Chapter 2

Mathematical Preliminaries

Throughout the thesis, the notation used adheres to standard conventions. The symbol $\mathbb{R}$ represents the set of real numbers, while $\mathbb{R}_+$ denotes the set of positive real numbers, i.e., $\mathbb{R}_+ = \{z \in \mathbb{R} \mid z > 0\}$, $\mathbb{R}_{\geq 0}$ set of non-negative real numbers. Similarly, $\mathbb{R}_{>k}$ and $\mathbb{R}_{\geq k}$ represent the sets of real numbers greater than k and greater than or equal to k, respectively, where k is a real constant. The collection of non-negative integers is denoted by $\mathbb{Z}_{\geq 0}$. The notation $\mathbb{R}^n$ denotes the n-dimensional Euclidean field. For any $p \geq 1$, the l_p-norm of a vector $y \in \mathbb{R}^n$ is defined as $\|y\|_p = (\sum_{i=1}^{n} |y_i|^p)^{\frac{1}{p}}$. When p is not explicitly mentioned, the norm is assumed to be the $2-$norm, denoted as $|y|$ which represents $\|y\|_2$. For matrices, we represent the transpose of matrix A as $A^\top$. Moreover, for a given vector $a = [a_1, a_2, \ldots, a_n] \in \mathbb{R}^n$, $\text{diag}(a)$ denotes the diagonal matrix with elements $a_i \in \mathbb{R}$, where $i = 1, 2, ..., n$. To specify the norm of a Lebesgue measurable function $\hat{l} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ in interval $[t_1, t_2)$, we utilize $\|\hat{l}\|_{[t_1,t_2)} = \text{ess sup}_{t \in [t_1,t_2)} \|l(\hat{t})\|$. Then, $\|\hat{l}\|_\infty = \|\hat{l}\|_{[0,\infty)}$. A set of essentially bounded and measurable functions $\hat{l}$ with $\|\hat{l}\|_\infty < +\infty$ is represented as $\mathcal{L}_\infty$. The notation $f \in C^p(X, Y)$ denotes a function which is p times continuously differentiable, where $f : X \to Y$, $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$. Consider a matrix E; we represent its null space and range space by $\mathcal{N}(E)$ and $\mathcal{R}(E)$, respectively. $\perp$ denotes orthogonality. In the context of a given set S, $\min\{S\}$ represents the smallest element within the set, whereas $\sup\{S\}$ denotes the supremum, which is the least upper bound, over the elements of S. An n-dimensional open ball B_δ with radius δ is defined as the set of points in n-dimensional Euclidean space that are at a distance less than δ from a fixed point z_0. Mathematically, it can be represented as $B_\delta = \{z \in \mathbb{R}^n : \|z - z_0\| < \delta\}$, where z_0 is the fixed point. The absolute value function is symbolized as $|\cdot|$. The transpose of a vector is represented as $(\cdot)^\top$, and $\lceil z \rceil$ stands for the ceiling function (smallest integer greater than or equal to z). $\mathrm{sign}(z) = \frac{z}{|z|}$, for $z \neq 0$ and $\mathrm{sign}(0) = 0$. l_∞ denotes the space of bounded functions. $\min(\cdot, \cdot)$ represents the minimum function and yields the minimum value at any given instant of time.

To grasp the insights conveyed in this research, it is imperative for the reader to have a foundational understanding of the language and terminology commonly used in the examination of stability within continuous and discrete-time dynamical systems. The definitions and concepts elucidated in this chapter hold pivotal significance throughout the subsequent sections of this thesis. The majority of the foundational information in this segment has been adjusted and rephrased from the renowned textbook by H. K. Khalil, titled "Nonlinear Systems" [35] and Wasim Haddad, titled "Nonlinear Dynamical Systems and Control: A Lyapunov-Based Approach" [36], with appropriate citations included to acknowledge the source of this material.

2.1 Comparison Functions

We recall some definitions related to comparison functions that will be helpful in providing important definitions of stability.

Definition 2.1 ***(Class $\mathcal{K}$ function)*** *[35]*
A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is class $\mathcal{K}$ function, if it is continuous, strictly increasing and $\alpha(0) = 0$. In addition, if $\lim\limits_{s\to\infty} \alpha(s) = \infty$, then it is $\mathcal{K}_\infty$ function.

Definition 2.2 ***(Class $\mathcal{L}$ function)*** *[35]*
A function $\iota : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be class $\mathcal{L}$ function, if it is continuous, strictly decreasing and $\lim\limits_{s\to\infty} \iota(s) = 0$.

Definition 2.3 ***(Class $\mathcal{KL}$ function)*** *[35]*
A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{KL}$ function if it is a class $\mathcal{K}$ function in its first argument and a class $\mathcal{L}$ function in its second argument.

Lemma 2.4 *[35] Let α_1 and α_2 be class $\mathcal{K}$ functions on $[0, a)$, α_3 and α_4 be a class $\mathcal{K}_\infty$ functions, and β be a class $\mathcal{KL}$ function. Denote the inverse of α_i by α_i^{-1}. Then,*

- *α_1^{-1} is defined on $[0,\, \alpha_1(a))$ and belongs to class $\mathcal{K}$.*

- α_3^{-1} *is defined on* $[0, \infty)$ *and belongs to class* $\mathcal{K}_\infty$.

- $\alpha_1 \circ \alpha_2$ *belongs to class* $\mathcal{K}$.

- $\alpha_3 \circ \alpha_4$ *belongs to class* $\mathcal{K}_\infty$.

Definition 2.5 ***(Generalized-$\mathcal{K}$ function)*** *[37]*
A map $\alpha : [0, a) \to \mathbb{R}_{\geq 0}$ *is a generalized-$\mathcal{K}$ function, if it is continuous, strictly increasing,* $\alpha(0) = 0$ *and satisfies:*

$$\begin{cases} \alpha(s_1) > \alpha(s_2), \textit{if } \alpha(s_1) > 0, s_1 > s_2 \\ \alpha(s_1) = \alpha(s_2), \textit{if } \alpha(s_1) = 0, s_1 > s_2. \end{cases}$$

Definition 2.6 ***(Generalized-$\mathcal{KL}$ function)*** *[37]*
A mapping $\Phi : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ *is a generalized-$\mathcal{KL}$ type ($\mathcal{GKL}$-function) if, for every fixed* $t \geq 0$*, the mapping* $s \mapsto \Phi(s, t)$ *is a generalized $\mathcal{K}$-function, and for each fixed* $s \geq 0$*, the mapping* $t \mapsto \Phi(s, t)$ *is continuous and decreases to zero as* $t \to T$ *for some* $T \leq \infty$*. Moreover, if* T *can be chosen a priori, then* Φ *is called predefined generalized $\mathcal{KL}$ ($\mathcal{PGKL}$) class function.*

2.2 Stability Notions

In the present section, the notion of Predefined/Prescribed-time stability and the related previous results are discussed. Consider the following autonomous dynamical system:

$$\dot{\zeta} = f(t, \zeta), \qquad \zeta(t_0) = \zeta_0, \tag{2.1}$$

where $\zeta \in \mathbb{R}^n$ denotes the system state, $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous on an open neighborhood $D \subseteq \mathbb{R}^n$, and $f(t, 0) = 0$, i.e., $\zeta = 0$ is an equilibrium point of (2.1), t_0 is the initial time, and ζ_0 is the initial state. First, various notions of stability for autonomous systems are reviewed.

Definition 2.7 *The origin of the system* (2.1) *is said to be:*

- ***Lyapunov Stable [35]****, if for every* $\epsilon > 0$*, there exists* $\delta(\epsilon) > 0$ *such that if* $\|\zeta(0)\| < \delta$*, then* $\|\zeta(t)\| < \epsilon$ *forall* $t \geq 0$.

- ***Asymptotic Stable [35]***, *(respectively globally asymptotic stable) if it is stable, and there exists* $c > 0$ *such that for all* $\|\zeta(0)\| < c$ *(respectively, for all* $\zeta(0) \in \mathbb{R}^n$*), the solution of* (2.1) *satisfies* $\lim_{t\to\infty}\zeta(t) = 0$.

- ***Exponentially stable [35]***, *(respectively globally exponentially stable), if there exist* $c, b, \beta > 0$ *such that* $\|\zeta(t)\| \leq be^{-\beta t}\|\zeta(0)\|$*, for all* $\|\zeta(0)\| < c$ *(respectively, for all* $\zeta(0) \in \mathbb{R}^n$*), and* $t \geq 0$.

- ***Finite-time stable (FTS) [38]***, *if it is stable and there exists an open neighborhood* (N) *of the origin such that for all* $\zeta(0) \in N\backslash\{0\}$, $\lim_{t\to T}\zeta(t) = 0$, *where* $T = T(\zeta(0)) < \infty$. *The origin is a globally finite-time stable if* $N = \mathbb{R}^n$.

- ***Fixed-time stable (FxTS) [39]***, *if it is globally FTS and the settling time function is uniformly bounded for all* $\zeta(0) \in \mathbb{R}^n$, *i.e.,* $\sup_{\zeta(0)\in\mathbb{R}^n} T(\zeta(0)) < \infty$.

- ***Predefined-time stable [40]***, *if it is FxTS and there exists a* $T_r > t_0$ *which is chosen a priori for some value of system parameter* $\eta > 1$ *and is independent of the initial conditions, and the condition:* $T_r \geq T_f$ *can be established for some chosen* $\eta > 1$*, where* T_f *denotes the actual convergence time in which the system trajectories converge to the origin.*

Remark 2.8 *In contrast to fixed-time stability, where the upper bound of the settling time is finite but determined implicitly by system parameters, predefined-time stability ensures that the bound* T_r *can be explicitly assigned by design, independently of the initial condition.*

2.3 Predefined-Time Stability

Consider the following nonautonomous differential equation

$$\dot{\zeta}(t) = \begin{cases} \frac{-\eta(e^{\zeta}-1)}{(e^{\zeta}(t_f - t))}, & \text{if } \ t_0 \leq t < t_f \\ 0, & \text{otherwise} \end{cases}$$

The solution to the above differential system is

$$\zeta(t) = \ln(C(t_f - t)^{\eta} + 1)$$

where integration constant $C = \frac{e^{\zeta(t_0)}-1}{(t_f-t_0)^{\eta}}$. Motivation of predefined-stability comes from the above non-autonomous differential equation, which makes both $\zeta(t)$ and $\dot{\zeta}(t) = 0$ tend to zero as $t \to t_f$.

Existence and uniqueness of the solution: [41]

$$\dot{\zeta} = u_1; \quad \zeta \in \mathbb{R} \tag{2.2}$$

$$u_1 = \begin{cases} -\frac{\eta\left(e^{\zeta}-1\right)}{e^{\zeta}\left(t_f-t\right)} & \text{if } t_0 \leq t < t_f \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

Let the initial condition be $\zeta(t_0) = \zeta_0$ and t_f is a prespecified time. Solution to (2.2) is: $\zeta = \ln\left(1 + C(t_f - t)^{\eta}\right)$, where, $C = \frac{e^{\zeta_0}-1}{(t_f-t_0)^{\eta}} \geq 0$, By substituting this solution in (2.3), we have,

$$u_1 = \begin{cases} \frac{-\eta\left(C\left(t_f-t\right)^{\eta-1}\right)}{\left(1+C\left(t_f-t\right)^{\eta}\right)}, & \text{if } t_0 \leq t < t_f \\ 0, & \text{otherwise} \end{cases} \tag{2.4}$$

From (2.4), for $C \geq 1, |u_1| < \eta C, \forall t \geq t_0$ and for $C < 1, |u_1| \leq \eta, \forall t \geq t_0$, Hence, u_1 is continuous in ζ, $\forall$ fixed t, while measurable in t, $\forall$ fixed ζ and bounded by a quantity that is integrable over a finite-time interval. So, for $t < t_f$, all the conditions for the existence of solution are satisfied for (2.2) in Carathéodory sense, therefore, a solution is augmented till t_f. For $t > t_f$, a classical solution exists, and hence, we can say that the solution exists for $[t_0, \infty)$. The above analysis is the same as when solutions are analyzed in the sense of Filippov for differential equations with discontinuous right-hand side [42]. Hence, the existence and uniqueness of the solution are confirmed for the predefined-time dynamics.

Next, we discuss Lyapunov conditions adapted for predefined-time stability, adapted from [40].

Lemma 2.9 *[40] Consider the dynamics* (2.1)*, let $D \subseteq \mathbb{R}^n$ be a domain containing the equilibrium point $\zeta = 0$. Let $\gamma_1(\zeta), \gamma_2(\zeta)$ be continuous positive definite functions on D. Assume that there exists a real-valued continuously differentiable Lyapunov function candidate $V : \mathbb{R}_{\geq 0} \times D \to \mathbb{R}_{\geq 0}$ which satisfies:*

- $\gamma_1(\zeta) \leq V(t, \zeta) \leq \gamma_2(\zeta)$, $\forall \zeta \in D \setminus \{0\}, \forall t$
- $V(t, 0) = 0, \forall t$

- *For* $V \neq 0$, $\dot{V} \leq \begin{cases} \frac{-\eta(e^V-1)}{e^V(t_f-t)}, & \textit{if} \quad t_0 \leq t < t_f \\ 0, & \textit{otherwise.} \end{cases}$

where $\eta > 1$*. Then, the origin of* (2.1) *is predefined-time stable, and* $T_r = t_f - t_0$ *(*t_f *can be chosen a priori).*

An alternate way to achieve the stability of the equilibrium point of a system in a prespecified-time is given in [43].

Lemma 2.10 *[43] Consider the system* (2.1)*. Let* $D \subset \mathbb{R}^n$ *contains* $\zeta = 0$ *and* $\gamma_1(\zeta)$ *and* $\gamma_2(\zeta)$ *be continuous positive definite functions. Assume that there exists a continuously differentiable scalar function* $V : D \to \mathbb{R}_{\geq 0}$*, and* $\eta > 1$ *such that*

- $\gamma_1(\zeta) \leq V(\zeta) \leq \gamma_2(\zeta)$ *,* $\forall \zeta \in D \setminus \{0\}, \forall t \in [t_0, \infty)$,
- $V(0) = 0, \forall t \in [t_0, \infty)$,
- $\frac{d}{dt}V(\zeta) \leq \begin{cases} \frac{-\eta V(\zeta)}{(t_f-t)}, & \textit{if} \;\; t_0 \leq t < t_f \\ 0, & \textit{otherwise.} \end{cases}$*, for* $V \neq 0$,

Then the origin of system (2.1) *is prescribed-time stable and* $T_r = t_f - t_0$ *(*t_f *can be chosen a priori).*

2.4 Small-Disturbance Input to State Stability (SD-ISS)

Consider the following dynamics

$$\dot{\zeta}(t) = g(\zeta(t), d(t)), \tag{2.5}$$

where, $\zeta \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open subset of $\mathbb{R}^n$, and inputs $d(\cdot) : \mathbb{R}_{>0} \to \mathbb{R}^m$, $g : \mathcal{D} \times \mathbb{R}^m \to \mathbb{R}^n$.

Definition 2.11 *[44] The function* $V : \mathcal{D} \to \mathbb{R}$ *is a size function if it is : (a) continuous, (b)* $V(0) = 0$ *and* $V(\zeta) > 0$*,* $\forall \zeta \in \mathcal{D} \setminus \{0\}$*, (c) coercive i.e.* $\|\zeta_k\| \to \infty, V(\zeta_k) \to \infty$*.*

Definition 2.12 *[45] Dynamics (2.5) is SD-ISS, if $\exists$ a size function $V(\cdot)$, a $d_0 \in \mathbb{R}$, a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$, and a class $\mathcal{K}$ function γ such that $\forall$ inputs d, locally bounded by d_0, $x(0) \in \mathcal{D}$, $\zeta(t)$ remains in $\mathcal{D}$ and follows*

$$V(\zeta) \leq \beta(V(\zeta(0), t)) + \gamma(\|d\|_\infty), \ \forall t \geq 0. \tag{2.6}$$

Definition 2.13 *[44] A function $V \in C^1(\mathcal{D}, \mathbb{R})$ is a SD-ISS Lyapunov function for dynamics (2.5) if (a) V is a size function, (b) $\exists$ a $\mathcal{K}$ class functions $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ such that if $\|\sigma\| \leq \alpha_1(V(\zeta))$, then*

$$\dot{V}(\zeta, \sigma) \leq -\alpha_2(V(\zeta)). \tag{2.7}$$

Lemma 2.14 *[45] Dynamics (2.5) is a SD-ISS iff it exhibits a small disturbance ISS Lyapunov function.*

Definition 2.15 *[46] The system (2.5) is PUBST-input to state stable (PUBST-ISS) if $\exists$ a $\mathcal{PGKL}$ class function $\tilde{\beta}$ with $\tilde{\beta}(r,t) = 0$, for $t \geq t_f$, where t_f is the predefined-time convergence, which is chosen in advance and a class $\mathcal{K}$ function $\tilde{\gamma}$ so that $\forall \zeta(t(0))$ and $d \in \mathcal{L}_\infty$, the solution $\zeta(t)$ exists and satisfies: $\|\zeta(t)\| \leq \tilde{\beta}(\|\zeta(t(0))\|, t_f - t) + \tilde{\gamma}(\|d\|_\infty)$, $\forall t \geq 0$.*

Definition 2.16 *[46] A continuous function $V : \mathbb{R}_{\geq 0} \times \mathcal{D} \to \mathbb{R}$ is said as PUBST-ISS Lyapunov function for system (2.5) if it satisfies: (i) $\gamma_1(\|\zeta\|) \leq V(t,\zeta) \leq \gamma_2(\|\zeta\|)$, $\forall \zeta \in \mathbb{R}^n$, where $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$, (ii) $\dot{V}(t,\zeta) \leq -\tilde{\beta}(\|\zeta(t(0))\|, t_f - t) - \gamma_3(\|\zeta\|), \forall \|\zeta(t)\| \geq \gamma_4(\|d\|)$, $\forall t \geq 0$, where $\gamma_3, \gamma_4 \in \mathcal{K}$ and $\tilde{\beta}$ is in $\mathcal{PGKL}$ class function and $\tilde{\beta}(\|\zeta\|, t_f - t) \mapsto \mu_1 \frac{e^V - 1}{e^V (t_f - t)}$, $\gamma_3(\|\zeta\|) \mapsto \mu_2 V^\rho$, where $\mu_1 > 1, \mu_2 > 0$ and $\rho > 0$.*

Lemma 2.17 *[46] Consider the system (2.5). It is a small disturbance PUBST-ISS with input as d if it exhibits a PUBST-ISS Lyapunov function.*

2.5 Various Notions of Convexity

Different notions of convexity, and first and second order conditions for convexity are summarized below:

Definition 2.18 *A continuously differentiable function $g : \mathcal{D} \to \mathbb{R}$, where $\mathcal{D} \subseteq \mathbb{R}^n$ is a convex set, is called*

- ***Convex*** *if for all* $x, y \in \mathcal{D}$, $g(\beta x+(1-\beta)y) \leq \beta g(x)+(1-\beta)g(y)$, *where* $\beta \in (0,1)$.

- ***Strictly Convex*** *if for all* $x, y \in \mathcal{D}$, $g(\beta x+(1-\beta)y) < \beta g(x)+(1-\beta)g(y)$, *where* $\beta \in (0,1)$.

- μ***- Strongly Convex*** *if for all* $x, y \in \mathcal{D}$, *there exists* $\mu > 0$ *such that* $g(y) \geq g(x) + \nabla g(x)^\top (y-x) + \frac{\mu}{2}\|x-y\|^2$.

Lemma 2.19 ***First-Order Conditions:*** *A continuously differentiable function* $g : \mathcal{D} \to \mathbb{R}$, *where* $\mathcal{D} \subseteq \mathbb{R}^n$ *is a convex set, is called*

- ***Convex*** *if and only if for all* $x, y \in \mathcal{D}$, $g(y) \geq g(x) + \nabla g(x)^\top (y-x)$.

- μ***- Strongly Convex*** *if and only if there exists there exists* $\mu > 0$ *such that* $(\nabla g(x) - \nabla g(y))^\top (x-y) \geq \mu \|x-y\|^2$, $\forall x, y \in \mathcal{D}$.

Second-Order Conditions: *A twice continuously differentiable function* $g : \mathcal{D} \to \mathbb{R}$, *where* $\mathcal{D} \subseteq \mathbb{R}^n$ *is a convex set, is called*

- ***Convex*** *if and only if for all* $x \in \mathcal{D}$, $\nabla^2 g(x) \geq 0$.

- ***Strictly Convex*** *if and only if for all* $x \in \mathcal{D}$, $\nabla^2 g(x) > 0$.

- ***Strongly Convex*** *if and only if there exists there exists* $\mu > 0$ *such that* $\nabla^2 g(x) \geq \mu I$, *for all* $x \in \mathcal{D}$.

Polyak- Lojasiewicz (PL) Inequality

Lemma 2.20 *[47] Consider the problem:* $\min_{\zeta \in \mathcal{D}} \mathfrak{f}(\zeta)$, *where* $\mathcal{D} \subset \mathbb{R}^n$ *and* $\mathfrak{f}$ *is continuously differentiable and a global minimizer* ζ^* *exists then following inequality holds*

$$\|\nabla \mathfrak{f}(\zeta)\|^2 \geq \alpha(\mathfrak{f}(\zeta) - \mathfrak{f}(\zeta^*)), \ \forall \zeta \in \mathcal{D}, \tag{2.8}$$

where, $\alpha > 0$.