_{p_1}M_1 + \cdots + T_{p_k}M_k = \mathbb{R}^{2n}$, *where* $T_{p_i}M_i$ *denotes the tangent space at* p_i *of* M_i. *If* μ_i *is a smooth measure on* M_i, $i = 1, \ldots, k$, *then* $\mu_1 \natural \cdots \natural \mu_k$ *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}^{2n}$.

Let $M_1, \ldots, M_k$ be connected real-analytic submanifolds of $\mathbb{R}^{2n}$. Let $\Sigma_k : M_1 \times \cdots \times M_k \to \mathbb{R}^{2n}$ be the map given by

$$\Sigma_k(p_1, \ldots, p_k) = p_1 + \cdots + p_k.$$

Observe that Σ_k is an analytic map. We need the following lemma to prove Theorem 3.3.2.

Lemma 3.3.3. *There exists a smooth function $\varphi_k : M_1 \times \cdots \times M_k \to \mathbb{C}$ such that, for a Borel set $E \subseteq \mathbb{R}^{2n}$,*

$$\mu_1 \natural \cdots \natural \mu_k(E) = (\varphi_k \, \mu_1 \times \cdots \times \mu_k) \, (\Sigma_k^{-1}(E)),$$

i.e., $\mu_1 \natural \cdots \natural \mu_k$ is the push-forward of the measure $\varphi_k \, \mu_1 \times \cdots \times \mu_k$ by Σ_k.

Proof. Let $E \subseteq \mathbb{R}^{2n}$ be a Borel set. We will prove the result by induction on k.

First, consider $k = 2$. Observe that by Definition 3.2.1,

$$\mu_1 \natural \mu_2(E) = (\varphi_2 \, \mu_1 \times \mu_2) \, (\Sigma_2^{-1}(E)), \tag{3.2}$$

where $\varphi_2 : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{C}$ is given by $\varphi_2((x_1, y_1), (x_2, y_2)) = e^{\pi i(x_1 \cdot y_2 - y_1 \cdot x_2)}$.

Assume that there exists a function $\varphi_{k-1} : M_1 \times \cdots \times M_{k-1} \to \mathbb{C}$ such that

$$\mu_1 \natural \cdots \natural \mu_{k-1}(E) = (\varphi_{k-1} \, \mu_1 \times \cdots \times \mu_{k-1}) \, (\Sigma_{k-1}^{-1}(E)).$$

Then

$$\begin{aligned}
&((\mu_1 \natural \cdots \natural \mu_{k-1}) \natural \mu_k)(E) \\
&= \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} \chi_E(x + x_k, y + y_k) e^{\pi i(x \cdot y_k - y \cdot x_k)} \, d(\mu_1 \natural \cdots \natural \mu_{k-1})(x, y) \, d\mu_k(x_k, y_k) \\
&= \int_{\mathbb{R}^{2n}} \cdots \int_{\mathbb{R}^{2n}} \chi_E(x_1 + \cdots + x_k, y_1 + \cdots + y_k) e^{\pi i((x_1 + \cdots + x_{k-1}) \cdot y_k - (y_1 + \cdots + y_{k-1}) \cdot x_k)} \\
&\quad \varphi_{k-1}((x_1, y_1), \ldots, (x_{k-1}, y_{k-1})) \, d\mu_1(x_1, y_1) \ldots d\mu_k(x_k, y_k) \\
&= (\varphi_k \, \mu_1 \times \cdots \times \mu_k)(\Sigma_k^{-1}(E)),
\end{aligned}$$

where $\varphi_k : M_1 \times \cdots \times M_k \to \mathbb{C}$ is given by

$$\varphi_k((x_1, y_1), \ldots, (x_k, y_k))$$
$$= \varphi_{k-1}((x_1, y_1), \ldots, (x_{k-1}, y_{k-1}))e^{\pi i((x_1+\cdots+x_{k-1})\cdot y_k-(y_1+\cdots+y_{k-1})\cdot x_k)}.$$

□

Theorem 3.3.2 now follows by the argument in [40, Theorem 5.1]. Here, we give a more streamlined proof using the coarea formula.

Since $M_1, \ldots, M_k$ are connected real-analytic submanifolds of $\mathbb{R}^{2n}$, it follows that $M_1 \times \cdots \times M_k$ is a connected real-analytic submanifold of $\mathbb{R}^{2nk}$. Let $\tau_{(M_1\times\cdots\times M_k)}$ denote the Riemannian measure on $M_1 \times \cdots \times M_k$.

Let $p_i \in M_i$, $i = 1, \ldots, k$ be such that

$$T_{p_1}M_1 + \cdots + T_{p_k}M_k = \mathbb{R}^{2n}.$$

Then the rank of Σ_k at the point $(p_1, \ldots, p_k)$ is $2n$. Therefore, the critical set of Σ_k is a proper analytic subvariety of $M_1 \times \cdots \times M_k$ and hence has $\tau_{(M_1\times\cdots\times M_k)}$ measure zero (see [66]).

Let μ_i be a smooth measure on M_i, for $i = 1, \ldots, k$. Then $\mu_1 \times \cdots \times \mu_k$ is a smooth measure on $M_1 \times \cdots \times M_k$, and so $\varphi_k\, \mu_1 \times \cdots \times \mu_k$ is a smooth measure on $M_1 \times \cdots \times M_k$.

The proof of Theorem 3.3.2 now follows from the following lemma.

Lemma 3.3.4. *Let M and N be Riemannian manifolds. Let τ and ν denote the Riemannian measures on M and N, respectively. Let $\mu = \psi\tau$ be a smooth measure on M. Suppose $f : M \to N$ is a differentiable map. If f is a submersion everywhere* *except on a set of τ-measure zero, then the push-forward of μ by f is absolutely continuous with respect to ν.*

Proof. Let $Z = \{x \in M \mid f \text{ is not a submersion at } x\}$. Then $\tau(Z) = 0$, and so $\mu(Z) = 0$. Let $\mathcal{J}_f$ denote the normal Jacobian of f, i.e., the absolute value of the determinant of df restricted to the orthogonal complement of its kernel. Then $\mathcal{J}_f$ is strictly positive on the set of regular points of f, i.e., on $M \setminus Z$.

Let $x \in N \setminus f(Z)$. By the inverse function theorem, $f^{-1}(x)$ is a Riemannian manifold. Let σ_x denote the Riemannian measure on $f^{-1}(x)$. Let $f_*\mu$ denote the push-forward of μ by f. Let $U \subseteq N$ be a Borel set. By the coarea formula (see [2, p159]), we have

$$\begin{aligned} f_*\mu(U) &= \mu(f^{-1}(U)) \\ &= \mu(f^{-1}(U) \setminus Z) \\ &= \int_{f^{-1}(U)\setminus Z} d\mu \\ &= \int_{f^{-1}(U)\setminus Z} \psi \, d\tau \\ &= \int_{U\setminus f(Z)} \int_{f^{-1}(x)} \frac{\psi(y)}{\mathcal{J}_f(y)} \, d\sigma_x(y) \, d\nu(x). \end{aligned}$$

Therefore $f_*\mu$ is absolutely continuous with respect to ν, and

$$\frac{df_*\mu}{d\nu}(x) = \int_{f^{-1}(x)} \frac{\psi(y)}{\mathcal{J}_f(y)} \, d\sigma_x(y).$$

□

This completes the proof of Theorem 3.3.2. We need the following lemma to prove Theorem 3.1.1.

Lemma 3.3.5. *Let M be a connected real-analytic submanifold of $\mathbb{R}^{2n}$. Assume that M is not contained in any affine hyperplane. Then there exist points $p_1, \ldots, p_k \in M$*

such that

$$T_{p_1}M + \cdots + T_{p_k}M = \mathbb{R}^{2n}.$$

Proof. Suppose $\sum_{p\in M} T_pM \neq \mathbb{R}^{2n}$. Then there exists a hyperplane Y in $\mathbb{R}^{2n}$ such that $\sum_{p\in M} T_pM \subseteq Y$.

Fix $p \in M$. Let $q \in M$. Since M is a connected manifold, it follows that M is path connected. Therefore, there exists a differentiable map $\xi : [0,1] \to M$ such that $\xi(0) = p$ and $\xi(1) = q$. By the fundamental theorem of calculus,

$$q = p + \int_0^1 \dot{\xi}(t)\, dt.$$

Since $\dot{\xi}(t) \in Y$ for $t \in [0,1]$ and Y is closed, it follows that $q \in p + Y$. This implies that M is contained in the affine hyperplane $p + Y$, which is a contradiction.

Hence $\sum_{p\in M} T_pM = \mathbb{R}^{2n}$. By the Steinitz exchange lemma, there exist finitely many points $p_1, \ldots, p_k \in M$ such that $T_{p_1}M + \cdots + T_{p_k}M = \mathbb{R}^{2n}$. □

By possibly adding one more point, we can assume that the integer k obtained in Lemma 3.3.5 is even.

We are now in a position to prove Theorem 3.1.1. Let $\widetilde{M} = \{-p \mid p \in M\}$. Then $\widetilde{M}$ is a connected real-analytic submanifold of $\mathbb{R}^{2n}$. Observe that if $p \in M$, $T_pM = T_{(-p)}\widetilde{M}$. Therefore, by Lemma 3.3.5,

$$T_{p_1}M + T_{(-p_2)}\widetilde{M} + \cdots + T_{p_{k-1}}M + T_{(-p_k)}\widetilde{M} = \mathbb{R}^{2n}. \tag{3.3}$$

Let μ be a smooth measure on M. Let $\widetilde{\mu}$ denote the push-forward of μ by the map which sends $p \in M$ to $-p$. Then, by Theorem 3.3.2 and equation (3.3), it follows that $(\mu \natural \widetilde{\mu})^{k/2}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{2n}$. Therefore $W((\mu \natural \widetilde{\mu})^{k/2})$ is a compact operator.

Observe that $W(\widetilde{\mu}) = W(\mu)^*$, where $W(\mu)^*$ is the adjoint of the operator $W(\mu)$. By Theorem 3.2.2,

$$W((\mu \natural \widetilde{\mu})^{k/2}) = (W(\mu)W(\widetilde{\mu}))^{k/2} = (W(\mu)W(\mu)^*)^{k/2}.$$

Since $W(\mu)W(\mu)^*$ is self-adjoint and $(W(\mu)W(\mu)^*)^{k/2}$ is compact, it follows from Theorem 2.2.4 that $W(\mu)W(\mu)^*$ is compact. By Theorem 2.2.5, it follows that $W(\mu)$ is compact. This completes the proof of Theorem 3.1.1.

3.4 Curve in $\mathbb{R}^2$

In this section, we prove the following result, which states that if $n = 1$, the assumption of real-analyticity can be removed from Theorem 3.1.1.

Theorem 3.4.1. *Suppose M is a finite type connected smooth hypersurface of $\mathbb{R}^2$. Let μ be a smooth measure on M. Then $W(\mu)$ is a compact operator.*

We need the following lemma about the absolute continuity of twisted convolution of measures supported on smooth curves, where at least one of the curves is of finite type.

Lemma 3.4.2. *Let $\gamma : [a, b] \to \mathbb{R}^2$ be a finite type unit-speed simple smooth curve. Let $\delta : [c, d] \to \mathbb{R}^2$ be a unit-speed simple smooth curve. Let μ and ν be smooth measures on $\mathrm{Im}(\gamma)$ and $\mathrm{Im}(\delta)$, respectively. Then $\mu \natural \nu$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$.*

Proof. Let $E \subseteq \mathbb{R}^2$ be a Borel set. Then, by equation (3.2),

$$\mu \natural \nu(E) = (\varphi_2 \, \mu \times \nu) \, (\Sigma_2^{-1}(E)),$$

where $\Sigma_2 : \mathrm{Im}(\gamma) \times \mathrm{Im}(\delta) \to \mathbb{R}^2$ is the map given by

$$\Sigma_2(z, w) = z + w.$$

Define $S : [a, b] \times [c, d] \to \mathbb{R}^2$ by

$$S(s, t) = \gamma(s) + \delta(t).$$

Let

$$T = \{(s, t) \in [a, b] \times [c, d] \mid \dot{\gamma}(s) = \pm\dot{\delta}(t)\}.$$

Observe that T is the critical set of S. We claim that T has area zero. Suppose T has a positive area. Then, by Fubini's theorem, there exists $t \in [c, d]$ such that the set

$$Z = \{s \in [a, b] \mid \dot{\gamma}(s) = \pm\dot{\delta}(t)\}$$

has positive length. By the Lebesgue differentiation theorem [12, Theorem 3.21], there exists $s' \in [a, b]$ such that s' is a point of density of Z. Then there exist distinct $s_j \in Z$ such that the sequence $\{s_j\}$ converges to s', and so the sequence $\{\dot{\gamma}(s_j)\}$ converges to $\dot{\gamma}(s')$. Let $\vec{v}$ be a unit vector in $\mathbb{R}^2$ perpendicular to $\dot{\delta}(t)$. Then

$$\langle \dot{\gamma}(s_j) \, , \, \vec{v} \rangle = \langle \pm\dot{\delta}(t) \, , \, \vec{v} \rangle = 0, \quad j = 1, 2, \ldots.$$

Hence, all the coefficients in the Taylor expansion of $\langle \dot{\gamma}(s) \, , \, \vec{v} \rangle$ about $s = s'$ are zero. This contradicts the fact that γ is a curve of finite type. Therefore T has area zero.

Since $\mathrm{Im}(\gamma)$ and $\mathrm{Im}(\delta)$ are smooth submanifolds of $\mathbb{R}^2$, it follows that $\mathrm{Im}(\gamma)\times\mathrm{Im}(\delta)$ is a smooth submanifold of $\mathbb{R}^4$. Let τ denote the Riemannian measure on $\mathrm{Im}(\gamma)\times\mathrm{Im}(\delta)$. Since $\gamma\times\delta$ is a smooth map and the set T has area zero, it follows that $(\gamma\times\delta)(T)$ has τ-measure zero.

Observe that $(\gamma\times\delta)(T)$ is the critical set of Σ_2. The proof now follows from Lemma 3.3.4. □

We are now ready to prove Theorem 3.4.1. Suppose M is a finite type hypersurface of $\mathbb{R}^2$. Let μ be a smooth measure on M. Let $\widetilde{\mu}$ denote the push-forward of μ by the map which sends $p \in M$ to $-p$. It follows from Lemma 3.4.2 that $\mu\natural\widetilde{\mu}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$. Therefore $W(\mu\natural\widetilde{\mu})$ is compact. Observe that $W(\widetilde{\mu}) = W(\mu)^*$. By Theorem 3.2.2,

$$W(\mu\natural\widetilde{\mu}) = W(\mu)W(\widetilde{\mu}) = W(\mu)W(\mu)^*.$$

Since $W(\mu)W(\mu)^*$ is compact, it follows from Theorem 2.2.5 that $W(\mu)$ is compact. This completes the proof of Theorem 3.4.1.

3.5 Conclusion

We conclude that the Weyl transform of a smooth measure supported on a finite type real-analytic submanifold of $\mathbb{R}^{2n}$ is compact.

However, the result about the decay of the Fourier transform of a smooth measure supported on a smooth submanifold of finite type was without the additional assumption of real-analyticity. We needed this additional assumption in order to prove Theorem 3.3.2. But in Theorem 3.4.1, we proved that when $n = 1$, we do not need this addition assumption of real-analyticity. The argument used here is difficult to generalize to higher dimensions. However, we will prove Theorem 3.1.1 for an arbitrary smooth submanifold of finite type in Chapter 4.