Chapter 2

L^p_μ-spectra of pseudo-differential operators associated with the Bessel operator

2.1 Introduction

The Fourier transform theory was recently used by some authors to study the spectral properties of pseudo-differential operators. By utilizing the theory of the Fourier transform, Catchpole [10], Wong [72–74], and Weder [70] studied the spectral properties of a class of pseudo-differential operators.

The pseudo-differential operator associated with a certain class of symbols H^m, H^m_0 and their application in L^p-Sobolev type space were discussed by Pathak and Pandey [39, 40]. Pathak and Pandey [42], investigated the minimal and maximal pseudo-differential operator associated with the Bessel operator. In 2001, Pathak and Upadhyay [43] found the L^p_μ-boundedness of the pseudo-differential operators associated with the Bessel operator. Pathak and Pandey [48] also studied pseudo-differential operators $h_{\mu,a}$ in 2007 and showed that the kernel $K(x, y)$ of the pseudo-differential operators is a continuous function.

Based on these results, our main objective in this chapter is to study the spectral properties of pseudo-differential operators by exploiting the theory of the Hankel transform. For the present chapter, our consideration of the problems are the following:

1. The minimal and maximal pseudo-differential operator exploiting the theory of the Hankel transform will be observed.

2. The spectra of pseudo-differential operators and some results of the spectrum and essential spectrum of pseudo-differential operators will be investigated by the Hankel transform technique.

3. The essential spectra of pseudo-differential operators are related to the Sobolev space, and the heat equation will be considered.

The entire chapter is organized in the following manner:

Section 2.1 is introductory, and it gives brief descriptions of pseudo-differential operators. In Section 2.2, we studied the properties of the minimal and maximal pseudo-differential operator and some concepts of the spectra of pseudo-differential operators. A characterization of the spectra of the pseudo-differential operator $h_{\mu,\sigma}$ is given in Section 2.3, and it is shown that, under certain conditions, the spectrum, essential spectrum, and range of symbol coincide. In Section 2.4, some applications of spectra of pseudo-differential operators are given.

2.2 Minimal and Maximal pseudo-differential operators

In this section, the results of the minimal, maximal pseudo-differential operator are investigated by exploiting the theory of the Hankel transform.

Let X and Y be complex Banach space with norms denoted by $\| \ \|_X$ and $\| \ \|_Y$ respectively. We are concerned with linear operators A mapping a dense subspace of X, usually denoted by $D(A)$ into Y. We call $D(A)$ the domain of the operator A.

Definition 2.2.1. The operator $h_{\mu,\sigma}$ is said to be closable if for any sequence $< x_k >$ of vectors in $D(h_{\mu,\sigma})$ such that $< x_k > \to 0$ in $L^p_\mu(I = (0, \infty))$, for $1 < p < \infty$ and $h_{\mu,\sigma}(x_k) \to y$ as $k \to \infty$, then we have $y = 0$. Thus an operator $h_{\mu,\sigma}$ has a closed extension iff $h_{\mu,\sigma}$ is closable.

Definition 2.2.2. Let σ be a symbol in H^m and $h_{\mu,\sigma}$ be its associated pseudo-differential operator. Suppose there exists a linear operator $h^*_{\mu,\sigma} : H_\mu(I) \to H_\mu(I)$ such that

$$\langle h_{\mu,\sigma}\phi, \psi\rangle = \langle \phi, h^*_{\mu,\sigma}\psi\rangle \qquad \text{for all } \ \phi, \psi \in H_\mu(I). \tag{2.2.1}$$

Then $h^*_{\mu,\sigma}$ is called a formal adjoint of the operator $h_{\mu,\sigma}$. Let $u \in H'_\mu(I)$, then $h_{\mu,\sigma}u$ is defined by the relation

$$\langle h_{\mu,\sigma}u, \phi\rangle = \langle u, \overline{h^*_{\mu,\sigma}\bar{\phi}}\rangle \tag{2.2.2}$$

where $\bar{\phi}$ is the complex conjugate of ϕ.

Definition 2.2.3. The domain $D(h^0_{\mu,\sigma})$ of $h^0_{\mu,\sigma}$ consists of all functions u in $L^p_\mu(I)$ for which there exists a sequence ϕ_k in $H_\mu(I)$ such that $\phi_k \to u$ in $L^p_\mu(I)$, $1 < p < \infty$ and $h_{\mu,\sigma}\phi_k \to f$ in $L^p_\mu(I)$ for some $f \in L^p_\mu(I)$ as $k \to \infty$. For any $u \in D(h^0_{\mu,\sigma})$ define

$h^0_{\mu,\sigma}u = f$, then $h^0_{\mu,\sigma}$ is called the minimal pseudo-differential operator of $h_{\mu,\sigma}$.

Remark: The closure in $L^p(I)$ for $1 \leq p < \infty$, of $h_{\mu,\sigma} : H_\mu(I) \to H_\mu(I)$ is denoted by $h_{\mu,\sigma p}$ and called the minimal operator.

Definition 2.2.4. The domain $D(h^1_{\mu,\sigma})$ is the set of all functions $u \in L^p_\mu(I)$, $1 < p < \infty$, such that there exist a function $f \in L^p_\mu(I)$, satisfying

$$\langle u, h^*_{\mu,\sigma}\phi\rangle = \langle f, \phi\rangle, \quad \phi \in H_\mu. \tag{2.2.3}$$

For any $u \in D(h^1_{\mu,\sigma})$ and $h^1_{\mu,\sigma}u = f$,

$$\langle u, h^*_{\mu,\sigma}\phi\rangle = \langle h^1_{\mu,\sigma}u, \phi\rangle, \quad \phi \in H_\mu.$$

Then $h^1_{\mu,\sigma}$ is called maximal operator of $h_{\mu,\sigma}$.

Theorem 2.2.5. *Let $\sigma \in H^m$. Then for any $1 < p < \infty$, the pseudo-differential operator $h_{\mu,\sigma} : H_\mu(I) \to H_\mu(I)$ is closable in $L^p_\mu(I)$.*

Proof. We can show this theorem with the help of [42, p. 135]. □

Theorem 2.2.6. *Let $h^1_{\mu,\sigma}$ be the maximal pseudo-differential operator and $h^0_{\mu,\sigma}$ be the minimal pseudo-differential operator. Then $h^1_{\mu,\sigma}$ is a closed extension of $h^0_{\mu,\sigma}$, for $1 < p < \infty$.*

Proof. The proof of this theorem is obvious from [42, p. 138]. □

Theorem 2.2.7. *Let $\sigma \in H^m$. Then we have $h^t_{\mu,\sigma}\psi = h^*_{\mu,\sigma}\psi$ for all $\psi \in H_\mu(I)$.*

Proof. See Lemma 3.7 in Pathak et al. [42], for the proof of Theorem 2.2.7. □

Theorem 2.2.8. *Let $\sigma \in H^m$ be a symbol, $\phi \in H_\mu(I)$ and λ be a complex number. Then the pseudo-differential operator associated with the symbol σ can be expressed as in the following form:*

$$(h_{\mu,\sigma} - \lambda)\phi(x) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)(\sigma(\xi) - \lambda)(h_\mu \phi)(\xi) d\xi, \tag{2.2.4}$$

for all $x, \xi \in I = (0, \infty)$.

Proof. Let

$$(h_{\mu,\sigma} - \lambda)\phi(x) = (h_{\mu,\sigma}\phi)(x) - \lambda\phi(x).$$

Then by definition of the pseudo-differential operator (1.3.15) and the inversion formula of the Hankel transform, we can write

$$(h_{\mu,\sigma} - \lambda)\phi(x) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)\sigma(\xi)(h_\mu\phi)(\xi) d\xi - \lambda \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)(h_\mu\phi)(\xi) d\xi.$$

Then, we have

$$(h_{\mu,\sigma} - \lambda)\phi(x) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)(\sigma(\xi) - \lambda)(h_\mu\phi)(\xi) d\xi.$$

□

Theorem 2.2.9. *Let $\sigma \in H^m$ be a symbol and ϕ be in $H_\mu(I)$. Then for $\xi \in I$ and $\mu \geq -1/2$*

$$(h_{\mu,\sigma}\phi)(x) = \int_0^\infty \Big(\int_0^\infty h_\mu(\xi^{\mu+1/2}\sigma(\xi))(z) D_\mu(x, y, z) dz \Big) \phi(y) dy. \tag{2.2.5}$$

Proof. In view of [43], and the definition of the pseudo-differential operator, we have

$$(h_{\mu,\sigma}\phi)(x) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)\sigma(\xi)(h_\mu\phi)(\xi) d\xi.$$

From (1.3.1), we can write

$$(h_{\mu,\sigma}\phi)(x) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)\sigma(\xi)\left(\int_0^\infty (y\xi)^{1/2} J_\mu(y\xi)\phi(y)dy\right) d\xi.$$

Exploiting the Fubini's theorem, we have

$$(h_{\mu,\sigma}\phi)(x) = \int_0^\infty \left(\int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)(y\xi)^{1/2} J_\mu(y\xi)\sigma(\xi)d\xi\right)\phi(y)dy.$$

From [43, p. 145], we get

$$\begin{aligned}(h_{\mu,\sigma}\phi)(x) = \int_0^\infty \Bigg(\int_0^\infty \xi^{\mu+1/2}\Big(\int_0^\infty (z\xi)^{1/2} J_\mu(z\xi)\\ \times D_\mu(x,y,z)dz\Big)\sigma(\xi)d\xi\Bigg)\phi(y)dy. \end{aligned} \tag{2.2.6}$$

Using the Fubini's theorem, we have

$$\begin{aligned}(h_{\mu,\sigma}\phi)(x) = \int_0^\infty \Bigg(\int_0^\infty \left(\int_0^\infty \xi^{\mu+1/2}(z\xi)^{1/2} J_\mu(z\xi)\sigma(\xi)d\xi\right)\\ \times D_\mu(x,y,z)dz\Bigg)\phi(y)dy. \end{aligned} \tag{2.2.7}$$

Therefore, we get

$$(h_{\mu,\sigma}\phi)(x) = \int_0^\infty \left(\int_0^\infty h_\mu(\xi^{\mu+1/2}\sigma(\xi))(z)D_\mu(x,y,z)dz\right)\phi(y)dy.$$

$\square$

Lemma 2.2.10. *For any $v \in L^p_\mu(I)$, $1 \le p < \infty$ we find the following relations.*

$$g_k \# h_{\mu,\sigma} v = h_{\mu,\sigma}(g_k \# v) = h_{\mu,\sigma} g_k \# v, \tag{2.2.8}$$

where g_k is a member of the test function space $D(I)$.

Proof. Using (1.3.4), we have

$$\begin{aligned}(g_k\#h_{\mu,\sigma}v)(x) &= \int_0^\infty g_k(x,y)(h_{\mu,\sigma}v)(y)dy\\ &= \int_0^\infty g_k(x,y)\left(\int_0^\infty (y\eta)^{1/2}J_\mu(y\eta)\sigma(\eta)(h_\mu v)(\eta)d\eta\right)dy\\ &= \int_0^\infty g_k(x,y)(y\eta)^{1/2}J_\mu(y\eta)dy\int_0^\infty \sigma(\eta)(h_\mu v)(\eta)d\eta.\end{aligned}$$

Now using (1.3.5), we can write

$$\begin{aligned}(g_k\#h_{\mu,\sigma}v)(x) &= \int_0^\infty (y\eta)^{1/2}J_\mu(y\eta)\left(\int_0^\infty g_k(z)D_\mu(x,y,z)dz\right)dy\\ &\qquad\times\int_0^\infty \sigma(\eta)(h_\mu v)(\eta)d\eta.\\ &= \int_0^\infty g_k(z)dz\left(\int_0^\infty (y\eta)^{1/2}J_\mu(y\eta)D_\mu(x,y,z)dy\right)\\ &\qquad\times\int_0^\infty \sigma(\eta)(h_\mu v)(\eta)d\eta.\end{aligned}$$

Exploiting (1.3.6), we obtain

$$\begin{aligned}(g_k\#h_{\mu,\sigma}v)(x) &= \int_0^\infty \eta^{-\mu-1/2}(x\eta)^{1/2}J_\mu(x\eta)(z\eta)^{1/2}J_\mu(z\eta)g_k(z)dz\\ &\qquad\times\int_0^\infty \sigma(\eta)(h_\mu v)(\eta)d\eta\\ &= \int_0^\infty (x\eta)^{1/2}J_\mu(x\eta)\sigma(\eta)\eta^{-\mu-1/2}(h_\mu g_k)(\eta)(h_\mu v)(\eta)d\eta.\end{aligned}$$

From (1.3.7), we get

$$\begin{aligned}(g_k\#h_{\mu,\sigma}v)(x) &= \int_0^\infty (x\eta)^{1/2}J_\mu(x\eta)\sigma(\eta)h_\mu(g_k\#v)(\eta)d\eta\\ &= h_{\mu,\sigma}(g_k\#v).\end{aligned}$$

In similar way, using (1.3.4) we have

$$\begin{aligned}
(h_{\mu,\sigma}g_k\#v)(x) &= \int_0^\infty (h_{\mu,\sigma}g_k)(x,y)v(y)dy \\
&= \int_0^\infty \left(\int_0^\infty (h_{\mu,\sigma}g_k)(z)D_\mu(x,y,z)dz\right) v(y)dy \\
&= \int_0^\infty \left(\int_0^\infty \left(\int_0^\infty (z\eta)^{1/2}J_\mu(z\eta)\sigma(\eta)(h_\mu g_k)(\eta)d\eta\right)\right. \\
&\qquad \left. \times D_\mu(x,y,z)dz\right) v(y)dy.
\end{aligned}$$

Using the Fubini's theorem, we have

$$\begin{aligned}
(h_{\mu,\sigma}g_k\#v)(x) &= \int_0^\infty \left(\int_0^\infty \left(\int_0^\infty (z\eta)^{1/2}J_\mu(z\eta)D_\mu(x,y,z)dz\right)\right. \\
&\qquad \left. \times \sigma(\eta)(h_\mu g_k)(\eta)d\eta\right) v(y)dy \\
&= \int_0^\infty \left(\int_0^\infty \eta^{-\mu-1/2}(x\eta)^{1/2}J_\mu(x\eta)(y\eta)^{1/2}J_\mu(y\eta)\right. \\
&\qquad \left. \times \sigma(\eta)(h_\mu g_k)(\eta)d\eta\right) v(y)dy.
\end{aligned}$$

Therefore

$$\begin{aligned}
(h_{\mu,\sigma}g_k\#v)(x) &= \int_0^\infty \eta^{-\mu-1/2}(x\eta)^{1/2}J_\mu(x\eta)\sigma(\eta)(h_\mu g_k)(\eta)d\eta \\
&\qquad \times \int_0^\infty (y\eta)^{1/2}J_\mu(y\eta)v(y)dy \\
&= \int_0^\infty (x\eta)^{1/2}J_\mu(x\eta)\sigma(\eta)\eta^{-\mu-1/2}(h_\mu g_k)(\eta)(h_\mu v)(\eta)d\eta.
\end{aligned}$$

From (1.3.7), we have

$$(h_{\mu,\sigma}g_k\#v)(x) = \int_0^\infty (x\eta)^{1/2}J_\mu(x\eta)\sigma(\eta)\eta^{-\mu-1/2}h_\mu(g_k\#v)(\eta)d\eta$$
$$= h_{\mu,\sigma}(g_k\#v).$$

Similarly, we have the following results

$$g_k\#h_{\mu,\sigma}v = h_{\mu,\sigma}g_k\#v = h_{\mu,\sigma}(g_k\#v). \tag{2.2.9}$$

□

Theorem 2.2.11. *Let $\sigma \in H^m$ be a symbol, then $h^0_{\mu,\sigma} = h^1_{\mu,\sigma}$ for $1 \le p < \infty$.*

Proof. Since $h^1_{\mu,\sigma}$ is a closed extension of $h^0_{\mu,\sigma}$ for $1 \le p < \infty$ [see Theorem 2.2.6], then we have only to show that $D(h^1_{\mu,\sigma}) \subseteq D(h^0_{\mu,\sigma})$, where D is domain.
Let us suppose $u \in D(h^1_{\mu,\sigma})$, then by the definitions of the maximal operator, there is an $f \in L^p_\mu(I)$ and $h^1_{\mu,\sigma}u = f$, then

$$\langle u, h^1_{\mu,\bar{\sigma}}\phi\rangle = \langle f, \phi\rangle, \qquad \phi \in H_\mu(I), \tag{2.2.10}$$

this implies

$$\langle u, h^1_{\mu,\bar{\sigma}}\phi\rangle = \langle h^1_{\mu,\sigma}u, \phi\rangle, \qquad \phi \in H_\mu(I).$$

Let $g_k \in D(I)$ (test function space) and for $k = 1, 2, 3, \ldots$

Define $$\phi_k(y)(x) = g_k(y, x), \qquad y \in I. \tag{2.2.11}$$

Then by (2.2.10)

$$\langle u, (h^1_{\mu,\bar{\sigma}}\phi_k)(y)\rangle = \langle f, \phi_k(y)\rangle, \tag{2.2.12}$$

and using (2.2.11), we have

$$\langle f, \phi_k(y)\rangle = (g_k \# f)(y), \qquad \phi_k \in H_\mu(I). \tag{2.2.13}$$

Let $(h_{\mu,\tilde{\sigma}}\phi_k)(y)$ be the complex conjugate of $(h_{\mu,\bar{\sigma}}\phi_k)(y)$. Then

$$\langle u, (h_{\mu,\bar{\sigma}}\phi_k)(y)\rangle = \langle h^0_{\mu,\sigma}u, \phi_k(y)\rangle = (g_k \# h^0_{\mu,\sigma})(y), \tag{2.2.14}$$

then from (2.2.13) and (2.2.14), we get

$$(g_k \# h^0_{\mu,\sigma}u)(y) = (g_k \# f)(y), \qquad y \in I. \tag{2.2.15}$$

Now applying the Hankel transform, for any $w \in L^p_\mu(I)$, $1 \leq p < \infty$ and using Lemma 2.2.10, we have

$$(g_k \# h^0_{\mu,\sigma}w) = (h^0_{\mu,\sigma}g_k) \# w = h^0_{\mu,\sigma}(g_k \# w). \tag{2.2.16}$$

From (2.2.15) and (2.2.16), we have

$$h^0_{\mu,\sigma}(g_k \# u) = (g_k \# f), \qquad k = 1, 2, 3, \ldots \tag{2.2.17}$$

Hence from (2.2.17) implies that, $g_k \# u \to u$ and $h^0_{\mu,\sigma}(g_k \# u) \to f$ in $L^p_\mu(I)$ as $k \to \infty$. For each $M > 0$, define $u_M(x) = u(x)$ if $x \leq M$ and $u_M(x) = 0$ if $x > M$. Then for each $M > 0$, $u_M \in D(I)$ test function space. Now from the definition of the minimal and maximal operator, we find the following, for each k

$$(g_k \# u_M) \to (g_k \# u) \quad \text{in} \quad L^p_\mu(I) \quad \text{as} \quad M \to \infty$$

and

$$h^0_{\mu,\sigma}(g_k \# u_M) \to h^0_{\mu,\sigma}(g_k \# u) \quad \text{in} \quad L^p_\mu(I) \quad \text{as} \quad M \to \infty,$$

then $u \in D(h^0_{\mu,\sigma})$, which completes the proof. □

2.3 Spectral properties of pseudo-differential operators

This section devoted the results of the spectrum and essential spectrum of the pseudo-differential operator involving the Hankel transform and discussed its properties. To study the spectral properties of the pseudo-differential operator, we define some definitions with the help of Schechter [58].

Definition 2.3.1. The resolvent set $\rho(h_{\mu,\sigma})$ of the pseudo-differential operator $h_{\mu,\sigma} : H_\mu(I) \to H_\mu(I)$, to be the set of all complex numbers λ for which the operator $h_{\mu,\sigma} - \lambda : H_\mu(I) \to H_\mu(I)$ is one to one and onto. The spectrum $\sum(h_{\mu,\sigma})$ of the pseudo-differential operator associated with the Hankel transform is defined to be the complement of $\rho(h_{\mu,\sigma})$ in the set of all complex numbers $\mathbb{C}$.

Definition 2.3.2. The essential spectrum $\sum_e(h_{\mu,\sigma})$ of the operator $h_{\mu,\sigma} : H_\mu(I) \to H_\mu(I)$ is defined to be the set of all complex number λ, if there is a sequence $< x_n >$ of elements of $D(h_{\mu,\sigma})$ such that $||x_n|| = 1$, $||(h_{\mu,\sigma} - \lambda)x_n|| \to 0$ and $< x_n >$ has no convergent subsequence, then $\lambda \in \sum_e(h_{\mu,\sigma})$.

Theorem 2.3.3. *Let σ be a symbol and $\lambda \in \mathbb{C}$ and if σ is bounded away from λ, then $\lambda \in \rho(h_{\mu,\sigma})$.*

Proof. Let $M > 0$, be such that

$$|\sigma(\xi) - \lambda| > M, \qquad \xi \in I = (0, \infty). \tag{2.3.1}$$

Let $\phi \in H_\mu(I)$, then we have

$$h_{\mu,\sigma}\phi = \lambda\phi.$$

This implies that

$$(h_{\mu,\sigma} - \lambda)\phi = 0. \tag{2.3.2}$$

Using Theorem 2.2.8, (2.3.2) becomes

$$\int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)(\sigma(\xi) - \lambda)(h_\mu\phi)(\xi)d\xi = 0.$$

Then

$$h_\mu^{-1}[(\sigma(\xi) - \lambda)(h_\mu\phi)(\xi)](x) = 0.$$

Using the property of the inversion formula of Hankel transform (1.3.2), the above expression becomes

$$(\sigma(\xi) - \lambda)(h_\mu\phi)(\xi) = 0 \;\text{ for }\; x, \xi \in I. \tag{2.3.3}$$

From (2.3.1) and (2.3.3), we yield

$$(h_\mu\phi)(\xi) = 0 \;\text{ as }\; \xi \in I \;\text{ and }\; \sigma(\xi) \neq \lambda.$$

Therefore, using (1.3.1), we get

$$\phi = 0.$$

This proves that $(h_{\mu,\sigma} - \lambda)$ is one to one. To prove that $(h_{\mu,\sigma} - \lambda)$ is onto. We take any function ψ in $H_\mu(I)$ and using Theorem 2.2.8, then there is a function ϕ such that

$$h_\mu^{-1}[(\sigma(\xi)-\lambda)(h_\mu\phi)(\xi)](x)=\psi(x).$$

By applying the Hankal transform, we get

$$(\sigma(\xi)-\lambda)(h_\mu\phi)(\xi)=(h_\mu\psi)(\xi).$$

This implies

$$(h_\mu\phi)(\xi)=\frac{(h_\mu\psi)(\xi)}{(\sigma(\xi)-\lambda)},\quad \xi\in I=(0,\infty), \tag{2.3.4}$$

is in $H_\mu(I)$. By (2.3.4) and the Hankel inversion formula (1.3.2), we get

$$(h_{\mu,\sigma}-\lambda)\phi=\psi$$

Therefore, $(h_{\mu,\sigma}-\lambda)$ is onto. □

Theorem 2.3.4. *Let $\sigma\in H^m$ and $\lambda\in\mathbb{C}$, and if σ is not bounded away from λ. Then we prove that $\lambda\in\sum_e(h_{\mu,\sigma p})$ for $1\le p<\infty$.*

Proof. Let $<\xi_k>$ be a sequence of elements of $I=(0,\infty)$ and $<\epsilon_k>$ be a sequence of positive number, such that $\epsilon_k\to 0$ as $k\to\infty$. We see that $D(I)\subset H_\mu(I)\subset L^p_\mu(I)$ and $D(I)$(test function space) is dense in $L^p_\mu(I)$ for $1\le p<\infty$ and $\mu\ge -1/2$, so $H_\mu(I)$ is dense in $L^p_\mu(I)$.
Now we consider $\psi\in D(I)$, such that

$$\int_0^\infty x^{-\mu-1/2}\psi(x)dx=1,\ \text{for } \mu\ge -1/2\ \text{ and }\ x\in I. \tag{2.3.5}$$

We define

$$\phi_k(x)=A_\mu^{-1}\epsilon_k^{1/p}x^{-\mu-1/2}(x\xi_k)^{1/2}J_\mu(x\xi_k)\psi(\epsilon_k x), \tag{2.3.6}$$

where $|(x\xi_k)^{1/2}J_\mu(x\xi_k)| < A_\mu$.

Let $\sigma \in H^m$ which is not bounded away from $\lambda \in \mathbb{C}$ for $\xi \in I$.

Therefore, we have

$$h_{\mu,\sigma}\phi_k(x) - \lambda\phi_k(x) = \int_0^\infty (x\xi)^{1/2}J_\mu(x\xi)\sigma(\xi)(h_\mu\phi_k)(\xi)d\xi - \lambda\phi_k(x). \tag{2.3.7}$$

By (1.3.1), we have

$$h_{\mu,\sigma}\phi_k(x) - \lambda\phi_k(x) = \int_0^\infty (x\xi)^{1/2}J_\mu(x\xi)\sigma(\xi)\left(\int_0^\infty (y\xi)^{1/2}J_\mu(y\xi)\phi_k(y)dy\right)d\xi$$
$$- \lambda\phi_k(x).$$

By the Fubini's theorem of order of integration, we get

$$h_{\mu,\sigma}\phi_k(x) - \lambda\phi_k(x) = \int_0^\infty \left(\int_0^\infty (x\xi)^{1/2}J_\mu(x\xi)(y\xi)^{1/2}J_\mu(y\xi)\sigma(\xi)d\xi\right)\phi_k(y)dy$$
$$- \lambda\phi_k(x).$$

From [43], the last expression becomes

$$\begin{aligned}h_{\mu,\sigma}\phi_k(x) - \lambda\phi_k(x) &= \int_0^\infty \left(\int_0^\infty \left(\int_0^\infty \xi^{\mu+1/2}(z\xi)^{1/2}J_\mu(z\xi)D_\mu(x,y,z)dz\right)\right. \\ &\qquad \left.\times\, \sigma(\xi)d\xi\right)\phi_k(y)dy - \lambda\phi_k(x). \\ &= \int_0^\infty \left(\int_0^\infty h_\mu(\xi^{\mu+1/2}\sigma(\xi))D_\mu(x,y,z)dz\right)\phi_k(y)dy \\ &\qquad - \lambda\phi_k(x). \\ &= \int_0^\infty h_\mu(\xi^{\mu+1/2}\sigma(\xi))(x,y)\phi_k(y)dy - \lambda\phi_k(x).\end{aligned}$$

By definition of the Hankel Convolution, we find

$$h_{\mu,\sigma}\phi_k(x) - \lambda\phi_k(x) = \left(h_\mu\left(\xi^{\mu+1/2}\sigma(\xi)\right)\#\phi_k\right)(x) - \lambda\phi_k(x).$$

Therefore using L^p_μ-norm, we get

$$\begin{aligned}||h_{\mu,\sigma}\phi_k(x) - \lambda\phi_k(x)||_p &\leq ||(h_\mu(\xi^{\mu+1/2}\sigma(\xi))\#\phi_k)(x)||_p + ||\lambda\phi_k(x)||_p \\ &\leq ||h_\mu(\xi^{\mu+1/2}\sigma(\xi))||_1||\phi_k||_p + |\lambda|||\phi_k(x)||_p \\ &= (||h_\mu(\xi^{\mu+1/2}\sigma(\xi))||_1 + |\lambda|)||\phi_k||_p.\end{aligned}$$

Using (2.3.6), we obtain

$$\begin{aligned}&||h_{\mu,\sigma}\phi_k(x) - \lambda\phi_k(x)||_p \\ &\quad\leq \Big(||h_\mu(\xi^{\mu+1/2}\sigma(\xi))||_1 + |\lambda|\Big)\Bigg(\int_0^\infty (|A_\mu^{-1}\epsilon_k^{1/p}x^{-\mu-1/2} \\ &\quad\quad \times (x\xi_k)^{1/2}J_\mu(x\xi_k)\psi(\epsilon_k x)|^p dx\Bigg)^{1/p} \quad (2.3.8) \\ &\quad\leq \epsilon_k^{\mu+1/2}\Big(||(h_\mu(\xi^{\mu+1/2}\sigma(\xi))||_1 + |\lambda|\Big)\left(\int_0^\infty |x^{-\mu-1/2}\psi(x)|^p dx\right)^{1/p}. \quad (2.3.9)\end{aligned}$$

Using [43, Lemma 2.1], the above integral is convergent. If we take $\epsilon_k \to 0$ as $k \to \infty$ for $\mu + 1/2 > 0$, then we get

$$||h_{\mu,\sigma}\phi_k(x) - \lambda\phi_k(x)||_p \to 0 \quad \text{as} \quad k \to \infty. \tag{2.3.10}$$

Hence the complex number $\lambda \in \sum_e(h_{\mu,\sigma p})$. □

Lemma 2.3.5. *For all $k \in \mathbb{N}_0$ and ϵ_k be a sequence of positive numbers, there exists positive constant A depending on ϵ_k and s. Then we have the following inequalities;*

$$|(\xi^{-1}D_\xi)^s \left(\xi^{-\mu-1/2}(h_\mu\phi_k)(\xi)\right)| \leq A_{\epsilon_k,s}\epsilon_k^{1/p-1-2s}\int_0^a u^{2s}\psi(u)du, \tag{2.3.11}$$

where $\phi_k \in H_\mu(I)$ and $\psi \in D(I)$.

Proof. Let

$$\phi_k(x) = \epsilon_k^{1/p}(x\rho)^{1/2}J_\mu(x\rho)x^{-\mu-1/2}\psi(\epsilon_k x). \tag{2.3.12}$$

Then

$$\begin{aligned}(h_\mu\phi_k)(x) &= \epsilon_k^{1/p}\int_0^\infty (x\xi)^{1/2}J_\mu(x\xi)(x\rho)^{1/2}J_\mu(x\rho)x^{-\mu-1/2}\psi(\epsilon_k x)dx \\ &= \epsilon_k^{1/p}\int_0^\infty (x\xi)^{-\mu}J_\mu(x\xi)(x\xi)^{\mu+1/2}(x\rho)^{1/2}J_\mu(x\rho)x^{-\mu-1/2}\psi(\epsilon_k x)dx.\end{aligned}$$

Therefore, we get

$$\xi^{-\mu-1/2}(h_\mu\phi_k)(x) = \epsilon_k^{1/p}\int_0^\infty (x\xi)^{-\mu}J_\mu(x\xi)(x\rho)^{1/2}J_\mu(x\rho)\psi(\epsilon_k x)dx.$$

$$(\xi^{-1}D_\xi)^s\xi^{-\mu-1/2}(h_\mu\phi_k)(x) = \epsilon_k^{1/p}\int_0^\infty (x\rho)^{1/2}J_\mu(x\rho)(\xi^{-1}D_\xi)^s[(x\xi)^{-\mu}J_\mu(x\xi)]\psi(\epsilon_k x)dx.$$

Using formula [83, p. 129]

$$\begin{aligned}(\xi^{-1}D_\xi)^s\xi^{-\mu-1/2}(h_\mu\phi_k)(x) &= \epsilon_k^{1/p}\int_0^\infty (x\rho)^{1/2}J_\mu(x\rho)\xi^{-s}(-x)^s(x\xi)^{-\mu}J_{\mu+s}(x\xi)\psi(\epsilon_k x)dx \\ &= (-1)^s\epsilon_k^{1/p}\int_0^\infty (x\rho)^{1/2}J_\mu(x\rho)(x\xi)^{-\mu-s}J_{\mu+s}(x\xi)x^{2s}\psi(\epsilon_k x)dx.\end{aligned}$$

Therefore, by substitution we get

$$(\xi^{-1}D_\xi)^s\xi^{-\mu-1/2}(h_\mu\phi_k)(x) = \epsilon_k^{1/p-1-2s}\int_0^\infty (u\rho/\epsilon_k)^{1/2}J_\mu(u\rho/\epsilon_k)(u\xi/\epsilon_k)^{-\mu-s} \times J_{\mu+s}(u\xi/\epsilon_k)u^{2s}\psi(u)du.$$

$$|(\xi^{-1}D_\xi)^s\xi^{-\mu-1/2}(h_\mu\phi_k)(x)| \le A_{\epsilon_k,s}\epsilon_k^{1/p-1-2s}\int_0^\infty u^{2s}\psi(u)du.$$

Since ψ is a function of compact support and $1/p-1-2s>0$, we have

$$|(\xi^{-1}D_\xi)^s\xi^{-\mu-1/2}h_\mu\phi_k(x)| < \infty.$$

□

Theorem 2.3.6. *Let $1 \le p < \infty$ and $\sigma \in H^m$, then $\dfrac{1}{\sigma(\xi)}$ is a multiplier in $L^p_\mu(I), I=(0,\infty)$ for the resolvent set $\rho(h_{\mu,\sigma p})$ of $h_{\mu,\sigma p}$.*

Proof. Without loss of generality, suppose that if $0 \in \rho(h_{\mu,\sigma})$, then there is a constant $M>0$ such that

$$||\phi||_p \le M||(h_{\mu,\sigma}\phi)||_p \quad \phi \in H_\mu(I). \tag{2.3.13}$$

Therefore, from the above equation, we get

$$|\sigma(\xi)| \ge 1/M, \qquad \xi \in I. \tag{2.3.14}$$

Let us consider $\xi \in I$, $\psi \in D(I)$ be such that $||\psi||_p = 1$ and choose sequence ϕ_k from (2.3.12).

Then the pseudo-differential operator with symbol $\sigma \in H^m$ is,

$$(h_{\mu,\sigma}\phi_k)(x) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)\sigma(\xi)(h_\mu\phi_k)(\xi)d\xi.$$

From [43, Lemma 2.1], we have

$$\begin{aligned}(h_{\mu,\sigma}\phi_k)(x) = \int_0^\infty & (x\xi)^{1/2} J_\mu(x\xi)(1+x^2)^{-l}(1-S_{\mu,\xi})^l \\ & \times (\sigma(\xi)(h_\mu\phi_k)(\xi))d\xi. \end{aligned} \tag{2.3.15}$$

Taking bionomial expansion in (2.3.15), we find

$$\begin{aligned}(h_{\mu,\sigma}\phi_k)(x) = \int_0^\infty & (x\xi)^{1/2} J_\mu(x\xi)(1+x^2)^{-l}\sum_{r=0}^{l}\binom{l}{r} \\ & \times (-1)^r S^r_{\mu,\xi}\left(\sigma(\xi)(h_\mu\phi_k)(\xi)\right)d\xi. \end{aligned} \tag{2.3.16}$$

Now using (1.3.10), we get

$$\begin{aligned}(h_{\mu,\sigma}\phi_k)(x) &= \sum_{r=0}^{l}\binom{l}{r}(-1)^r\int_0^\infty (x\xi)^{1/2}J_\mu(x\xi)(1+x^2)^{-l}\Bigg(\sum_{j=0}^{r} b_j\xi^{2j+\mu+1/2} \\ &\qquad \times (\xi^{-1}D_\xi)^{r+j}(\xi^{-\mu-1/2}\sigma(\xi)(h_\mu\phi_k)(\xi))\Bigg)d\xi \qquad (2.3.17) \\ &= \sum_{r=0}^{l}\binom{l}{r}(-1)^r\sum_{j=0}^{r} b_j(1+x^2)^{-l}\int_0^\infty (x\xi)^{1/2}J_\mu(x\xi)\xi^{2j+\mu+1/2} \\ &\qquad \times (\xi^{-1}D_\xi)^{r+j}(\xi^{-\mu-1/2}\sigma(\xi)(h_\mu\phi_k)(\xi))d\xi. \end{aligned}$$

Using the Leibniz formula, we get

$$(h_{\mu,\sigma}\phi_k)(x) = \sum_{r=0}^{l}\binom{l}{r}(-1)^r\sum_{j=0}^{r} b_j(1+x^2)^{-l}\int_0^\infty (x\xi)^{1/2}J_\mu(x\xi)\xi^{2j+\mu+1/2}\sum_{s=0}^{r+j}\binom{r+j}{s}$$
$$\times\left((\xi^{-1}D_\xi)^{r+j-s}\sigma(\xi)\right)(\xi^{-1}D_\xi)^s(\xi^{-\mu-1/2}(h_\mu\phi_k)(\xi))d\xi \tag{2.3.18}$$
$$= \sum_{r=0}^{l}\sum_{j=0}^{r}\sum_{s=0}^{r+j}\binom{r+j}{s}\binom{l}{r}(-1)^r b_j(1+x^2)^{-l}\int_0^\infty (x\xi)^{1/2}J_\mu(x\xi)\xi^{2j+\mu+1/2}$$
$$\times\left((\xi^{-1}D_\xi)^{r+j-s}\sigma(\xi-\eta+\eta)\right)(\xi^{-1}D_\xi)^s(\xi^{-\mu-1/2}(h_\mu\phi_k)(\xi))d\xi.$$

With the help of the Taylor expansion, the above expression can be obtained as

$$(h_{\mu,\sigma}\phi_k)(x) = \sum_{r=0}^{l}\sum_{j=0}^{r}\sum_{s=0}^{r+j}\binom{r+j}{s}\binom{l}{r}(-1)^r b_j(1+x^2)^{-l}\int_0^\infty (x\xi)^{1/2}J_\mu(x\xi)\xi^{2j+\mu+1/2}$$
$$\times(\xi^{-1}D_\xi)^{r+j-s}\sum_{p'=0}^{n}\frac{(\xi-\eta)^{p'}}{p'!}\sigma^{p'}(\eta)(\xi^{-1}D_\xi)^s(\xi^{-\mu-1/2}(h_\mu\phi_k)(\xi))d\xi \tag{2.3.19}$$

$$= \sum_{r=0}^{l}\sum_{j=0}^{r}\sum_{s=0}^{r+j}\sum_{p'=0}^{n}\frac{\sigma^{p'}(\eta)}{p'!}\binom{r+j}{s}\binom{l}{r}(-1)^r b_j(1+x^2)^{-l}\int_0^\infty (x\xi)^{1/2}J_\mu(x\xi)$$
$$\times\xi^{2j+\mu+1/2}(\xi^{-1}D_\xi)^{r+j-s}(\xi-\eta)^{p'}(\xi^{-1}D_\xi)^s(\xi^{-\mu-1/2}(h_\mu\phi_k)(\xi))d\xi \tag{2.3.20}$$

$$= \sum_{r=0}^{l}\sum_{j=0}^{r}\sum_{s=0}^{r+j}\sum_{p'=0}^{n}\sum_{m=0}^{p'}\frac{\sigma^{p'}(\eta)}{p'!}\binom{r+j}{s}\binom{l}{r}(-1)^r b_j(-\eta)^{p'-m}(1+x^2)^{-l}$$
$$\times\int_0^\infty (x\xi)^{1/2}J_\mu(x\xi)\xi^{2j+\mu+1/2}(\xi^{-1}D_\xi)^{r+j-s}\xi^m$$
$$\times(\xi^{-1}D_\xi)^s(\xi^{-\mu-1/2}(h_\mu\phi_k)(\xi))d\xi.$$

Since $\sigma \in H^m$, we get the following approximation

$$|(h_{\mu,\sigma}\phi_k)(x)| \leq A_\mu \left(\sum_{r=0}^{l} \sum_{j=0}^{r} \sum_{s=0}^{r+j} \binom{r+j}{s} \binom{l}{r} (-1)^r b_j \sum_{p'=0}^{n} \frac{\sigma^{p'}(\eta)}{p'!} \sum_{m=0}^{p'} (-\eta)^{p'-m} \right)$$
$$\times |(1+x^2)^{-l}| \sup_{\xi \in I} |(\xi^{-1}D_\xi)^s \left(\xi^{-\mu-1/2}(h_\mu\phi_k)(\xi)\right)|$$
$$\times \int_0^\infty (1+\xi)^{2j+\mu+1/2} D_{r,j,s}(1+\xi)^{m-2r-2j+2s} d\xi,$$

where $D_{r,j,s} > 0$ is a constant.

Using the Lemma 2.3.5, we get

$$|(h_{\mu,\sigma}\phi_k)(x)| \leq A_\mu \left(\sum_{r=0}^{l} \sum_{j=0}^{r} \sum_{s=0}^{r+j} \sum_{p'=0}^{n} \sum_{m=0}^{p'} D_{r,j,s} \binom{r+j}{s} \binom{l}{r} (-1)^r b_j \right) |(1+x^2)^{-l}|$$
$$\times \sup |(\xi^{-1}D_\xi)^s (\xi^{-\mu-1/2}(h_\mu\phi_k)(\xi))|$$
$$\times \frac{\sigma^{p'}(\eta)}{p'!}(-\eta)^{p'-m} \int_0^\infty (1+\xi)^{\mu+1/2+m-2r+2s} d\xi.$$

When we choose $m < 2r - 2s - \mu - 3/2$, then the above expression is convergent

$$|(h_{\mu,\sigma}\phi_k)(x)| \leq A_\mu \left(\sum_{r=0}^{l} \sum_{j=0}^{r} \sum_{s=0}^{r+j} \sum_{p'=0}^{n} \sum_{m=0}^{p'} D_{r,j,s} A'_{\epsilon_k,s,m,r,s} \epsilon_k^{\frac{1}{p}-1-2s} \binom{r+j}{s} \binom{l}{r} b_j \right)$$
$$\times |(1+x^2)^{-l}| \frac{\sigma^{p'}(\eta)}{p'!}(-\eta)^{p'-m}. \tag{2.3.21}$$

Case 1. $p' \neq 0$

$$||(h_{\mu,\sigma}\phi_k)(x)||_p \leq \sum_{r=0}^{l} \sum_{j=0}^{r} \sum_{s=0}^{r+j} \epsilon_k^{1/p-1-2s} A''_{\mu,r,j,s,m,\epsilon_k} ||(1+x^2)^{-l}||_p$$
$$\times \frac{\sigma^{p'}(\eta)}{p'!}(-\eta)^{p'-m},$$

since $\epsilon_k \to 0$ as $k \to \infty$, $p < \frac{1}{1+2s}$,

this implies

$$||(h_{\mu,\sigma}\phi_k)(x)||_p \to 0.$$

Case 2. $p' = 0$, (2.3.21) yields

$$||(h_{\mu,\sigma}\phi_k)(x)||_p \leq \sum_{r=0}^{l}\sum_{j=0}^{r}\sum_{s=0}^{r+j} e^{\epsilon_k^{1/p-1-2s}} A''_{\mu,r,j,s,m,\epsilon_k}||(1+x^2)^{-l}||_p|\sigma(\eta)|$$

as $k \to \infty$, we have $e^{\epsilon_k^{1/p-1-2s}} \to 1$ and $||(1+x^2)^{-l}||_p < \infty$.

This implies,

$$||(h_{\mu,\sigma}\phi_k)(x)||_p \to |\sigma(\eta)|. \tag{2.3.22}$$

Using (2.3.13), we obtain

$$1 = ||\phi_k||_p \leq M||(h_{\mu,\sigma}\phi_k)(x)||_p, \qquad k = 1, 2, 3, \ldots \tag{2.3.23}$$

From (2.3.22) and (2.3.23), we find

$$\frac{1}{\sigma(\eta)} \leq M. \tag{2.3.24}$$

This implies $\frac{1}{\sigma(\eta)}$ is a multiplier in $L^p_\mu(I)$. □

Theorem 2.3.7. *Let $1 \leq p < \infty$ and $\sigma \in H^m$, then $\lambda \in \mathbb{C}$ is in the resolvent set $\rho(h_{\mu,\sigma p})$ of $h_{\mu,\sigma p}$ iff $\dfrac{1}{(\sigma(\xi) - \lambda)}$ is a multiplier in $L^p_\mu(I), I = (0, \infty)$.*

Proof. Using the technique of Theorem 2.3.6 and Schechter [58], and by exploiting the theory of Hankel transform, we can show that $1/(\sigma(\xi) - \lambda)$ for $\lambda \neq 0$ is a multiplier in $L^p_\mu(I)$. $\square$

Theorem 2.3.8. *Let $1 < p < \infty$ and σ be a strongly Carleman symbol of class $S^m_{\rho,0}$, $m > 0$, $0 < \rho < 1$. Suppose that $\sigma \in S^m_{\rho+1,0}$ is such that σ is strongly Carleman is satisfied for some $b > m/4$. Then*

$$\sum(h_{\mu,\sigma}) = \sum_e(h_{\mu,\sigma}) = \{\sigma(\xi) : \xi \in I\} \quad \textit{for} \quad \left|\frac{1}{p} - \frac{1}{2}\right| < \frac{4b-m}{2-\rho}. \tag{2.3.25}$$

Proof. With the help of the (1.3.14), we define a class of symbol

$$S^m_{\rho+1,0} = \{\sigma(\xi) \in C_0^\infty(I) : |(\xi^{-1}D_\xi)^\alpha\sigma(\xi)| \leq C_\alpha(1+\xi)^{m-\rho\alpha-\alpha}\}.$$

This implies that

$$(\xi^{-1}D_\xi)^\alpha\sigma(\xi) = O(\xi^{m-(\rho+1)\alpha}) \ \text{ as } \ \xi \to \infty.$$

Since $\sigma(\xi) \neq \lambda$ for $\xi \in I$ and put $\tau = \sigma - \lambda$, we get

$$|(\xi^{-1}D_\xi)^\alpha\sigma(\xi)| = |(\xi^{-1}D_\xi)^\alpha\tau(\xi)| = O(\xi^{m-(\rho+1)\alpha}),$$

as $\xi \to \infty$.

Therefore, we get

$$\frac{(\xi^{-1}D_\xi)^\alpha\tau(\xi)}{\tau(\xi)} = O(\xi^{m-(\rho+1)\alpha-b}),$$

where $\tau(\xi) = O(\xi^b), \quad \xi > 0$.

Then

$$(\xi^{-1}D_\xi)^\alpha(1/\tau(\xi)) = O(\xi^{m-(\rho+1)\alpha-2b}) \text{ as } \xi \to \infty.$$

Thus

$$1/\tau(\xi) \in S^{m-2b}_{\rho+1,0}.$$

This implies, that

$$1/(\sigma(\xi)-\lambda) \in S^{m-2b}_{\rho+1,0}.$$

Considering the concept of the Fefferman [19], we get

$$1/(\sigma(\xi)-\lambda) \text{ is an } L^p_\mu \text{ multiplier if } \left|\frac{1}{p}-\frac{1}{2}\right| < \frac{4b-m}{2-\rho}, \quad \rho \neq 2.$$

Then by Theorem 2.3.7,

$$\lambda \notin \sum(h_{\mu,\sigma}) \quad \text{for} \quad \left|\frac{1}{p}-\frac{1}{2}\right| < \frac{4b-m}{2-\rho}, \quad \rho \neq 2.$$

Therefore

$$\sum(h_{\mu,\sigma}) \subseteq \{\sigma(\xi) : \xi \in I\} \;\; \text{for} \;\; \left|\frac{1}{p}-\frac{1}{2}\right| < \frac{4b-m}{2-\rho}, \quad \rho \neq 2.$$

By Theorem 2.3.4, we have

$$\{\sigma(\xi) : \xi \in I\} \subseteq \sum_e (h_{\mu,\sigma}) \;\; \text{for } 1 < p < \infty.$$

Since

$$\sum_e (h_{\mu,\sigma}) \subseteq \sum(h_{\mu,\sigma}) \quad \text{for } 1 < p < \infty.$$

It follows from the above equation we say that (2.3.25), is valid for our case

$$i.e \qquad \left|\frac{1}{p}-\frac{1}{2}\right| < \frac{4b-m}{2-\rho}, \qquad \rho \neq 2.$$

$\square$

Theorem 2.3.9. *Let $\sigma \in H^m$ and $\sigma(\xi) \neq \lambda$, where λ is a complex number for all $\xi \in I = (0,\infty)$. If $k \in \mathbb{N}$ and $(\sigma-\lambda)^k = \tau$, then $h_{\mu,\tau} = h^k_{\mu,(\sigma-\lambda)}$ for all $\phi \in H_\mu(I)$.*

Proof. We define the pseudo-differential operator with a given symbol $\sigma \in H^m$

$$(h_{\mu,(\sigma-\lambda)}\phi)(x) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)(\sigma(\xi)-\lambda)(h_\mu\phi)(\xi)d\xi, \quad x, \xi \in I. \tag{2.3.26}$$

This implies that

$$(h_{\mu,(\sigma-\lambda)}\phi)(x) = h_\mu^{-1}[(\sigma(\xi)-\lambda)(h_\mu\phi)](x). \tag{2.3.27}$$

Therefore by property of the Hankel transform, (2.3.27) becomes

$$h_\mu[(h_{\mu,(\sigma-\lambda)}\phi)(x)](\xi) = (\sigma(\xi)-\lambda)(h_\mu\phi)(\xi). \tag{2.3.28}$$

Again, the pseudo-differential operator (2.3.26) is

$$h_{\mu,(\sigma-\lambda)}[(h_{\mu,(\sigma-\lambda)}\phi)](x) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)(\sigma(\xi)-\lambda)h_\mu[h_{\mu,(\sigma-\lambda)}\phi(x)](\xi)d\xi.$$

From (2.3.28), we have

$$\begin{aligned}
(h^2_{\mu,(\sigma-\lambda)}\phi)(x) &= \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)\,(\sigma(\xi)-\lambda)\,[(\sigma(\xi)-\lambda)(h_\mu\phi)(x)](\xi)d\xi \\
&= \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)(\sigma(\xi)-\lambda)^2(h_\mu\phi(x))(\xi)d\xi \\
&= (h_{\mu,(\sigma-\lambda)^2}\phi)(x).
\end{aligned}$$

Now repeating this process, we get

$$\begin{aligned}
(h^k_{\mu,(\sigma-\lambda)}\phi)(x) &= (h_{\mu,(\sigma-\lambda)^k}\phi)(x), \qquad \text{for all } x \in I. \\
h^k_{\mu,(\sigma-\lambda)} &= h_{\mu,(\sigma-\lambda)^k}.
\end{aligned} \tag{2.3.29}$$

□

2.4 Applications

In this section, we give some applications of the essential spectrum of pseudo-differential operators associated with the Bessel operator in the Sobolev space. Also we discuss the solution of the heat equation in form of the pseudo differential operator and find the essential spectrum of the pseudo differential operator associated with the solution of the heat equation.

Theorem 2.4.1. *Let $\sigma \in H^m$ and $\lambda \in \mathbb{C}$, and if σ is not bounded away from λ. Then we prove that $\lambda \in \sum_e(J_{s,p})$ for $1 \leq p < \infty$, where $J_{s,p}$ is a pseudo-differential operator in L^p_μ sobolev space and is defined by $J_{s,p}u = [h_\mu^{-1}(1+\xi^2)^{-s/2}h_\mu]u$, $0 < s < \infty$ where $u \in L^p_\mu(I = (0, \infty))$.*

Proof. $J_{s,p}$ is a pseudo-differential operator define above, then proceeding as in proof of Theorem 2.3.4, we have

$$\begin{aligned} ||J_{s,p}\phi_k(x) - \lambda\phi_k(x))||_p \leq \epsilon_k^{\mu+1/2}(||h_\mu(\xi^{\mu+1/2}(1+\xi^2)^{-s/2})||_1 + |\lambda|) \\ \times \left(\int_0^\infty |x^{-\mu-1/2}\psi(x)|^p dx\right)^{1/p}, \end{aligned} \tag{2.4.1}$$

where ψ and ϕ_k is defined in (2.3.5) and (2.3.6) respectively.

Now we have to find

$$\begin{aligned} ||h_\mu(\xi^{\mu+1/2}(1+\xi^2)^{-s/2})||_1 &= \int_0^\infty |(x\xi)^{1/2}J_\mu(x\xi)\xi^{\mu+1/2}(1+\xi^2)^{-s/2}|d\xi \\ &\leq A_\mu \int_0^\infty \xi^{\mu+1/2}(1+\xi^2)^{-s/2}d\xi. \end{aligned}$$

Replacing $(1+\xi^2)=u$ in the right side of the above equation, we get

$$\begin{aligned}||h_\mu(\xi^{\mu+1/2}\sigma(\xi))||_1 &\leq A_\mu\int_1^\infty (u-1)^{\mu/2+1/4}u^{-s/2}du/2\sqrt{(u-1)}.\\ &\leq A_\mu\int_1^\infty (u-1)^{\mu/2-1/4}u^{-s/2}du.\end{aligned}$$

Again taking $u=1/v$, we get

$$||h_\mu(\xi^{\mu+1/2}\sigma(\xi))||_1 \leq A_\mu\int_0^1 (1-v)^{\mu/2+3/4-1}v^{-\mu/2-3/4+s/2-1}dv.$$

Then by definition of the Beta function, we estimate the above integral

$$||h_\mu(\xi^{\mu+1/2}\sigma(\xi))||_1 \leq \frac{A_\mu}{2}B(\mu/2+3/4,-\mu/2-3/4+s/2) \tag{2.4.2}$$

for $\mu>-3/2$ and $s>\mu+3/2$, and B denotes the Beta function.

For $\mu\geq -1/2$, by (2.3.5), (2.4.1) and (2.4.2), we obtain

$$||J_{s,p}\phi_k(x)-\lambda\phi_k(x))||_p\to 0 \quad \text{as} \quad \epsilon_k\to 0 \quad \text{and} \quad k\to\infty.$$

Therefore, Theorem 2.3.4 yields

$$\lambda\in\sum_e(J_{s,p}).$$

□

Theorem 2.4.2. *consider the heat equation*

$$\frac{\partial u}{\partial t}=\Delta u \tag{2.4.3}$$

where $u\in H_\mu(I=(0,\infty))$ and $u(0,x)=f(x)$ and $\Delta=S_{\mu,x}=\frac{d^2}{dx^2}+\frac{1-4\mu^2}{4x^2}$. Then *the solution of the heat equation can be expressed in form of the pseudo-differential operator*

$$u(x,t) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)(h_\mu f)(\xi) e^{-\xi^2 t} d\xi \tag{2.4.4}$$

for any given time $t > 0$.

Proof. Applying the Hankel transform on (2.4.3), we have

$$\frac{\partial(h_\mu u(\xi,t))}{\partial t} = h_\mu(\Delta u) = h_\mu(S_{\mu,x}u).$$

Using (1.3.9), we get

$$(h_\mu u)(\xi,t) = C_1 e^{-\xi^2 t}. \tag{2.4.5}$$

Putting $t = 0$ in (2.4.5), we obtain

$$(h_\mu u)(\xi,0) = C_1. \tag{2.4.6}$$

Therefore

$$\begin{aligned}(h_\mu u)(\xi,0) &= \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi) u(x,0) dx && (2.4.7)\\ &= \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi) f(x) dx \\ &= (h_\mu f)(\xi). && (2.4.8)\end{aligned}$$

From (2.4.6) and (2.4.7), the expression (2.4.5), becomes

$$(h_\mu u)(\xi,t) = (h_\mu f)(\xi) e^{-\xi^2 t}. \tag{2.4.9}$$

Taking the inverse Hankel transform of (2.4.9), we get (2.4.4).

We denote (2.4.4) as $(h_{\mu,\sigma_t}f)(x)$, which represent the pseudo-differential operator with a symbol $\sigma_t(\xi) = e^{-\xi^2 t} \in H^m$, for any given time $t > 0$. □

Lemma 2.4.3. *Let $\sigma_t(\xi) = e^{-\xi^2 t} \in H^m$, for any given $t > 0$. Then*

$$||h_\mu(\xi^{\mu+1/2}e^{-\xi^2 t})||_1 \leq A_\mu ||(1+x^2)^{-k}||_1 b_k \frac{1}{2} t^{2k-\mu/2-3/4}\Gamma(k+\mu/2+3/4),$$

$k \geq 1$, $x \in I = (0,\infty)$, *and for* $\mu > -1/2$.

Proof. By definition of the Hankel transform, we have

$$\begin{aligned} h_\mu(\xi^{\mu+1/2}\sigma_t(\xi)) &= \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)\xi^{\mu+1/2}\sigma_t(\xi)d\xi \\ &= \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)\xi^{\mu+1/2}e^{-\xi^2 t}d\xi. \end{aligned}$$

Let k be a positive integer greater than 1, then by using (1.3.9) we get

$$\begin{aligned} h_\mu(\xi^{\mu+1/2}\sigma_t(\xi)) &= (1+x^2)^{-k}\int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)(1-S_{\mu,\xi})^k(\xi^{\mu+1/2}e^{-\xi^2 t})d\xi \\ &= (1+x^2)^{-k}\int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)\sum_{r=0}^{k}(-1)^r\binom{k}{r}S^r_{\mu,\xi}(\xi^{\mu+1/2}e^{-\xi^2 t})d\xi \\ &= (1+x^2)^{-k}\sum_{r=0}^{k}(-1)^r\binom{k}{r}\int_0^\infty (x\xi)^{1/2} J_\mu(x\xi)\sum_{j=0}^{r} b_j \xi^{2j+\mu+1/2} \\ &\times (\xi^{-1}D_\xi)^{r+j}(e^{-\xi^2 t})d\xi, \end{aligned}$$

since $(\xi^{-1}D_\xi)^{r+j}e^{-\xi^2 t} = (-2t)^{r+j}e^{-\xi^2 t}$, therefore

$$\begin{aligned} h_\mu(\xi^{\mu+1/2}e^{-\xi^2 t}) = (1+x^2)^{-k}\sum_{r=0}^{k}(-1)^r\binom{k}{r}\sum_{j=0}^{r} b_j(-2)^{r+j}\int_0^\infty (x\xi)^{1/2} J_\mu(x\xi) \\ \times\, \xi^{2j+\mu+1/2}t^{r+j}e^{-\xi^2 t}d\xi. \end{aligned}$$

Therefore,

$$|h_\mu(\xi^{\mu+1/2}e^{-\xi^2 t})| \leq A_\mu|(1+x^2)^{-k}|\sum_{r=0}^{k}\sum_{j=0}^{r} b_j 2^{r+j}t^{r+j}\int_0^\infty \xi^{2j+\mu+1/2}e^{-\xi^2 t}d\xi. \quad (2.4.10)$$

Substitute $u = t^{\frac{2j+\mu+3/2}{2}}\xi^{2j+\mu+3/2}$, we get

$$\begin{aligned}\int_0^\infty \xi^{2j+\mu+1/2}e^{-\xi^2 t}d\xi &= \int_0^\infty (2j+\mu+\frac{3}{2})t^{\frac{2j+\mu+3/2}{2}}e^{-u^{\frac{2}{2j+\mu+3/2}}}du \\ &=(2j+\mu+\frac{3}{2})t^{\frac{2j+\mu+3/2}{2}}\int_0^\infty e^{-u^{\frac{2}{2j+\mu+3/2}}}du.\end{aligned}$$

Replace $y = u^{\frac{2}{2j+\mu+3/2}}$, we obtain

$$\begin{aligned}\int_0^\infty \xi^{2j+\mu+1/2}e^{-\xi^2 t}d\xi &=\frac{1}{2}t^{-j-\mu/2-3/4}\int_0^\infty e^{-y}y^{\frac{2j+\mu+3/2}{2}-1}dy \\ &= \frac{1}{2}t^{-j-\mu/2-3/4}\Gamma(j+\mu/2+3/4),\end{aligned}$$

for $j+\mu/2+3/4 > 0$. Therefore (2.4.10) becomes

$$\begin{aligned}||h_\mu(\xi^{\mu+1/2}e^{-\xi^2 t})||_1 &\leq A_\mu||(1+x^2)^{-k}||_1\sum_{r=0}^{k}\sum_{j=0}^{r} b_j 2^{r+j}t^{2k}\frac{1}{2}t^{-j-\mu/2-3/4}\Gamma(j+\mu/2+3/4) \\ &\leq A_\mu||(1+x^2)^{-k}||_1 b_k\frac{1}{2}t^{2k-\mu/2-3/4}\Gamma(k+\mu/2+3/4).\end{aligned}$$

Thus for $k > 1$, we have

$$||h_\mu(\xi^{\mu+1/2}e^{-\xi^2 t})||_1 \leq B_{\mu,t,k}||(1+x^2)^{-k}||_1 < \infty.$$

$\square$

Theorem 2.4.4. *If $\lambda \in \mathbb{C}$ and $\sigma_t(\xi)$ is a symbol belonging to the class H^m, then $h_{\mu,\sigma_t}u$ is a pseudo-differential operator generated by the heat equation for $u \in H_\mu(I)$ and $\mu > -1/2$. if $\sigma_t(\xi)$ is not bounded away from λ for $\xi \in I$. Then $\lambda \in \sum_e(h_{\mu,\sigma_t})$.*

Proof. In view of Theorem 2.3.4, Theorem 2.4.1, and Lemma 2.4.3 we can obtain the desired result. □

2.5 Conclusions

The spectral theory of a class of pseudo-differential operators was introduced by Schechter [58], Catchpole [10], Weder [70], and Wong [72–74] by exploiting the theory of the Fourier transform. Recently authors found different papers about the class of spectra of pseudo-differential operators which are given as follows:

In 2012, Lorinszi et al. [35] analyzed the spectral analysis of pseudo-differential operators by a combination of functional integration methods and direct analysis. In 2016, Abels and Pfeuffer [1] discussed some spectral invariance results for non-smooth pseudo-differential operators with coefficients in Holder space. In 2018, Pilipovic et al. [53] investigated the spectral properties of a class of global infinite order pseudo-differential operators and obtain the asymptotic behavior of the spectral counting functions of such operators. Rodriguez [56] considered spectral properties of pseudo-differential operators on unit circle T and Verediere [69] determined the spectral theory of pseudo-differential operators of degree 0 and got the application to forced linear waves.

In the present chapter, by exploiting the theory of Hankel transform technique, authors introduced the L^p_μ-spectra of pseudo-differntial operators associated with the Bessel operator and obtained some interesting results about the essential spectrum of pseudo-differential operators. Some application related to the essential spectrum of pseudo-differential operators involving the Hankel transform in the Sobolev space, and in the heat equation are obtained.

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