Chapter 2

Ćirić quasi-contraction operators with common fixed point results

This chapter focuses on presenting some common fixed point results for a pair of Ćirić-type quasi-contraction operators in the complete metric space. Here we establish the existence and uniqueness of the common fixed point results to apply them in stability analysis. As an application, we will also find under which conditions the common fixed point problem satisfies well-posedness, Ulam–Hyers stability, and Ostrowski's property.

2.1 Introduction

Definition 2.1. Let (X, d) be a metric space. Then, $T : X \to X$ is said to be a Picard operator on X if:

(i) $Fix(T) = \{x^*\}$;

(ii) $\{T^n(x)\} \to x^*$ as $n \to \infty$ for all $x \in X$.

For example, any Banach contraction on a complete metric space is a Picard operator.

Definition 2.2. Let (X, d) be a metric space. Then, $T : X \to X$ is said to be a weakly Picard operator if, for all $x \in X$, the sequence $\{T^n(x)\}$ converges and the limit (which may depend on x) is a fixed point of T.

Definition 2.3. $T : X \to X$ is said to be a ψ-weakly Picard operator on the metric space (X, d) if:

(i) T is a weakly Picard operator;

(ii) $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing and continuous at 0 with $\psi(0) = 0$;

(iii) $d(x, T^\infty(x)) \leq \psi(d(x, T(x)))$, for all $x \in X$, where $T^\infty(x) = \lim\limits_{n\to\infty} T^n(x)$.

In particular, if T is a Picard operator and $x^* \in X$ denotes its unique fixed point, then T is said to be ψ-Picard operator if

$$d(x, x^*) \leq \psi(d(x, T(x))) \quad \text{for all} \quad x \in X.$$

In both cases, if $\psi(t) = ct$, then T is called a c-weakly Picard operator, respectively c-Picard operator. For example, any (Banach) k-contraction T on a complete metric space (X, d) is $\frac{1}{1-k}$-Picard operator, while any graphic k-contraction T is a $\frac{1}{1-k}$-weakly Picard operator, see [74].

Definition 2.4. [74] Let $T : X \to X$ be an operator on the metric space (X, d). Then, T is called graphic k-contraction if there exists $k \in (0, 1)$ such that

$$d(T(x), T^2(x)) \leq kd(x, T(x)) \quad \text{for all} \quad x \in X.$$

Definition 2.5. [84] Let $T : X \to X$ be a weakly Picard operator on the metric space (X, d) such that $Fix(T) = \{x^*\}$. Then, T is said to be β-quasi-contraction if there exists $\beta \in (0,1)$ such that

$$d(T(x), x^*) \leq \beta d(x, x^*) \text{ for all } x \in X.$$

Theorem 2.6. [74] *Let* (X, d) *be a complete metric space and the closed graph of* $T : X \to X$ *be a graphic* k*-contraction. Then the following conclusions hold:*

(i) the Picard sequence $\{T^n(x)\}$ *converges to a fixed point* x^* *of* T;

(ii) T *is* $\frac{1}{1-k}$*-weakly Picard operator;*

(iii) $Fix(T) = \{x : T(x) = x\}$ *satisfy the well-posed condition of Reich and Zaslavski, i.e.,* $Fix(T) = \{x^*\}$ *and for any sequence* $\{v_n\} \subseteq X$ *with* $d(v_n, T(v_n)) \to 0$ *as* $n \to \infty$*, implies that* $v_n \to x^*$ *as* $n \to \infty$*;*

(iv) $Fix(T) = \{x : T(x) = x\}$ *has Ulam-Hyers stablity, i.e., there exists* $c > 0$ *such that for all* $\epsilon > 0$ *and for all* $\tilde{x} \in X$ *satisfying* $d(\tilde{x}, T(\tilde{x})) \leq \epsilon$*, implies that* $d(\tilde{x}, x^*) \leq c\epsilon$*.*

Theorem 2.7. [74] *Let* $T : X \to X$ *be a* β*-quasi-contraction on the metric space* (X, d) *such that* $Fix(T) = \{x^*\}$*. Then,* T *has the Ostrowski stability property, i.e.,* $Fix(T) = \{x^*\}$ *and for any* $\{w_n\} \subset X$ *with* $d(w_{n+1}, T(w_n)) \to 0$ *implies that* $w_n \to x^*$ *as* $n \to \infty$*.*

2.1.1 Delineation

The current chapter is structured as follows: Section 2.2 focuses on establishing the existence, uniqueness, and approximation results for the common fixed point of Ćirić quasi-contraction type operators. In the same framework, we obtain sufficient conditions assuring that the common fixed point problem is well-posed and has the

Ulam–Hyers stability, as well as the Ostrowski property for the considered problem. In Section 2.3, we discuss an application to a hierarchical system of nonlinear variational inequality problems.

2.2 Common fixed point results

Theorem 2.8. *Let $S, T : X \to X$ be two operators on a complete metric space (X, d) and suppose that there exists $k \in (0, \frac{1}{3})$ and for all $x, y \in X$, S and T satisfy the condition:*

$$d(S(x), T(y)) \leq k \max\Big\{d(x,y), d(x,S(x)), d(x,T(y)), d(y,S(x)), d(y,T(y))\Big\}. \tag{2.1}$$

Then the following conclusions hold:

(i) $ComFix(S,T) = Fix(S) = Fix(T) = \{x^*\}$;

(ii) for any $x_0 \in X$, the sequence $\{x_n\}$ given by
$x_{2n+1} = S(x_{2n})$, $x_{2n+2} = T(x_{2n+1})$ for all $n \in \mathbb{N}$, converges to x^* as $n \to \infty$;

(iii) for any $y_0 \in X$, the sequence $\{y_n\}$ given by
$y_{2n+1} = T(y_{2n})$, $y_{2n+2} = S(y_{2n+1})$ for all $n \in \mathbb{N}$, converges to x^* as $n \to \infty$.

Proof. (i) Let us consider $x^* \in Fix(S)$ therefore, $S(x^*) = x^*$. Thereby, from (2.1) we have

$$\begin{aligned} d(x^*, T(x^*)) &= d(S(x^*), T(x^*)) \\ &\leq k \max\{d(x^*,x^*), d(x^*,S(x^*)), d(x^*,T(x^*)), d(x^*,S(x^*)), d(x^*,T(x^*))\} \end{aligned}$$

$$= k\max\{0, d(x^*, T(x^*))\}.$$

As such $d(x^*, T(x^*)) \leq kd(x^*, T(x^*))$. If $d(x^*, T(x^*)) \neq 0$, then $k \geq 1$, which is a contradiction. Therefore, $d(x^*, T(x^*)) = 0$, which implies $x^* = T(x^*)$, and so $x^* \in Fix(T)$, that is $Fix(S) \subseteq Fix(T)$.
Similarly, it can be shown that $Fix(T) \subseteq Fix(S)$. Therefore, $Fix(S) = Fix(T)$.
Now we prove that S and T have at most one common fixed point. If possible let $x^*, y^* \in Fix(S) \cap Fix(T)$ then by (2.1) we have $d(x^*, y^*) \leq kd(x^*, y^*)$. If $d(x^*, y^*) \neq 0$ then $k \geq 1$, which is a contradiction. Therefore, $d(x^*, y^*) = 0$, which implies $x^* = y^*$. Hence, we conclude that $ComFix(S, T) = Fix(S) = Fix(T) = \{x^*\}$.
(ii) For an arbitrary $x_0 \in X$, we consider the sequence $\{x_n\}$ given by $x_{2n+1} = S(x_{2n})$, $x_{2n+2} = T(x_{2n+1})$ for all $n \in \mathbb{N}$.
Now,

$$\begin{aligned}
d(x_1, x_2) = d(S(x_0), T(x_1)) &\leq k\max\{d(x_0, x_1), d(x_0, S(x_0)), d(x_0, T(x_1)), d(x_1, S(x_0)), d(x_1, T(x_1))\} \\
&= k\max\{d(x_0, x_1), d(x_0, x_2), d(x_1, x_2)\}, \\
&\qquad \text{(here if } \max\{d(x_0, x_1), d(x_0, x_2), d(x_1, x_2)\} = d(x_1, x_2) \\
&\qquad \text{then } k \geq 1\text{, which is a contradiction)} \\
&\leq k\max\{d(x_0, x_1), d(x_0, x_2)\} \\
&\leq k\{d(x_0, x_1) + d(x_0, x_2)\} \\
&\leq k\{d(x_0, x_1) + d(x_0, x_1) + d(x_1, x_2)\}.
\end{aligned}$$

Therefore, $d(x_1, x_2) \leq \frac{2k}{1-k} d(x_0, x_1)$. Now, using the principle of mathematical induction, we get

$$d(x_n, x_{n+1}) \leq (\frac{2k}{1-k})^n d(x_0, x_1). \tag{2.2}$$

Since $k \in (0, \frac{1}{3})$, therefore $\frac{2k}{1-k} < 1$. Now, taking the limit as $n \to \infty$ on both sides of (2.2) we get

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

Let us consider $m > n$.

Now,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \{(\frac{2k}{1-k})^n + (\frac{2k}{1-k})^{n+1} + \cdots + (\frac{2k}{1-k})^{m-1}\} d(x_0, x_1) \\ &= (\frac{2k}{1-k})^n \{1 + \frac{2k}{1-k} + \cdots + (\frac{2k}{1-k})^{m-n-1}\} d(x_0, x_1) \\ &= (\frac{1-k}{1-3k}) \{(\frac{2k}{1-k})^n - (\frac{2k}{1-k})^m\} d(x_0, x_1). \end{aligned}$$

The right hand side goes to 0 as $m, n \to \infty$, therefore, $\lim_{n,m\to\infty} d(x_n, x_m) = 0$, which implies $\{x_n\}$ is a Cauchy sequence in X. Since X is complete therefore, $\{x_n\}$ is convergent and let it converge to x^*.

Now,

$$\begin{aligned} d(x^*, S(x^*)) &\leq d(x^*, x_{2n+2}) + d(x_{2n+2}, S(x^*)) \\ &\leq d(x^*, x_{2n+2}) + d(T(x_{2n+1}), S(x^*)) \\ &\leq d(x^*, x_{2n+2}) + k \max\Big\{d(x_{2n+1}, x^*), d(x_{2n+1}, T(x_{2n+1})), d(x_{2n+1}, S(x^*)), \\ &\qquad d(x^*, T(x_{2n+1})), d(x^*, S(x^*))\Big\} \\ &\leq d(x^*, x_{2n+2}) + k \max\Big\{d(x_{2n+1}, x^*), d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, S(x^*)), \\ &\qquad d(x^*, x_{2n+2}), d(x^*, S(x^*))\Big\}. \end{aligned}$$

Now, taking the limit as $n \to \infty$ on both sides we get, $d(x^*, S(x^*)) \leq kd(x^*, S(x^*))$. If $d(x^*, S(x^*)) \neq 0$, then $k \geq 1$, which is a contradiction. Therefore, we have $d(x^*, S(x^*)) = 0$, which implies $x^* = S(x^*)$.

Again,

$$\begin{aligned} d(x^*, T(x^*)) &\leq d(x^*, y_{2n+1}) + d(y_{2n+1}, T(x^*)) \\ &= d(x^*, y_{2n+1}) + d(S(y_{2n}), T(x^*)) \\ &\leq d(x^*, y_{2n+1}) + k \max\Big\{d(y_{2n}, x^*), d(y_{2n}, S(y_{2n})), d(y_{2n}, T(x^*)), \end{aligned}$$

$$d(x^*, S(y_{2n})), d(x^*, T(x^*))\Big\}$$

$$\leq d(x^*, y_{2n+1}) + k \max\Big\{d(y_{2n}, x^*), d(y_{2n}, y_{2n+1}), d(y_{2n}, T(x^*)), d(x^*, y_{2n+1}), d(x^*, T(x^*))\Big\}.$$

Now taking the limit as $n \to \infty$ on both sides we get, $d(x^*, T(x^*)) \leq kd(x^*, T(x^*))$. If $d(x^*, T(x^*)) \neq 0$, then $k \geq 1$, which is a contradiction. Therefore, we have $d(x^*, T(x^*)) = 0$, which implies $x^* = T(x^*)$.

Hence, $x_{2n+1} = S(x_{2n})$, $x_{2n+2} = T(x_{2n+1})$ for all $n \in \mathbb{N}$, as such the sequence $\{x_n\}$ converge to x^* as $n \to \infty$.

(iii) Similarly for an arbitrary $y_0 \in X$, the sequence $\{y_n\}$ given by $y_{2n+1} = T(y_{2n})$, $y_{2n+2} = S(y_{2n+1})$ for all $n \in \mathbb{N}$, it can be shown that $\{y_n\}$ is a Cauchy sequence in X and it converges to y^*.

Further,

$$\begin{aligned}
d(y^*, T(y^*)) &\leq d(y^*, y_{2n+2}) + d(y_{2n+2}, T(y^*)) \\
&= d(y^*, y_{2n+2}) + d(T(y_{2n+1}), T(y^*)) \\
&\leq d(y^*, y_{2n+2}) + k \max\Big\{d(y_{2n+1}, y^*), d(y_{2n+1}, T(y_{2n+1})), d(y_{2n+1}, T(y^*)), \\
&\qquad\qquad d(y^*, T(y_{2n+1})), d(y^*, T(y^*))\Big\} \\
&\leq d(y^*, y_{2n+2}) + k \max\Big\{d(y_{2n+1}, y^*), d(y_{2n+1}, y_{2n+2}), d(y_{2n+1}, T(y^*)), \\
&\qquad\qquad d(y^*, y_{2n+2}), d(y^*, T(y^*))\Big\}.
\end{aligned}$$

Now taking the limit as $n \to \infty$ on both sides we get, $d(y^*, T(y^*)) \leq kd(y^*, T(y^*))$. If $d(y^*, T(y^*)) \neq 0$, then $k \geq 1$, which is a contradiction. Therefore, $d(y^*, T(y^*)) = 0$, which implies $y^* = T(y^*)$.

Since, $ComFix(S, T) = Fix(S) = Fix(T)$, therefore, $x^* = y^*$.

Hence $y_{2n+1} = T(y_{2n})$, $y_{2n+2} = S(y_{2n+1})$ for all $n \in \mathbb{N}$, as such the sequence $\{y_n\}$ converge to x^* as $n \to \infty$. □

Example 2.9. Suppose $S, T : [0, 2] \to [0, 2]$ be two operators such that

$$S(x) = \begin{cases} \frac{x}{10}, & \text{if } \ 0 \leq x \leq 1 \\ \frac{x}{5}, & \text{if } \ x > 1 \end{cases}$$

and

$$T(y) = \frac{y^2 + y}{10}, \ y \in [0, 2].$$

Choose $x = \frac{999}{1000}$ and $y = \frac{1001}{1000}$, then $d(S(x), T(y)) = 0.1004001$ and $d(x, y) = 0.002$. Therefore, $d(S(x), T(y)) \leq k\, d(x, y)$ imply that $k \geq 50.20005$, which is a contradiction. Therefore, we can not assume the condition that there exists $k < 1$ such that $d(S(x), T(y)) \leq k\, d(x, y) \quad \text{for all} \ \ x, y \in [0, 2]$.

On the other hand, the pair S and T satisfies the condition (2.1) for any $k \in (\frac{3}{10}, \frac{1}{3})$. Moreover S and T have $ComFix(S, T) = 0$.

Theorem 2.10. *Let $S, T : X \to X$ be two operators on a complete metric space (X, d) such that there exists $k \in (0, \frac{1}{3})$ and for all $x, y \in X$, satisfying the condition:*

$$d(S(x), T(y)) \leq k \max \Big\{ d(x, y), d(x, S(x)), d(x, T(y)), d(y, S(x)), d(y, T(y)) \Big\}.$$

Then, we have the following conclusions:

(i) if $k < \frac{3-\sqrt{5}}{4}$, then S and T are graphic contractions;

(ii) if $k < \frac{1}{4}$, then S and T are β-quasi-contractions;

(iii) if $k < \frac{3-\sqrt{5}}{4}$, then S and T are c-Picard operators, with $c = \frac{(1-k)^2}{1-4k-k^2}$.

Proof. Let us consider x to be an arbitrary element in X. We have

$$d(S^2(x), S(x)) \leq d(S^2(x), T(x^*)) + d(S(x), T(x^*))$$

$$\begin{aligned}
&\leq k \max\Big\{d(S(x),x^*), d(S(x),S^2(x)), d(S(x),T(x^*)), d(x^*,S^2(x)), d(x^*,T(x^*))\Big\}+\\
&\qquad k \max\Big\{d(x,x^*), d(x,S(x)), d(x,T(x^*)), d(x^*,S(x)), d(x^*,T(x^*))\Big\}\\
&\leq k \max\Big\{d(S(x),x^*), d(S(x),S^2(x)), d(x^*,S^2(x))\Big\}+\\
&\qquad k \max\Big\{d(x,x^*), d(x,S(x)), d(x^*,S(x))\Big\}\\
&\leq k[d(S(x),x^*) + d(S(x),S^2(x)) + d(x^*,S^2(x))]+\\
&\qquad k[d(x,x^*) + d(x,S(x)) + d(x^*,S(x))].
\end{aligned}$$

Therefore,

$$d(S^2(x),S(x)) \leq k[d(x,x^*) + 2d(S(x),x^*) + d(S^2(x),x^*) + d(S(x),S^2(x)) + d(x,S(x))]. \tag{2.3}$$

Now,

$$\begin{aligned}
d(S^2(x),x^*) = d(S^2(x),S(x^*)) &\leq k \max\{d(S(x),x^*), d(S^2(x),S(x)), d(S^2(x),x^*)\}\\
&\leq k[d(S(x),x^*) + d(S^2(x),S(x)) + d(S^2(x),x^*)].
\end{aligned}$$

Therefore,

$$d(S^2(x),x^*) \leq \frac{k}{1-k}[d(S(x),x^*) + d(S^2(x),S(x))]. \tag{2.4}$$

Now using relations (2.3) and (2.4) we get

$$\begin{aligned}
d(S^2(x),S(x)) \leq k\Big[&d(x,x^*) + 2d(S(x),x^*) + \frac{k}{1-k}[d(S(x),x^*) + d(S^2(x),S(x))]+\\
&d(S(x),S^2(x)) + d(S(x),x)\Big].
\end{aligned}$$

Therefore,

$$d(S^2(x),S(x)) \leq \frac{k(1-k)}{1-2k}\Big[d(x,x^*) + \frac{2-k}{1-k}d(S(x),x^*) + d(S(x),x)\Big]. \tag{2.5}$$

Again,

$$d(S(x),x^*) = d(S(x),S(x^*)) \leq k \max\{d(x,x^*), d(S(x),x), d(S(x),x^*)\}$$

$$\leq k[d(x,x^*)+d(S(x),x)+d(S(x),x^*)].$$

Therefore,

$$d(S(x),x^*) \leq \frac{k}{1-k}[d(x,x^*)+d(S(x),x)]. \tag{2.6}$$

Again,

$$d(x,x^*) \leq d(S(x),x)+d(S(x),x^*). \tag{2.7}$$

Therefore, from (2.6) and (2.7) we have

$$d(S(x),x^*) \leq \frac{2k}{1-2k}d(S(x),x) \tag{2.8}$$

and

$$d(x,x^*) \leq \frac{1}{1-2k}d(S(x),x). \tag{2.9}$$

Therefore, using the relations (2.5), (2.8) and (2.9) we obtain

$$d(S^2(x),S(x)) \leq \frac{2k}{(1-2k)^2}d(S(x),x). \tag{2.10}$$

If $k < \frac{3-\sqrt{5}}{4}$, then $\frac{2k}{(1-2k)^2} < 1$, i.e., S is a graphic contraction.

(ii) Again, from (2.6) we have

$$\begin{aligned} d(S(x),x^*) &\leq \frac{k}{1-k}[d(x,x^*)+d(S(x),x)] \\ &\leq \frac{k}{1-k}[d(x,x^*)+d(S(x),x^*)+d(x,x^*)]. \end{aligned}$$

Therefore,

$$d(S(x),x^*) \leq \frac{2k}{1-2k}d(x,x^*). \tag{2.11}$$

If $k < \frac{1}{4}$ then $\frac{2k}{1-2k} < 1$, and so is a β-quasi-contraction.

(iii) Again, from the relation (2.10), we can say that S is graphic $\tau = \frac{2k}{(1-2k)^2}$ contraction.

Therefore, from the Theorem 2.6, we obtain that S is a $\frac{1}{1-\tau} = \frac{4k^2-4k+1}{4k^2-6k+1}$ Picard operator.

Hence the desired results.

□

Theorem 2.11. *Let $S, T : X \to X$ be two operators on the complete metric space (X, d) such that there exists $k \in (0, \frac{1}{3})$ and for all $x, y \in X$, satisfying the condition:*

$$d(S(x), T(y)) \leq k \max\Big\{d(x,y), d(x,S(x)), d(x,T(y)), d(y,S(x)), d(y,T(y))\Big\}.$$

Then, the following conclusions hold:

- *(i) if $k < \frac{3-\sqrt{5}}{4}$, then $Fix(S) = \{x : S(x) = x\}$ and $Fix(T) = \{x : T(x) = x\}$ satisfy well-posed conditions of Reich and Zaslavski;*
- *(ii) if $k < \frac{3-\sqrt{5}}{4}$, then $Fix(S) = \{x : S(x) = x\}$ and $Fix(T) = \{x : T(x) = x\}$ have Ulam-Hyers stability;*
- *(iii) if $k < \frac{1}{4}$, then S and T have the Ostrowski property.*

Proof. (i) The operators S and T are both graphic contractions for ($k < \frac{3-\sqrt{5}}{4}$) [see Theorem 2.10]. Therefore, by the Theorem 2.6, we can conclude that $Fix(S)$ and $Fix(T)$ satisfy Reich and Zaslavski conditions for well-posedness.

(ii) Also, from the Theorem 2.6 and Theorem 2.10, we can conclude that $Fix(S)$ and $Fix(T)$ have Ulam-Hyers stability when $k < \frac{3-\sqrt{5}}{4}$.

(iii) The operators S and T are both β-quasi-contractions for ($k < \frac{1}{4}$) [see Theorem

2.10]. Therefore, by the Theorem 2.7, we can conclude that $k < \frac{1}{4}$, as such S and T have the Ostrowski property. □

2.3 An application to nonlinear variational inequality problems

Let us construct the following operational problem: find $(x, y) \in X \times Y$ which satisfies the successive relations

$$\begin{cases} x = S(y) \\ y = T(x) \\ (x, y) = h(x, y), \end{cases} \tag{2.12}$$

where $S : Y \to X$, $T : X \to Y$ and $h = (h_1, h_2) : X \times Y \to X \times Y$ are given operators and X, Y are non-empty and closed subsets of a complete metric space (M, d).

We consider the following hypotheses:

(i) there exists $\beta \in (0, \dfrac{1}{3})$ such that $d(h_1(x_1, y_1), S(y_2)) \leq \beta \max\{d(y_1, y_2), d(x_1, h_1(x_1, y_1)), d(x_1, S(y_2))\}$;

(ii) there exists $\lambda \in (0, \dfrac{1}{3})$ such that $d(h_2(x_1, y_1), T(x_2)) \leq \lambda \max\{d(x_1,x_2), d(y_1, h_2(x_1, y_1)), d(y_1, T(x_2))\}$.

Let us define the metric $\tilde{d}$ on $X \times Y$ for $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ by

$$\tilde{d}(z_1, z_2) = \max\{d(x_1, x_2), d(y_1, y_2)\}.$$

We denote $t : X \times Y \to X \times Y$, $t(x,y) = (S(y), T(x))$ and $k = \max\{\beta, \lambda\}$.
From the above constructions, our problem becomes a common fixed point problem of the following form

$$(x,y) = t(x,y) = h(x,y).$$

Therefore, we have

$$\begin{aligned}
\tilde{d}(h(z_1), t(z_2)) &= \max\Big\{d(h_1(x_1,y_1), S(y_2)), d(h_2(x_1,y_1), T(x_2))\Big\} \\
&\leq \max\Big\{\beta \max\{d(y_1,y_2), d(x_1, h_1(x_1,y_1)), d(x_1, S(y_2))\}, \\
&\qquad\qquad \lambda \max\{d(x_1,x_2), d(y_1, h_2(x_1,y_1)), d(y_1, T(x_2))\}\Big\} \\
&\leq k \max\Big\{ \max\{d(x_1,x_2), d(y_1,y_2)\}, \max\{d(x_1, h_1(x_1,y_1)), d(y_1, h_2(x_1,y_1))\}, \\
&\qquad\qquad \max\{d(x_1, S(y_2)), d(y_1, T(x_2))\}\Big\} \\
&\leq k \max\Big\{\tilde{d}(z_1,z_2), \tilde{d}(z_1, h(z_1)), \tilde{d}(z_1, t(z_2))\Big\} \\
&\leq k \max\Big\{\tilde{d}(z_1,z_2), \tilde{d}(z_1, h(z_1)), \tilde{d}(z_1, t(z_2)), \tilde{d}(z_2, h(z_1)), \tilde{d}(z_2, t(z_2))\Big\}.
\end{aligned}$$

Thus h and t satisfy the condition (2.1). Therefore, using Theorem 2.8, we can get a common fixed point of h and t, i.e., the existence and uniqueness of the solutions, and also can apply all the stability properties using the Theorem 2.11.
For example, the above abstract model can be applied in the case of a hierarchical system of nonlinear variational inequality problems, which is defined as follows:
Find $(x^*, y^*) \in Fix(S)$ such that

$$\begin{cases} \langle aT_1(y^*) + x^* - y^*, x - x^* \rangle \ \geq 0, \\ \langle bT_2(x^*) + y^* - x^*, y - y^* \rangle \ \geq 0, \\ \text{for all } (x,y) \in Fix(s), \end{cases} \tag{2.13}$$

where $S : X \times Y \to X \times Y$ is given by $S(x,y) = (S_1(x), S_2(y))$, where $S_1 : X \to X$, $S_2 : Y \to Y$, $T_1 : X \to Y$, $T_2 : Y \to X$ are given operators, $a, b > 0$ and X, Y

are two non-empty closed and convex subsets of a real Hilbert space H.

It is known that problem is equivalent to the following problem:

Find $(x^*, y^*) \in X \times Y$ such that

$$\begin{cases} y^* = P_{Fix(S_2)}(I - aT_1)(x^*), \\ x^* = P_{Fix(S_1)}(I - bT_2)(y^*), \\ \quad (x^*, y^*) \in Fix(S), \end{cases} \tag{2.14}$$

where, for a non-empty, closed and convex set, $C \subset H$, the symbol P_C denotes the metric projection onto C, i.e., $P_C(u) = \{v \in C : \|u - v\| = \inf_{c \in C} \|u - c\|\},\ u \in H$. Notice that (2.14) is exactly the type of problem modeled by system (2.12). Thus, imposing adequate assumptions on S, T_1, T_2, on the parameters $a, b > 0$ and on the given sets X, Y we can obtain existence, uniqueness and stability results for the hierarchical system of nonlinear variational inequality problems (2.13). For complementary and related results about the hierarchical system of nonlinear variational inequality problems, we refer, [86, 87].

2.4 Concluding remarks

In this chapter, we have generalized the common fixed point problem for a pair of operators satisfying quasi-contraction-type metric conditions. We have proved the existence and uniqueness of the common fixed point problem using this contraction. After that, with the help of graphical contraction and β-quasi contraction, we have proved the sufficient condition for the common fixed point problem to be well-posed and to have Ulam–Hyers stability and Ostrowski property.

We can construct a similar type of common fixed point theory in a complete metric space (X,d) for a pair of operators using convex contraction instead of graphic contraction. We recall that, an operator $T : X \to X$ on (X,d), is said to be (l,k)-convex contraction if there exist $l,k \in \mathbb{R}_+$ with $l+k<1$ such that

$$d(T^2x,T^2y) \leq ld(Tx,Ty)+kd(x,y) \quad \text{for all } x,y \in X.$$

Since any continuous (l,k)-convex contraction on a complete metric space (X,d) is a Picard operator, (see, [26, 68] for related results and extensions), it will be interesting to see if one can deduce a similar type of common fixed point theory in a complete metric space (X,d) for a pair of self-operators satisfying the generalized quasi-contraction [62]. We recall that a mapping T is said to be a generalized quasi-contraction if there exists $k \in [0,1)$ such that for all $x,y \in X$,

$$\begin{aligned} d(Tx,Ty) \leq k\max\{&d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx),\\ &d(T^2x,x),d(T^2x,Tx),d(T^2x,y),d(T^2x,Ty)\}. \end{aligned}$$

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