Chapter 4

Global exponential stability of inertial Cohen-Grossberg neural networks with time-varying delays

4.1 Introduction

Diverging from traditional neural networks with first-order neuronal state variables, Babcock and Westervelt [61] in 1986 introduced electronic neural networks incorporating inductors into the neural circuit, resulting in a system with inertial characteristics. These networks, known as INNs, are characterized by second-order derivative neuronal state variables represented by second-order differential equations. This departure from the conventional first-order and fractional-order state variables brings about a distinctive dynamical framework. Including inertial terms in NNs introduces a heightened level of complexity in their behaviour, leading to chaos and bifurcation in dynamical problems. The study of INNs provides a unique perspective on neural network dynamics, showcasing the impact of inertial components on the system's overall behaviour. Cohen and Grossberg initially introduced Cohen-Grossberg neural networks (CGNNs) in their work [26]. Compared to Hopfield neural networks and Cellular neural networks, CGNNs, as outlined in the same reference [26], offer a broader application scope. They effectively illustrate a wide range of network structures in fields such as population ecology, neurobiology, and beyond. This chapter addresses the challenge of establishing global exponential stability in delayed Cohen-Grossberg inertial neural networks (CGINNs) by introducing a novel Lyapunov function, deviating from the conventional reduced-order approach. The newly formulated Lyapunov function, two distinct control strategies, and an inequality technique are employed to assess the global exponential stability of the considered second-order INNs. The dynamic behaviour of CGINNs in this study diverges from the reduced-order method due to variable substitution. The simplicity of the proposed method's inequalities facilitates a more straightforward achievement of the stability criterion for CGINNs as compared to existing results. Finally, a numerical example is presented to validate the efficacy of the proposed approach.

4.2 Model Description and Preliminaries

The model of CGINNs with variable delays is given by

$$\begin{aligned}\ddot{q}_i(t) =& - u_i\dot{q}_i(t) - a_i(q_i(t))\Bigg[k_i(q_i(t)) - \sum_{j=1}^{n} b_{ij}f_j(q_j(t)) \\ & - \sum_{j=1}^{n} c_{ij}f_j(q_j(t- \ p_{ij}(t)))\Bigg] + w_i(t), \end{aligned} \tag{4.1}$$

where $i \in I = \{1, 2, 3..., n\}$, n corresponds to the number of neurons, $q_i(t)$ denotes the i-th state of the neurons at time t, second order derivative $\ddot{q}_i(t)$ of state variable corresponds to inertial term of the system (4.1), $u_i > 0$ are constants, $a_i(.)$ represent the continuous amplification function and $k_i(.)$ stand for the behaved functions of the i-th neuron with the condition $k_i(0) = 0$, b_{ij}, c_{ij} represent the weight connections from the j-th neurons to the i-th neurons , $f_j(q_j(t))$ represent the activation functions of $j-$th neurons at t with criteria $f_j(0) = 0$, $p_{ij}(t)$ represent the time varying delays from j-th neurons to i-th neurons at time t, $w_i(t)$ are the input controls to attain stability.

This chapter makes the following assumptions for the proposed model (4.1).

Assumption 4.2.1. *For each* $i \in I$, $\exists$ $0 \leq \underline{a}_i \leq \overline{a}_i$ *and* $\overline{k}_i \geq \underline{k}_i \geq 0$ *for every* $q \in \mathbb{R}$, *such that*

$$\underline{a}_i \leq a_i(q(t)) \leq \overline{a}_i, \quad \underline{k}_i \leq \dot{k}_i(q(t)) \leq \overline{k}_i.$$

Assumption 4.2.2. *For every* $i \in I$, *there exist constants* $l_i > 0$ *such that the following inequality holds for any* q, $\overline{q} \in \mathbb{R}$,

$$|f_i(q) - f_i(\overline{q})| \leq l_i|q - \overline{q}|.$$

Assumption 4.2.3. *Let* $i, j \in I$, *then* $0 \leq p_{ij}(t) \leq p$ *and* $\dot{p}_{ij}(t) \leq \overline{p}_{ij} < 1.$

The initial conditions of (4.1) are as follows:

$$q_i(s) = \phi_i(s),\ \dot{q}_i(s) = \pi_i(s),\quad -p \le s \le 0, \tag{4.2}$$

where $i = 1, 2, 3..., n;$ $p = \max_{1\le i,j\le n} \{p_{ij}\}$ and $\phi_i(s),\ \pi_i(s)$ are the bounded and continuous functions.

Definition 4.2.1. *[113] The CGINNs (4.1) are globally exponentially stable if there exist constants $K \ge 1$ and $\lambda > 0$ such that*

$$\left[\sum_{i=1}^{n}(q_i^2(t) + \dot{q}_i^2(t))\right]^{1/2} \le K\|\phi(s)\|e^{-\lambda t},$$

for all $t \ge 0$, and arbitrary initial functions $\phi(s) = (\phi_i(s), \phi_2(s),\ ...,$ $\phi_n(s), \pi_1(s), \pi_2(s),..., \pi_n(s))^T,\quad s \in [-p, 0].$

4.3 Global exponential stability

This section aims to establish sufficient criteria for achieving stability in the proposed CGINN systems (4.1). The analysis of stability will be carried out through the design and implementation of two control schemes.

First and foremost, a suitable feedback control scheme is employed as part of the stability analysis. This feedback control scheme is designed to influence the dynamics of the CGINN systems and contribute to their stability.

$$w_i(t) = -\alpha_i q_i(t) - \beta_i \dot{q}_i(t), \tag{4.3}$$

where the constants α_i and β_i are termed as control gains and $\alpha_i > 0,\ \ \beta_i > 0,\ i \in I.$

Theorem 4.3.1. *If* $0 < \lambda < 0.5$, *followed by Assumptions* (4.2.1)-(4.2.3), *the CGINNs* (4.1) *attain exponential stability condition under the feedback control scheme* (4.3), *and also if the control gains* α_i *and* β_i *satisfy*

$$\alpha_i > \overline{k}_i\overline{a}_i + \lambda\delta_i + \lambda - \underline{k}_i\underline{a}_i + \frac{1}{2}\sum_{j=1}^{n}\overline{a}_i l_j(|b_{ij}| + |c_{ij}|) + \frac{3}{2}\sum_{j=1}^{n}\overline{a}_j l_i\left(\frac{|c_{ji}|}{(1-\overline{p}_{ij})}e^{2\lambda p} + |b_{ji}|\right),$$

$$\beta_i > \frac{1}{2}\overline{k}_i\overline{a}_i + \frac{1}{2}\sum_{j=1}^{n}\overline{a}_i l_j\left(|b_{ij}| + |c_{ij}|\right) - u_i + \frac{1}{2} + \lambda, \qquad \text{where } i \in I.$$

Proof. Let,

$$\begin{aligned} V(t) =& \frac{1}{2}\sum_{i=1}^{n}\left(\delta_i q_i^2(t) + \dot{q}_i^2(t)\right)e^{2\lambda t} + \frac{1}{2}\sum_{i=1}^{n}\left(q_i(t) + \dot{q}_i(t)\right)^2 e^{2\lambda t} \\ &+ \frac{3}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\overline{a}_{ij}|c_{ij}|l_j}{(1-\overline{p}_{ij})}\int_{t-p_{ij}(t)}^{t} q_j^2(s)e^{2\lambda(s+p)}ds, \end{aligned} \tag{4.4}$$

where δ_i is the positive quantity defined later.

Taking derivative along the equation (4.1), we have

$$\begin{aligned} \dot{V}(t) =& \lambda\sum_{i=1}^{n}\left(\delta_i q_i^2(t) + \dot{q}_i^2(t)\right)e^{2\lambda t} + \sum_{i=1}^{n}\left(\delta_i q_i(t) + \ddot{q}_i(t)\right)\dot{q}_i(t)e^{2\lambda t} \\ &+ \lambda\sum_{i=1}^{n}\Big(q_i(t) + \dot{q}_i(t)\Big)^2 e^{2\lambda t} + \sum_{i=1}^{n}(q_i(t) + \dot{q}_i(t))(\dot{q}_i(t) + \ddot{q}_i(t))e^{2\lambda t} \\ &+ \frac{3}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\overline{a}_{ij}|c_{ij}|l_j}{(1-\overline{p}_{ij})}\Big(q_j^2(t)e^{2\lambda(t+p)} - q_j^2(t-p_{ij}(t)) \\ &\times(1-\dot{p}_{ij}(t))e^{2\lambda(t-p_{ij}(t)+p)}\Big). \\ =& \sum_{i=1}^{n}(\lambda\delta_i + \lambda)q_i^2(t)e^{2\lambda t} + \sum_{i=1}^{n}(\delta_i + 2\lambda + 1)q_i(t)\dot{q}_i(t)e^{2\lambda t} + \sum_{i=1}^{n}(2\lambda + 1) \\ &\times\dot{q}_i^2(t)e^{2\lambda t} + \sum_{i=1}^{n}(2\dot{q}_i(t) + q_i(t))\ddot{q}_i(t)e^{2\lambda t} + \frac{3}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\overline{a}_i l_j|c_{ij}|}{(1-\overline{p}_{ij})} \end{aligned}$$

$$\times\left(q_j^2(t)e^{2\lambda(t+p)} - q_j^2(t-p_{ij}(t))(1-\dot{p}_{ij}(t))e^{2\lambda(t-p_{ij}(t)+p)}\right)$$

$$\begin{aligned}=&e^{2\lambda t}\Big[\sum_{i=1}^{n}(\lambda\delta_i+\lambda)q_i^2(t)+\sum_{i=1}^{n}(\delta_i+2\lambda+1)q_i(t)\dot{q}_i(t)+\sum_{i=1}^{n}(2\lambda+1)\dot{q}_i^2(t)\\&+\sum_{i=1}^{n}(2\dot{q}_i(t)+q_i(t))\Big\{-u_i\dot{q}_i(t)-a_i(q_i(t))\big[k_i(q_i(t))-\sum_{j=1}^{n}b_{ij}f_j(q_j(t))\\&-\sum_{j=1}^{n}c_{ij}f_j(q_j(t-p_{ij}(t)))\big]+w_i\Big\}\Big]+\frac{3}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\overline{a}_i l_j|c_{ij}|}{(1-\overline{p}ij)}\Big(q_j^2(t)e^{2\lambda(t+p)}\\&-q_j^2(t-p_{ij}(t))(1-\dot{p}_{ij}(t))e^{2\lambda(t-p_{ij}(t)+p)}\Big).\end{aligned}$$

Following from the Assumption (4.2.1) and using the Mean value theorem, we have

$$\begin{aligned}\dot{V}(t)=&e^{2\lambda t}\Big[\sum_{i=1}^{n}(\lambda\delta_i+\lambda)q_i^2(t)+\sum_{i=1}^{n}(\delta_i+2\lambda+1)q_i(t)\dot{q}_i(t)\\&+\sum_{i=1}^{n}(2\lambda+1)\dot{q}_i^2(t)+\sum_{i=1}^{n}(2\dot{q}_i(t)+q_i(t))\Big\{-u_i\dot{q}_i(t)-a_i(q_i(t))\\&\times\dot{k}_i(\theta q_i(t))q_i(t)+a_i(q_i(t))\sum_{j=1}^{n}b_{ij}f_j(q_j(t))+a_i(q_i(t))\\&\times\sum_{j=1}^{n}c_{ij}f_j(q_j(t-p_{ij}(t)))+w_i\Big\}\Big]+\frac{3}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\overline{a}_i l_j|c_{ij}|}{(1-\overline{p}_{ij})}\Big(q_j^2(t)e^{2\lambda(t+p)}\\&-q_j^2(t-p_{ij}(t))(1-\dot{p}_{ij}(t))e^{2\lambda(t-p_{ij}(t)+p)}\Big),\quad 0<\theta<1\end{aligned}$$

$$\begin{aligned}\text{or, }\dot{V}(t)\le&e^{2\lambda t}\Big\{\sum_{i=1}^{n}(\lambda\delta_i+\lambda)q_i^2(t)+\sum_{i=1}^{n}(\delta_i+2\lambda+1)q_i(t)\dot{q}_i(t)\\&+\sum_{i=1}^{n}(2\lambda+1)\dot{q}_i^2(t)-\sum_{i=1}^{n}u_iq_i(t)\dot{q}_i(t)-\sum_{i=1}^{n}2u_i\dot{q}_i^2(t)\\&-\sum_{i=1}^{n}u_iq_i(t)\dot{q}_i(t)-\sum_{i=1}^{n}2u_i\dot{q}_i^2(t)+\sum_{i=1}^{n}2\overline{k}_i\overline{a}_i|\dot{q}_i(t)||q_i(t)|\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{n} \underline{k}_i \underline{a}_i q_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n}(|q_i(t)| + 2|\dot{q}_i(t)|)|a_i(q_i(t))||b_{ij}||f_j(q_{ij}(t))| \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n}(|q_i(t) + 2|\dot{q}_i(t)|)|a_i(q_i(t))||c_{ij}||f_j(q_j(t - p_{ij}(t)))| \\
&+ \sum_{i=1}^{n}(q_i(t) + 2\dot{q}_i(t))(-\alpha_i q_i(t) - \beta_i \dot{q}_i(t))\Bigg\} \\
&+ \frac{3}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{3\overline{a}_i l_j |c_{ij}|}{2(1-\overline{p}_{ij})}\Bigg(q_j^2(t)e^{2\lambda(t+p)} - q_j^2(t - p_{ij}(t))(1 - \dot{p}_{ij}(t)) \\
&\times e^{2\lambda(t - p_{ij}(t) + p)}\Bigg),
\end{aligned} \tag{4.5}
$$

Using mean inequality and Assumptions (4.2.1) and (4.2.2), we have

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n}(|q_i(t)| + 2|\dot{q}_i(t)|)|a_i(q_i(t))||b_{ij}||f_j(q_j(t))| \\
&\quad\le \sum_{i=1}^{n}\sum_{j=1}^{n}(|q_i(t)| + 2|\dot{q}_i(t)|)\overline{a}_i l_j |b_{ij}||q_j(t)| \\
&\quad\le \sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|l_j\overline{a}_i\dot{q}_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|l_j\overline{a}_i q_j^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|l_j\overline{a}_i|q_i(t)||q_j(t)| \\
&\quad\le \sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|l_j\overline{a}_i\dot{q}_i^2(t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}|l_j\overline{a}_i q_i^2(t) + \frac{3}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{a}_j|b_{ji}|l_i q_i^2(t).
\end{aligned} \tag{4.6}
$$

Similarly, we have

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n}(|q_i(t)| + 2|\dot{q}_i(t)|)|a_i(q_i(t))||c_{ij}||f_j(q_j(t - p_{ij}(t)))| \\
&\le \sum_{i=1}^{n}\sum_{j=1}^{n}\overline{a}_i|c_{ij}|l_j(|q_i(t)| + 2|\dot{q}_i(t)|)|q_j(t - p_{ij}(t))| \\
&\le \sum_{i=1}^{n}\sum_{j=1}^{n}\overline{a}_i|c_{ij}|l_j\dot{q}_i^2(t) + \frac{3}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{a}_i|c_{ij}|l_j(q_j^2(t - p_{ij}(t))) \\
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{a}_i|c_{ij}|l_j q_i^2(t).
\end{aligned} \tag{4.7}
$$

Furthermore,

$$\sum_{i=1}^{n} 2\overline{k}_i\overline{a}_i|\dot{q}_i t||q_i(t)| \leq \sum_{i=1}^{n} \overline{k}_i\overline{a}_i\dot{q}_i^2(t) + \sum_{i=1}^{n} k_i\overline{a}_i q_i^2(t). \tag{4.8}$$

Again, from Assumption (4.2.3), we get

$$\begin{aligned}
&\frac{3}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\overline{a}_i l_j|c_{ij}|}{(1-\overline{p}_{ij})}\Big(q_j^2(t)e^{2\lambda(t+p)} - q_j^2(t-p_{ij}(t))(1-\dot{p}_{ij}(t))e^{2\lambda(t-p_{ij}(t)+p)}\Big)\\
&\leq \frac{3}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\overline{a}_i l_j|c_{ij}|}{(1-\overline{p}_{ij})}q_j^2(t)e^{2\lambda t}e^{2\lambda p} - \frac{3}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\overline{a}_i l_j|c_{ij}|q_j^2(t-p_{ij}(t))e^{2\lambda t}.
\end{aligned} \tag{4.9}$$

Now, using equations (4.6)-(4.9) in inequality (4.5), we have

$$\begin{aligned}
\dot{V}(t) \leq & e^{2\lambda t}\Bigg\{\sum_{i=1}^{n}\left(\delta_i + 2\lambda + 1 - u_i - 2\alpha_i - \beta_i\right) q_i(t)\dot{q}_i(t)\\
&- \sum_{i=1}^{n}\Bigg(\underline{k}_i\underline{a}_i - \overline{k}_i\overline{a}_i - \lambda\delta_i - \lambda + \alpha_i - \frac{1}{2}\sum_{j=1}^{n}\overline{a}_i l_j(|b_{ij}| + |c_{ij}|)\\
&- \frac{3}{2}\sum_{j=1}^{n}(\frac{\overline{a}_j l_i|c_{ji}|e^{2\lambda p}}{(1-\overline{p}_{ij})} + |b_{ji}|)\Bigg)q_i^2(t)\\
&- \sum_{i=1}^{n}(2u_i - 1 - 2\lambda - \overline{k}_i\overline{a}_i + 2\beta_i - \sum_{j=1}^{n}\overline{a}_i l_j(|b_{ij}| + |c_{ij}|))\dot{q}_i^2(t)\Bigg\}.
\end{aligned} \tag{4.10}$$

Choosing

$$\delta_i = 2\alpha_i + \beta_i + u_i - 1 - 2\lambda, \quad i \in I, \tag{4.11}$$

$$\begin{aligned}
\Gamma = \min_{1\leq i\leq n}\Bigg\{&\underline{k}_i\underline{a}_i - \overline{k}_i\overline{a}_i - \lambda\delta_i - \lambda + \alpha_i - \frac{1}{2}\sum_{j=1}^{n}\overline{a}_i l_j(|b_{ij}| + |c_{ij}|)\\
&- \frac{3}{2}\sum_{j=1}^{n}\left(\frac{\overline{a}_j l_i|c_{ji}|e^{2\lambda p}}{(1-\overline{p}_{ij})} + |b_{ji}|\right)\Bigg\},
\end{aligned}$$

$\Omega = \min_{1\leq i\leq n}\left\{2u_i - 1 - 2\lambda - \overline{k}_i\overline{a}_i + 2\beta_i - \sum_{j=1}^{n}\overline{a}_i l_j(|b_{ij}| + |c_{ij}|)\right\}$,

and using the conditions of Theorem 4.3.1, observe that for $\Gamma > 0$ and $\Omega > 0$,

$$\dot{V}(t) \leq e^{2\lambda t}\left\{-\Gamma\sum_{i=1}^{n} q_i^2(t) - \Omega\sum_{i=1}^{n}\dot{q}_i^2(t)\right\} < 0. \tag{4.12}$$

$L(t)$ is decreasing functional, and thus

$$\frac{1}{2}\sum_{i=1}^{n}[\delta_i q_i^2(t) + \dot{q}_i^2(t)]e^{2\lambda t} \leq V(0),\ \forall\ t\geq 0, \tag{4.13}$$

where

$$\begin{aligned} V(0) = \frac{1}{2}\sum_{i=1}^{n}\Bigg[&\Big((\delta_i\phi_i^2(s) + \pi_i^2(s)) + (\phi_i(s) + \pi_i(s))^2\Big) + \frac{3}{2}\sum_{j=1}^{n}\frac{|c_{ij}|\overline{a}_i l_j}{(1-\overline{p_{ij}})} \\ &\times\int_{-p_{ij}}^{0}\phi_i^2(\theta)e^{2\lambda(\theta+p)}d\theta\Bigg]_{s\in[-p,0]}. \end{aligned} \tag{4.14}$$

Considering $\underline{\delta} = min\,\{\delta_i, 1\}_{i\in\mathcal{N}}$, $\overline{\delta} = max\,\{\delta_i, 1\}_{i\in\mathcal{N}}$ and $v(0) \leq \frac{1}{2}(\overline{\delta} + \Delta)\|\phi(s)\|^2$, we obtain

$$\begin{aligned} &\sum_{i=1}^{n}\Bigg[\Big((\phi_i(s) + \pi_i(s))^2\Big) + \frac{3}{2}\sum_{j=1}^{n}\frac{|c_{ij}|\overline{a}_i l_j}{(1-\overline{p_{ij}})}\int_{-p_{ij}}^{0}\phi_i^2(\theta)e^{2\lambda(\theta+p)}d\theta\Bigg]_{s\in[-p,0]} \\ &\quad\leq \Delta\|\phi(s)\|^2. \end{aligned}$$

From equation (4.13), we have

$$\left[\sum_{i=1}^{n}(q_i^2(t)+\dot{q}_i^2(t))\right]^{1/2} \leq \sqrt{\frac{\overline{\delta}+\Delta}{\underline{\delta}}}\,\|\phi(s)\|e^{-\lambda t}, \quad t \geq 0. \tag{4.15}$$

Thus, the proof of Theorem 4.3.1 is completed. □

Remark 4.3.1. *Using the relation*

$$\delta_i = 2\alpha_i + \beta_i + u_i - 1 - 2\lambda$$

and the values of α_i *and* β_i *from Theorem 4.3.1, we have*

$$\delta_i \geq \frac{1}{(1-2\lambda)}\Bigg[\frac{5}{2}\overline{k}_i\overline{a}_i - 2\underline{k}_i\underline{a}_i + \frac{3}{2}\sum_{j=1}^{n}\overline{a}_i l_j((|b_{ij}|+|c_{ij}|)) + 3\sum_{j=1}^{n}\overline{a}_j l_i(\frac{|c_{ji}|e^{2\lambda p}}{(1-\overline{p}_{ij})}+|b_{ji}|)+\lambda-\frac{1}{2}\Bigg].$$

Remark 4.3.2. *The utilization of the Lyapunov function method is a well-known approach for achieving stabilization in neural networks (NNs). Typically, most Lyapunov functions involve the model's state variables under consideration. In this context, equation* (4.1) *is treated as a second-order differential system for the purpose of stabilization investigation. A newly developed Lyapunov function has been introduced, as represented by Equation* (4.3)*. Notably, this functional incorporates both the state variables, denoted as* $q_i(t)$*, and their derivatives, i.e.,* $\dot{q}_i(t)$*, from the addressed model* (4.1)*. This departure from the conventional Lyapunov function marks a distinctive feature of the present approach.*

Remark 4.3.3. *If the amplification functions* $a_i(q_i(t)) = 1$*, and* $k_i(q_i(t)) = k_i q_i(t)$*, where* $k_i > 0$ *for* $i \in I$*, then the addressed CGINN model* (4.1) *becomes Cellular or*

Hopfield type INNs.

$$\ddot{q}_i(t) = -u_i\dot{q}_i(t) - k_i q_i(t) - \sum_{j=1}^{n} b_{ij} f_j(q_j(t)) - \sum_{j=1}^{n} c_{ij} f_j((q_j(t - p_{ij}(t)))) \\ + w_i(t),$$

where $i \in I$. *The above equation is investigated by using a reduced-order technique [114, 115, 116, 117]. However, obtaining suitable control gain in actual practical applications is difficult to achieve stability results due to more complex conditions. However, in this chapter, a more general type of INNs is proposed along with simpler inequality given in Theorem 4.3.1, which can simplify the computational process as compared to the existing results [114, 115, 116, 117].*

Theorem 4.3.2. *If* $0 < \lambda < 0.5$, *followed by the Assumptions* (4.2.1)-(4.2.3), *the considered system* (4.1) *attains the exponential stability by using the adaptive controllers given below.*

$$\begin{aligned} w_i(t) &= -\alpha_i(t) q_i(t) - \beta_i(t)\dot{q}_i(t), \\ \dot{\alpha}_i(t) &= \psi_i(q_i^2(t) + 2q_i(t)\dot{q}_i(t))e^{2\lambda t}, \\ \dot{\beta}_i(t) &= \overline{\psi}_i(2\dot{q}_i^2(t) + q_i(t)\dot{q}_i(t))e^{2\lambda t}, \end{aligned} \tag{4.16}$$

where $\psi > 0$ *and* $\overline{\psi} > 0$ *are constants for each* $i \in I$.

Proof. Consider the Lyapunov functional as

$$V(t) = \frac{1}{2}\sum_{i=1}^{n}\left(\delta_i q_i^2(t) + \dot{q}_i^2(t)\right) e^{2\lambda t} + \frac{1}{2}\sum_{i=1}^{n}\left(q_i(t) + \dot{q}_i(t)\right)^2 e^{2\lambda t}$$

$$+\frac{3}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\overline{a}_i h_j|c_{ij}|}{(1-\overline{p}_{ij})}\int_{t-p_{ij}(t)}^{t}q_j^2(s)e^{2\lambda(s+p)}ds$$
$$+\frac{1}{2}\sum_{i=1}^{n}\frac{1}{\psi}(\alpha_i(t)-\overline{\alpha}_i)^2+\frac{1}{2}\sum_{i=1}^{n}\frac{1}{\psi}(\beta_i(t)-\overline{\beta}_i)^2, \tag{4.17}$$

where $\delta_i > 0$ and $\overline{\alpha}_i$, $\overline{\beta}_i$ are constants which will be described further.

A similar approach of Theorem 4.3.1, gives

$$\begin{aligned}\dot{V}(t) \leq & e^{2\lambda t}\Bigg\{\sum_{i=1}^{n}(\lambda\delta_i+\lambda)q_i^2(t)+\sum_{i=1}^{n}(\delta_i+2\lambda+1-u_i)q_i(t)\dot{q}_i(t)\\ &+\sum_{i=1}^{n}(2\lambda+1-2u_i)q_i^2(t)+\sum_{i=1}^{n}\overline{k}_i\overline{a}_i(q_i^2(t)+\dot{q}_i^2(t))\\ &-\sum_{i=1}^{n}\underline{k}_i\underline{a}_i q_i^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{n}(|q_i(t)|+2|\dot{q}_i(t)|)|a_i(q_i(t))||b_{ij}||f_j(q_j(t))|\\ &+\sum_{i=1}^{n}\sum_{j=1}^{n}(q_i(t)+2\dot{q}_i(t))|a_i(q_i(t))||c_{ij}||f_j(q_j(t-p_j(t)))|\\ &+\sum_{i=1}^{n}(-2\alpha_i(t)q_i(t)\dot{q}_i(t)-2\beta_i(t)\dot{q}_i^2(t)-\beta_i(t)q_i(t)\dot{q}_i(t))\Bigg\}\\ &+\frac{3}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\overline{a}_i l_j|c_{ij}|}{(1-\overline{p}_{ij})}\Big(q_j^2(t)e^{2\lambda(t+p)}-q_j^2(t-p_{ij}(t))(1-\dot{p}_{ij}(t))e^{2\lambda(t-p_{ij}(t)+p)}\Big)\\ &+\sum_{i=1}^{n}(\alpha_i(t)-\overline{\alpha}_i)(q_i^2(t)+2\dot{q}_i(t)q_i(t))e^{2\lambda t}+\sum_{i=1}^{n}(\beta_i(t)-\overline{\beta}_i)(2\dot{q}_i^2(t)\\ &+\dot{q}_i(t)q_i(t))e^{2\lambda t}.\end{aligned} \tag{4.18}$$

Moreover,

$$\Bigg\{\sum_{i=1}^{n}\big(-2\alpha_i(t)q_i(t)\dot{q}_i(t)-2\beta_i(t)\dot{q}_i^2(t)-\alpha_i(t)q_i^2(t)-\beta_i(t)q_i(t)\dot{q}_i(t)\big)$$
$$+\sum_{i=1}^{n}(\alpha_i(t)-\overline{\alpha}_i)(q_i^2(t)+2\dot{q}_i(t)q_i(t))+\sum_{i=1}^{n}(\beta_i(t)-\overline{\beta}_i)(2\dot{q}_i^2(t)+\dot{q}_i(t)q_i(t))\Bigg\}e^{2\lambda t}$$

$$=\Bigg\{\sum_{i=1}^{n}\big(-\overline{\alpha}_i q_i^2(t)-2\overline{\alpha}_i\dot{q}_i(t)q_i(t)-\overline{\beta}_i\dot{q}_i(t)q_i(t)-2\overline{\beta}_i\dot{q}_i^2(t)\big)\Bigg\}e^{2\lambda t}. \tag{4.19}$$

Using equations (4.6)-(4.9), (4.19) in the equation (4.18), we have

$$\begin{aligned}\dot{V}(t) \le & e^{2\lambda t}\Bigg\{\sum_{i=1}^{n}\Bigg(\lambda\delta_i+\lambda+\overline{k}_i\overline{a}_i-\underline{k}_i\underline{a}_i+\frac{1}{2}\sum_{j=1}^{n}\overline{a}_i l_j(|b_{ij}|+|c_{ij}|)\\ &+\frac{3}{2}\sum_{j=1}^{n}\overline{a}_j|b_{ij}|l_i+\frac{3}{2}\sum_{j=1}^{n}\frac{\overline{a}_j|c_{ji}|l_i}{(1-\overline{p}_{ij})}e^{2\lambda t}-\overline{\alpha}_i\Bigg)q_i^2(t)\\ &+\sum_{i=1}^{n}\Bigg(1+2\lambda-u_i+\overline{k}_i\overline{a}_i+\sum_{j=1}^{n}\overline{a}_i h_j(|b_{ij}||c_{ij}|-2\overline{\beta}_i\Bigg)\dot{q}_i^2(t)\\ &+\sum_{i=1}^{n}\big(\delta_i+1+2\lambda-u_i-2\overline{\alpha}_i-\overline{\beta}_i\big)q_i(t)\dot{q}_i(t)\Bigg\}.\end{aligned}$$

Choosing

$$\delta_i=2\overline{\alpha}_i+\overline{\beta}_i+u_i-1-2\lambda,$$

$$\begin{aligned}\overline{\alpha}_i=&\lambda\delta_i+\lambda+\overline{k}_i\overline{a}_i+\frac{1}{2}\sum_{j=1}^{n}\overline{a}_i h_j(|b_{ij}|+|c_{ij}|)\\ &+\frac{3}{2}\sum_{j=1}^{n}\overline{a}_j l_i(|b_{ij}|+\frac{|c_{ji}|}{(1-p_o)}e^{2\lambda p},\end{aligned} \tag{4.20}$$

$$2\overline{\beta}_i=1+2\lambda+\overline{k}_i\overline{a}_i\sum_{j=1}^{n}\overline{a}_i l_j(|b_{ij}|+|c_{ij}|), \tag{4.21}$$

we have

$$\dot{V}(t)\le e^{2\lambda t}\Bigg\{\sum_{i=1}^{n}(-\underline{k}_i\underline{a}_i)q_i^2(t)+\sum_{i=1}^{n}(-u_i)\dot{q}_i^2(t)\Bigg\}<0.$$

Thus,

$$\frac{1}{2}\sum_{i=1}^{n}\left(\delta_i q_i^2(t)+\dot{q}_i^2(t)\right)e^{2\lambda t}\leq L(0),\quad t\geq 0. \tag{4.22}$$

$$\sum_{i=1}^{n}(q_i^2(t)+\dot{q}_i^2(t))^{\frac{1}{2}}\leq \|\phi(s)\|e^{-\lambda t},\quad \text{for}\ \ t\geq 0. \tag{4.23}$$

From the equation (4.17), we have

$$\begin{aligned}V(0)=&\frac{1}{2}\sum_{i=1}^{n}\Bigg[\left((\delta_i\phi_i^2(s)+\pi_i^2(s))+(\phi_i(s)+\pi_i(s))^2\right)+\frac{3}{2}\sum_{j=1}^{n}\frac{|c_{ij}|\overline{a}_i l_j}{(1-\overline{p}_{ij})}\\&\times\int_{-p_{ij}(0)}^{0}\phi_i^2(\theta)e^{2\lambda(\theta+p)}d\theta+\frac{1}{\psi}(\alpha_i(0)-\overline{\alpha}_i)^2+\frac{1}{\psi}(\beta_i(0)-\overline{\beta}_i)^2\Bigg]_{s\in[-p,\ 0]}.\end{aligned} \tag{4.24}$$

Now let $\underline{\delta}=min\left\{\delta_i,1\right\}_{i\in\mathcal{N}}$, $\overline{\delta}=max\left\{\delta_i,1\right\}_{i\in\mathcal{N}}$, then

$$V(0)\leq\frac{1}{2}(\overline{\delta}+\Delta)\|\phi(s)\|^2, \tag{4.25}$$

which satisfies

$$\begin{aligned}&\sum_{i=1}^{n}\Bigg[\left((\phi_i(s)+\pi_i(s))^2\right)+\frac{3}{2}\sum_{j=1}^{n}\frac{|c_{ij}|\overline{a}_i l_j}{(1-\overline{p_{ij}})}\int_{-p_{ij}}^{0}\phi_i^2(\theta)e^{2\lambda(\theta+p)}d\theta\\&+\frac{1}{\psi}(\alpha_i(0)-\overline{\alpha}_i)^2+\frac{1}{\psi}(\beta_i(0)-\overline{\beta}_i)^2\Bigg]_{s\in[-p,0]}\leq\Delta\|\phi(s)\|^2.\end{aligned} \tag{4.26}$$

From the equation (4.22), we have

$$\left[\sum_{i=1}^{n}(q_i^2(t)+\dot{q}_i^2(t))\right]^{1/2}\leq\sqrt{\frac{\overline{\delta}+\Delta}{\underline{\delta}}}\|\ \phi(s)\|e^{-\lambda t},\quad t\geq 0. \tag{4.27}$$

By using Definition 4.2.1, it can be said that the system (4.1) is exponentially stabilized.

The proof of Theorem 4.3.2 is now completed. □

Remark 4.3.4. *Using* $\delta_i = 2\overline{\alpha}_i + \overline{\beta}_i + u_i - 1 - 2\lambda$ *and the values of* $\overline{\alpha}_i$ *and* $\overline{\beta}_i$ *from equations* (4.20) *and* (4.21), *we get*

$$\delta_i = \frac{1}{(1-2\lambda)}\Bigg[\frac{5}{2}\overline{k}_i\overline{a}_i + \frac{3}{2}\sum_{j=1}^{n}\overline{a}_i l_j(|b_{ij}| + |c_{ij}|) + 3\sum_{j=1}^{n}\overline{a}_j l_i(|b_{ij}| + \frac{|c_{ji}|}{(1-\overline{p}_{ij})}e^{2\lambda p}) + 3\lambda + \frac{1}{2}\Bigg] \geq 0.$$

Remark 4.3.5. *Different researchers have investigated the stability analysis of CGINNs with time delay by using a common approach viz., variable substitution to reduce the second-order CGINNs into first-order differential equations [118, 119, 120, 121, 122, 123], and then relevant stability criteria have been established by using the Lyapunov approach. The reduced-order approach undoubtedly doubles the system dimensions, and as a result, the complexity of the theoretical result obtained is higher. Unlike these works, in this chapter, the exponential stability criteria of CGINNs by newly developed Lyapunov functional can be done without reducing its order and providing simpler inequality. Hence, the results in this chapter are greatly reasonable.*

Remark 4.3.6. *In the article [121], the exponential stability of CGINNs is investigated using the reduction-order method, applying the LMI approach and removing the conditions of the assumptions of differentiability and monotonicity of the behaved functions. However, in this chapter, the exponential stability of CGINNs is investigated using a non-reduced order approach using a newly developed Lyapunov functional with a control scheme to get rid of the LMI approach, and it also reduces the complexity of the theoretical work.*
Stability analysis for CGINNs by using a non-reduced order technique together with suitable control schemes to get the result in a less conservative way by removing the

differentiability and monotonicity of behaved functions may be the future direction of this research work.

Remark 4.3.7. *Researchers have already investigated the exponential stability of CGINNs [118, 119, 120, 121, 122, 123]. Till now, there is no result with adaptive stabilization of CGINNs by non-reduction order approach. To fill this research gap, the adaptive control scheme* (4.16) *is successfully designed in the present chapter to investigate the exponential stability of CGINNs, which is ensured in Theorem 4.3.2.*

Remark 4.3.8. *The sign of the interconnection weight may be positive or negative, and thus, the conditions of the main results given in Theorems 4.3.1 and 4.3.2 are less conservative.*

4.4 Numerical Example

This section includes a numerical example aimed at confirming the effectiveness and reliability of the results obtained in Section 4.3.

Example 4.4.1. *Let us consider the CGINNs with time-varying delays as*

$$\begin{aligned}\ddot{q}_i(t) =& -u_i\dot{q}_i(t) - a_i(q_i(t))\Big[k_i(q_i(t)) - \sum_{j=1}^{2} b_{ij} f_j(q_j(t)) - \sum_{j=1}^{2} c_{ij} \\ & \times f_j(q_j(t-p_{ij}(t)))\Big] + w_i(t), \quad i = 1,2, \end{aligned} \tag{4.28}$$

having values of the parameters as $\lambda = 0.2, \quad u_1 = 1.5 \ , u_2 = 2.0,$

$a_1(q) = 1.1 - \frac{0.1}{q^2+1}, \quad a_2(q) = 1.0 - \frac{0.1}{q^2+1},$

$p_{11}(t) = p_{12}(t) = p_{21}(t) = p_{22}(t) = \frac{e^t}{1+e^t}$, $f_j(q_j(t)) = tan(q_j(t))$ *for* $j = 1,2$

$k_1(q) = k_2(q) = (0.9 + \frac{0.1}{q^2+1})q,$

$$B = [b_{ij}]_{2\times 2} = \begin{bmatrix} 2 & -0.1 \\ -6.5 & 2.4 \end{bmatrix}, \quad C = [c_{ij}]_{2\times 2} = \begin{bmatrix} -1.4 & -0.1 \\ -0.5 & -1.1 \end{bmatrix}.$$

The trajectories of the considered system (4.28) *in absence of controllers are given in Figure 4.1 with initial values given as*

$q_1(0) = -0.2, \quad q_2(0) = 0.5, \quad \dot{q}_1(0) = 0.4, \quad \dot{q}_2(0) = -0.4.$

By simple calculation, obtain $\underline{a}_1 = 1.0, \quad \overline{a}_1 = 1.1, \quad \underline{a}_2 = 0.9, \quad \overline{a}_2 = 1$ *and* $\underline{k}_i = 0.887, \quad \overline{k}_i = 1$ *for* $i = 1, 2.$

Therefore the Lipschitz constants are $l_1 = l_2 = 1, \quad \overline{p}_{ij} = \frac{1}{4}$, $p = 1.$

Thus the Assumptions 4.2.1-4.2.3 are satisfied.

Choosing $\delta_1 = 73, \delta_2 = 30, \alpha_1 = 40, \alpha_2 = 20, \beta_1 = 2, \beta_2 = 5,$

the feedback controllers are obtained as

$$w_1 = -40q_1(t) - 2\dot{q}_1(t), \quad w_2 = -20q_2(t) - 5\dot{q}_2(t). \tag{4.29}$$

By virtue of Theorem 4.3.1, the system (4.28) *is exponentially stable under a feedback control scheme. The state trajectories of* (4.28) *are depicted through Figure 4.2. Furthermore, choosing* $\psi_1 = 0.7, \quad \psi_2 = 0.3$ *and* $\overline{\psi}_1 = 0.5, \overline{\psi}_2 = 0.3,$ $\alpha_1(0) = \alpha_2(0) = \beta_1(0) = \beta_2(0) = 0.2,$ *by the virtue of Theorem 4.3.2, the stability of system* (4.28) *is obtained with the adaptive control scheme. The state trajectories of the mathematical model given in* (4.28) *are depicted in Figure 4.3. The time responses of control gains* $\alpha_i(t)$ *and* $\beta_i(t)$ *for* $i = 1, 2$ *are depicted through Figure 4.4.*

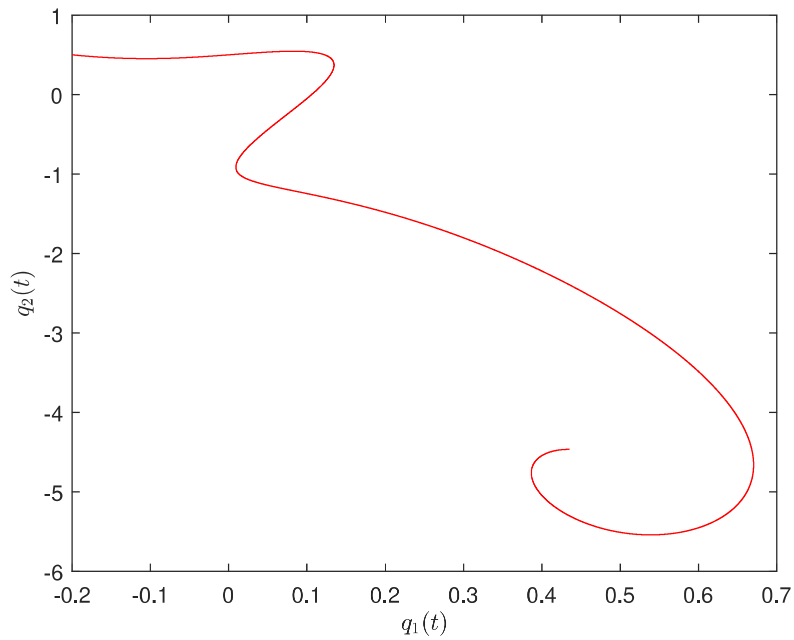

FIGURE 4.1: Phase trajectory in the absence of any control schemes of the system (4.28)

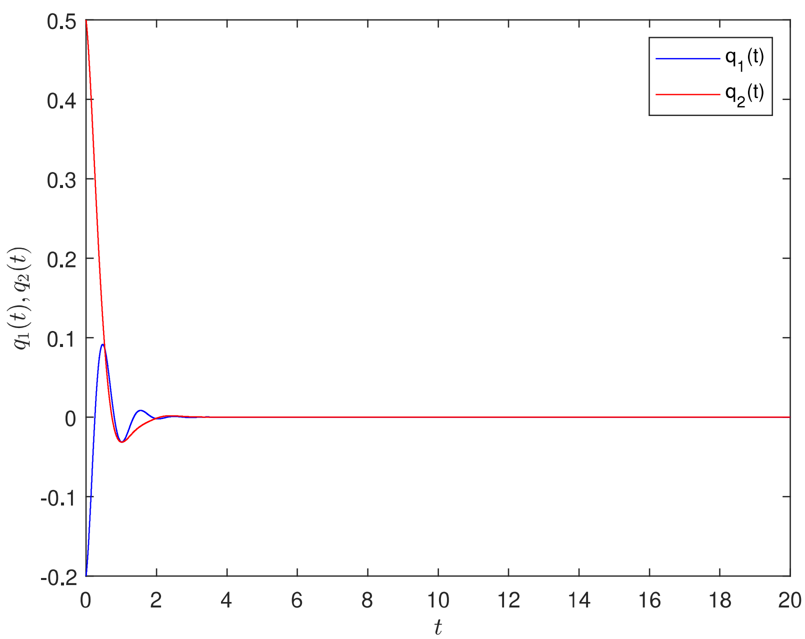

FIGURE 4.2: Stabilization of considered model (4.28) by feedback controller

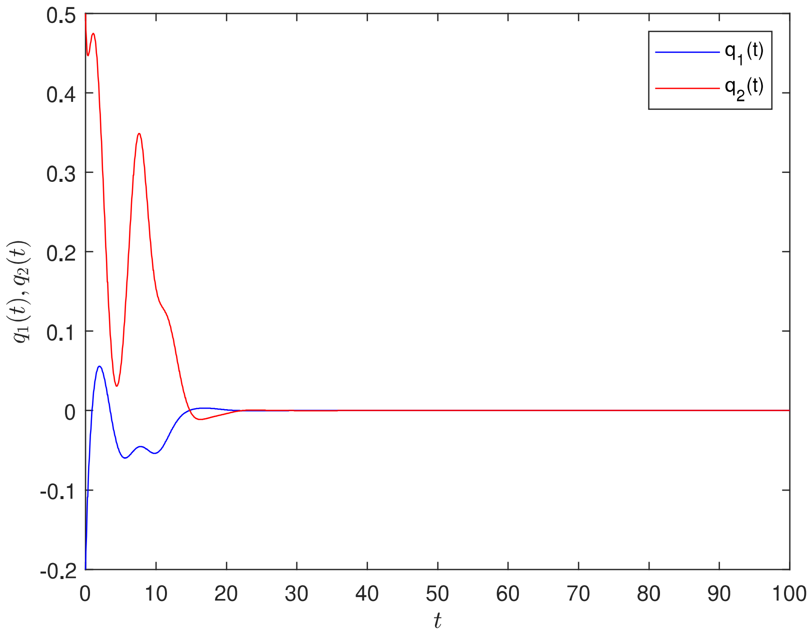

FIGURE 4.3: Stabilization of considered model (4.28) by adaptive controller

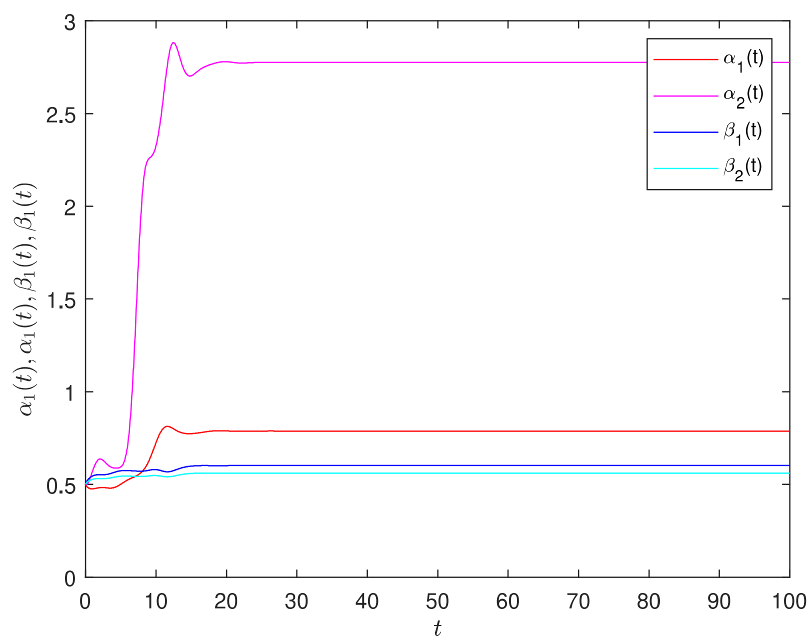

FIGURE 4.4: Evolution of adaptive control gains

4.5 Conclusion

The present chapter investigates the global exponential stability of second-order CGINNs with time-varying delays by designing novel control schemes and a newly developed Lyapunov functional. The present work is followed by a non-reduction order approach for directly solving the stability problem of CGINNs. The non-reduction order approach adopted in this chapter can simplify the analysis process and provide some simpler inequality conditions with multiple parameters to ensure the stability of the considered model. The present approach is different from the traditional reduced-order approach [118, 119, 120, 121, 122, 123]. The efficiency and effectiveness of the proposed theoretical work are validated through a numerical example.

****