

# Chapter 7

## Lagrange $\alpha$ -exponential synchronization of non-identical fractional-order complex-valued neural networks

### 7.1 Introduction

This chapter focuses on investigating Lagrange  $\alpha$ -exponential synchronization of non-identical fractional-order complex-valued neural networks (FOCVNNs). The study employs additional inequalities and the Lyapunov method to construct various favorable conditions for achieving Lagrange  $\alpha$ -exponential synchronization and  $\alpha$ -exponential convergence of these networks. Additionally, the chapter elucidates the structure of the  $\alpha$ -exponential convergence ball, demonstrating how the convergence rate correlates with the system's characteristics and the differential order. These

findings, which do not necessitate consideration of the existence and uniqueness of equilibrium points, contribute to the generalization and enhancement of previous works and may apply to mono-stable and multi-stable FOCVNNs. A notable aspect of the chapter is the graphical demonstration of the effectiveness of the proposed method through numerical simulations, particularly in synchronizing a specific case of the considered fractional order drive and response systems.

## 7.2 Mathematical model

The following FOCVNN is considered as a drive system as

$$D^\alpha \omega_{\hat{j}}(t) = -b_{\hat{j}} \omega_{\hat{j}}(t) + \sum_{\hat{k}=1}^n a_{\hat{j}\hat{k}} f_{\hat{k}}(\omega_{\hat{k}}(t)) + I'_{\hat{j}}(t), \quad 0 < \alpha < 1, \quad (7.1)$$

where  $\omega(t) = (\omega_1(t), \omega_2(t), \dots, \omega_n(t))^T \in \mathbf{C}^n$  is a state vector,  $\omega_{\hat{j}}(t) = \eta_{\hat{j}}(t) + i\nu_{\hat{j}}(t) \in \mathbf{C}$  with  $\eta_{\hat{j}}(t), \nu_{\hat{j}}(t) \in \mathbf{R}$  refer to the  $\hat{j}$ th unit state at time  $t$ , the charging time constant is denoted by  $B = \text{diag}\{b_1, b_2, \dots, b_n\}$ , where  $b_{\hat{j}} > 0$ . The connection weight of the neuron is denoted by  $A = (a_{\hat{j}\hat{k}})_{n \times n} \in \mathbf{C}^{n \times n}$ , where  $a_{\hat{j}\hat{k}} = a_{\hat{j}\hat{k}}^R + ia_{\hat{j}\hat{k}}^I \in \mathbf{C}$  with  $a_{\hat{j}\hat{k}}^R, a_{\hat{j}\hat{k}}^I \in \mathbf{R}$ , the activation function is represented by  $f(\omega(t)) = (f_1(\omega_1(t)), f_2(\omega_2(t)), \dots, f_n(\omega_n(t)))^T \in \mathbf{C}^n \rightarrow \mathbf{C}^n$ , the external input vector is denoted by  $I'(t) = (I'_1(t), I'_2(t), \dots, I'_n(t))^T \in \mathbf{C}^n$ .

The matrix form of system (7.1) is illustrated as

$$D^\alpha \omega(t) = -B\omega(t) + Af(\omega(t)) + I'(t). \quad (7.2)$$

The following are the real and imaginary components of the drive system (7.1) as

$$\begin{aligned} D^\alpha \eta_{\hat{j}}(t) &= -b_{\hat{j}} \eta_{\hat{j}}(t) + \sum_{\hat{k}=1}^n a_{\hat{j}\hat{k}}^R f_{\hat{k}}^R(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) - \sum_{\hat{k}=1}^n a_{\hat{j}\hat{k}}^I f_{\hat{k}}^I(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) + I_{\hat{j}}^R(t), \\ D^\alpha \nu_{\hat{j}}(t) &= -b_{\hat{j}} \nu_{\hat{j}}(t) + \sum_{\hat{k}=1}^n a_{\hat{j}\hat{k}}^R f_{\hat{k}}^I(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) + \sum_{\hat{k}=1}^n a_{\hat{j}\hat{k}}^I f_{\hat{k}}^R(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) + I_{\hat{j}}^I(t). \end{aligned} \quad (7.3)$$

The response system for (7.1) is taken as

$$D^\alpha u_{\hat{j}}(t) = -d_{\hat{j}} u_{\hat{j}}(t) + \sum_{\hat{k}=1}^n c_{\hat{j}\hat{k}} h_{\hat{k}}(u_{\hat{k}}(t)) + J_{\hat{j}}(t) + K_{\hat{j}}(t), \quad (7.4)$$

where  $u(t) = (u_1(t), u_2(t), \dots, u_n(t))^T \in \mathbf{C}^n$  represents the state vector,  $u_{\hat{j}}(t) = p_{\hat{j}}(t) + is_{\hat{j}}(t) \in \mathbf{C}$  with  $p_{\hat{j}}(t), s_{\hat{j}}(t) \in \mathbf{R}$  refer to the  $\hat{j}$ th unit state at time  $t$ , in neural network the charging time constant is denoted by  $D = \text{diag}\{d_1, d_2, \dots, d_n\}$ , where  $d_{\hat{j}} > 0$ , the connection weight of the neuron is represented by  $C = (c_{\hat{j}\hat{k}})_{n \times n} \in \mathbf{C}^{n \times n}$ , where  $c_{\hat{j}\hat{k}} = c_{\hat{j}\hat{k}}^R + ic_{\hat{j}\hat{k}}^I \in \mathbf{C}$  with  $c_{\hat{j}\hat{k}}^R, c_{\hat{j}\hat{k}}^I \in \mathbf{R}$ ,  $h(u(t)) = (h_1(u_1(t)), h_2(u_2(t)), \dots, h_n(u_n(t)))^T \in \mathbf{C}^n \rightarrow \mathbf{C}^n$  is the activation function,  $K(t)$  and  $J(t) = (J_1(t), J_2(t), \dots, J_n(t))^T \in \mathbf{C}^n$  are the control input and external input vector, respectively.

The matrix form of system (7.4) is written as

$$D^\alpha u(t) = -Du(t) + Ch(u(t)) + J(t) + K(t). \quad (7.5)$$

The real and imaginary components of the response system (7.4) of FOCVNN are as follows.

$$D^\alpha p_{\hat{j}}(t) = -d_{\hat{j}} p_{\hat{j}}(t) + \sum_{\hat{k}=1}^n c_{\hat{j}\hat{k}}^R h_{\hat{k}}^R(p_{\hat{k}}(t), s_{\hat{k}}(t)) - \sum_{\hat{k}=1}^n c_{\hat{j}\hat{k}}^I h_{\hat{k}}^I(p_{\hat{k}}(t), s_{\hat{k}}(t)) + J_{\hat{j}}^R(t)$$

$$\begin{aligned}
 & + K_{\hat{j}}^R(t), \\
 D^\alpha s_{\hat{j}}(t) = & -d_{\hat{j}}s_{\hat{j}}(t) + \sum_{\hat{k}=1}^n c_{\hat{j}\hat{k}}^R h_{\hat{k}}^I(p_{\hat{k}}(t), s_{\hat{k}}(t)) + \sum_{\hat{k}=1}^n c_{\hat{j}\hat{k}}^I h_{\hat{k}}^R(p_{\hat{k}}(t), s_{\hat{k}}(t)) + J_{\hat{j}}^I(t) \\
 & + K_{\hat{j}}^I(t). \tag{7.6}
 \end{aligned}$$

**Assumption 7.2.1.** Consider  $E(t) = q(t) + ir(t)$  and  $\tilde{E}(t) = \tilde{q}(t) + i\tilde{r}(t)$  for any  $q(t), r(t), \tilde{q}(t), \tilde{r}(t) \in \mathbf{R}$ .  $H(E(t)) \in \mathbf{C}$  is represented by the separation of real and imaginary components as  $H_{\hat{j}}(E(t)) = H_{\hat{j}}^R(q(t), r(t)) + iH_{\hat{j}}^I(q(t), r(t))$ , where  $H_{\hat{j}}^R(\cdot, \cdot), H_{\hat{j}}^I(\cdot, \cdot) : \mathbf{R}^2 \rightarrow \mathbf{R}$ , and satisfy

$$\begin{aligned}
 |H_{\hat{j}}^R(\tilde{q}(t), \tilde{r}(t)) - H_{\hat{j}}^R(q(t), r(t))| & \leq F_{\hat{j}}^{RR}|\tilde{q}(t) - q(t)| + F_{\hat{j}}^{RI}|\tilde{r}(t) - r(t)|, \\
 |H_{\hat{j}}^I(\tilde{q}(t), \tilde{r}(t)) - H_{\hat{j}}^I(q(t), r(t))| & \leq F_{\hat{j}}^{IR}|\tilde{q}(t) - q(t)| + F_{\hat{j}}^{II}|\tilde{r}(t) - r(t)|.
 \end{aligned}$$

**Assumption 7.2.2.** The external input vectors  $I_{\hat{j}}'(t), J_{\hat{j}}(t) \in \mathbf{C}, \hat{j} \in \mathbf{N}$  are described by the separation of real and imaginary components as  $I_{\hat{j}}'(t) = I_{\hat{j}}'^R(t) + iI_{\hat{j}}'^I(t), J_{\hat{j}}(t) = J_{\hat{j}}^R(t) + iJ_{\hat{j}}^I(t)$ , where  $I_{\hat{j}}'^R(t), I_{\hat{j}}'^I(t), J_{\hat{j}}^R(t), J_{\hat{j}}^I(t) : \mathbf{R} \rightarrow \mathbf{R}$  and satisfy  $|I_{\hat{j}}'^R(t)| \leq \hat{I}_{\hat{j}}'^R(t), |I_{\hat{j}}'^I(t)| \leq \hat{I}_{\hat{j}}'^I(t), |J_{\hat{j}}^R(t)| \leq \hat{J}_{\hat{j}}^R(t), |J_{\hat{j}}^I(t)| \leq \hat{J}_{\hat{j}}^I(t)$ .

**Assumption 7.2.3.** Let  $F = \text{diag}\{F_1, F_2, \dots, F_n\}$ , where  $F > 0$  be a diagonal matrix such that  $|H_{\hat{j}}(E) - H_{\hat{j}}(\tilde{E})| \leq F_{\hat{j}}|E - \tilde{E}|$ , for all  $E, \tilde{E}, H_{\hat{j}}(\cdot) \in \mathbf{C}$  with  $\hat{j} \in \mathbf{N}$ .

**Assumption 7.2.4.** For  $I_{\hat{j}}'(t), J_{\hat{j}}(t) \in \mathbf{C}, \hat{j} \in \mathbf{N}$ , there exist positive matrices  $\hat{I}' = (\hat{I}'_1, \hat{I}'_2, \dots, \hat{I}'_n)^T$  and  $\hat{J} = (\hat{J}_1, \hat{J}_2, \dots, \hat{J}_n)^T$  such that  $|I_{\hat{j}}'(t)| \leq \hat{I}'_{\hat{j}}, |J_{\hat{j}}(t)| \leq \hat{J}_{\hat{j}}$ .

**Remark 7.2.1.** The Assumptions 7.2.1 and 7.2.2 are defined for the activation functions  $f_{\hat{j}}(\omega_{\hat{j}}(t))$  and  $h_{\hat{j}}(u_{\hat{j}}(t))$  for NNs (7.1) and (7.4), respectively. Assumptions 7.2.1 and 7.2.3 have Remarkable similarities but they are not equal but depend on activation functions of NNs (7.1) and (7.4). If the activation functions follow Assumption 7.2.1, Assumption 7.2.3 is trivially satisfied [124]. If activation functions are not specified, splitting the real and imaginary components, we will use RVNNs to solve those, which will satisfy Assumption 7.2.3 [125].

The synchronization error is defined as  $E_{\hat{j}}(t) = u_{\hat{j}}(t) - \omega_{\hat{j}}(t)$ . We can derive the following error system from (7.1) and (7.4) as

$$D^\alpha E_{\hat{j}}(t) = -d_{\hat{j}}E_{\hat{j}}(t) + \sum_{\hat{k}=1}^n c_{\hat{j}\hat{k}} H_{\hat{k}}(E_{\hat{k}}(t)) - (d_{\hat{j}} - b_{\hat{j}})\omega_{\hat{j}}(t) + \sum_{\hat{k}=1}^n \left\{ c_{\hat{j}\hat{k}} h_{\hat{k}}(\omega_{\hat{k}}(t)) - a_{\hat{j}\hat{k}} f_{\hat{k}}(\omega_{\hat{k}}(t)) \right\} + J_{\hat{j}}(t) - I'_{\hat{j}}(t) + K_{\hat{j}}(t). \quad (7.7)$$

Now separating real and imaginary components of (7.7) as  $E(t) = q(t) + ir(t)$ , we get the error systems with the help of equations (7.3) and (7.6) as

$$\begin{aligned} D^\alpha q_{\hat{j}}(t) &= -d_{\hat{j}}q_{\hat{j}}(t) + \sum_{\hat{k}=1}^n c_{\hat{j}\hat{k}}^R H_{\hat{k}}^R(q_{\hat{k}}(t), r_{\hat{k}}(t)) - \sum_{\hat{k}=1}^n c_{\hat{j}\hat{k}}^I H_{\hat{k}}^I(q_{\hat{k}}(t), r_{\hat{k}}(t)) \\ &\quad - (d_{\hat{j}} - b_{\hat{j}})\eta_{\hat{j}}(t) + \sum_{\hat{k}=1}^n \left\{ c_{\hat{j}\hat{k}}^R h_{\hat{k}}^R(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) - a_{\hat{j}\hat{k}}^R f_{\hat{k}}^R(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) \right\} \\ &\quad - \sum_{\hat{k}=1}^n \left\{ c_{\hat{j}\hat{k}}^I h_{\hat{k}}^I(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) - a_{\hat{j}\hat{k}}^I f_{\hat{k}}^I(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) \right\} \\ &\quad + J_{\hat{j}}^R(t) - I_{\hat{j}}^R + K_{\hat{j}}^R(t), \\ D^\alpha r_{\hat{j}}(t) &= -d_{\hat{j}}r_{\hat{j}}(t) + \sum_{\hat{k}=1}^n c_{\hat{j}\hat{k}}^R H_{\hat{k}}^I(q_{\hat{k}}(t), r_{\hat{k}}(t)) + \sum_{\hat{k}=1}^n c_{\hat{j}\hat{k}}^I H_{\hat{k}}^R(q_{\hat{k}}(t), r_{\hat{k}}(t)) \\ &\quad - (d_{\hat{j}} - b_{\hat{j}})\nu_{\hat{j}}(t) + \sum_{\hat{k}=1}^n \left\{ c_{\hat{j}\hat{k}}^R h_{\hat{k}}^I(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) - a_{\hat{j}\hat{k}}^R f_{\hat{k}}^I(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) \right\} \\ &\quad + \sum_{\hat{k}=1}^n \left\{ c_{\hat{j}\hat{k}}^I h_{\hat{k}}^R(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) - a_{\hat{j}\hat{k}}^I f_{\hat{k}}^R(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) \right\} \\ &\quad + J_{\hat{j}}^I(t) - I_{\hat{j}}^I + K_{\hat{j}}^I(t), \end{aligned} \quad (7.8)$$

where  $q_{\hat{j}}(t) = p_{\hat{j}}(t) - \eta_{\hat{j}}(t)$  and  $r_{\hat{j}}(t) = s_{\hat{j}}(t) - \nu_{\hat{j}}(t)$ ,  $H_{\hat{k}}^R(q_{\hat{k}}(t), r_{\hat{k}}(t)) = h_{\hat{k}}^R(p_{\hat{k}}(t), s_{\hat{k}}(t)) - h_{\hat{k}}^R(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t))$ ,  $H_{\hat{k}}^I(q_{\hat{k}}(t), r_{\hat{k}}(t)) = h_{\hat{k}}^I(p_{\hat{k}}(t), s_{\hat{k}}(t)) - h_{\hat{k}}^I(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t))$ .

**Definition 7.2.1.** [126] NNs (7.1) and (7.4) are Lagrange uniformly stable (or uniform bounded), if any solution  $E(t) = E(t, t_0, E_0)$  and  $G > 0$ , there exists a constant

$K = K(G) > 0$ , such that  $\|E(t)\| < K, \forall E_0 \in \mathbf{C}_G = \{E_0 \in \mathbf{C} : \|E_0\| \leq G\}$  and  $t \geq t_0$ .

**Definition 7.2.2.** [106] If there exist constants  $\gamma > 0, \lambda > 0$  and for any  $G > 0$  there exists  $K = K(G) > 0$  such that  $\|E(t)\| < \gamma + Ke^{-\lambda t^\alpha}, \forall t \geq 0, E_0 \in \mathbf{C}_G$ , then NNs (7.1) and (7.4) are called globally Lagrange  $\alpha$ -exponentially stable. The networks (7.1) and (7.4) are also said to be globally uniform  $\alpha$ -exponentially convergent with rate  $\lambda$  in ball  $B(\gamma) = \{E(t) \in \mathbf{C}^n : \|E(t)\| \leq \gamma\}$ .

**Lemma 7.2.1.** [106] Let us consider  $V(t)$  be a real valued continuous function on the interval  $[0, \infty)$ , then there exist two constants  $W$  and  $\beta$  such that

$D^\alpha V(t) \leq -WV(t) + \beta, t \geq 0$ , then

$$V(t) - \frac{\beta}{W} \leq \left(V(0) - \frac{\beta}{W}\right) e^{-\frac{W}{\Gamma(\alpha+1)}t^\alpha}, \quad 0 < \alpha \leq 1.$$

**Remark 7.2.2.** If  $V(t) \geq \frac{\beta}{W} > 0$  with  $W > 0$ , then  $V(t)$  will exponentially approach to  $\frac{\beta}{W}$  as  $t$  increases.

**Remark 7.2.3.** Let us consider  $E_{\hat{j}}(t) = q_{\hat{j}}(t) + ir_{\hat{j}}(t)$ , where  $q_{\hat{j}}(t), r_{\hat{j}}(t) \in \mathbf{R}$ . If  $E_{\hat{j}}(t) \geq 0$ , then

$${}_a^C D_t^\alpha |E_{\hat{j}}(t)| = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{E_{\hat{j}}^{(1)}(\tau)}{(t-\tau)^\alpha} d\tau = {}_a^C D_t^\alpha E_{\hat{j}}(t)$$

and if  $E_{\hat{j}}(t) < 0$ , then

$${}_a^C D_t^\alpha |E_{\hat{j}}(t)| = -\frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{E_{\hat{j}}^{(1)}(\tau)}{(t-\tau)^\alpha} d\tau = -{}_a^C D_t^\alpha E_{\hat{j}}(t).$$

Therefore

$${}_a^C D_t^\alpha |E_{\hat{j}}(t)| = \text{sgn}(E_{\hat{j}}(t)) {}_a^C D_t^\alpha E_{\hat{j}}(t).$$

Similarly

$${}^C D_t^\alpha |q_{\hat{j}}(t)| = \text{sgn}(q_{\hat{j}}(t)) {}^C D_t^\alpha q_{\hat{j}}(t),$$

and

$${}^C D_t^\alpha |r_{\hat{j}}(t)| = \text{sgn}(r_{\hat{j}}(t)) {}^C D_t^\alpha r_{\hat{j}}(t).$$

### 7.3 Main results

**Theorem 7.3.1.** Based on Assumptions 7.2.1 and 7.2.2, if there exist constants  $A'_j > 0, B'_j > 0$ , s.t.

$$W_j^1 = d_j + k_j^R - \frac{1}{A'_j} \sum_{\hat{k}=1}^n \left( A'_{\hat{k}} |c_{\hat{k}\hat{j}}^R| + B'_{\hat{k}} |c_{\hat{k}\hat{j}}^I| \right) F_j^{RR} - \frac{1}{A'_j} \sum_{\hat{k}=1}^n \left( A'_{\hat{k}} |c_{\hat{k}\hat{j}}^I| + B'_{\hat{k}} |c_{\hat{k}\hat{j}}^R| \right) F_j^{IR} > 0, \quad (7.9)$$

$$W_j^2 = d_j + k_j^I - \frac{1}{B'_j} \sum_{\hat{k}=1}^n \left( A'_{\hat{k}} |c_{\hat{k}\hat{j}}^I| + B'_{\hat{k}} |c_{\hat{k}\hat{j}}^R| \right) F_j^{II} - \frac{1}{B'_j} \sum_{\hat{k}=1}^n \left( A'_{\hat{k}} |c_{\hat{k}\hat{j}}^R| + B'_{\hat{k}} |c_{\hat{k}\hat{j}}^I| \right) F_j^{RI} > 0, \quad (7.10)$$

then networks (7.1) and (7.4) are globally  $\alpha$ -exponentially stable in Lagrange sense based on controllers such that

$$\begin{aligned} K_j^R(t) &= (d_j - b_j) \eta_{\hat{j}}(t) - \sum_{\hat{k}=1}^n \left\{ c_{\hat{j}\hat{k}}^R h_{\hat{k}}^R(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) - a_{\hat{j}\hat{k}}^R f_{\hat{k}}^R(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) \right\} \\ &\quad + \sum_{\hat{k}=1}^n \left\{ c_{\hat{j}\hat{k}}^I h_{\hat{k}}^I(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) - a_{\hat{j}\hat{k}}^I f_{\hat{k}}^I(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) \right\} - k_j^R q_{\hat{j}}(t), \\ K_j^I(t) &= (d_j - b_j) \nu_{\hat{j}}(t) - \sum_{\hat{k}=1}^n \left\{ c_{\hat{j}\hat{k}}^R h_{\hat{k}}^I(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) - a_{\hat{j}\hat{k}}^R f_{\hat{k}}^I(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) \right\} \\ &\quad - \sum_{\hat{k}=1}^n \left\{ c_{\hat{j}\hat{k}}^I h_{\hat{k}}^R(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) - a_{\hat{j}\hat{k}}^I f_{\hat{k}}^R(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) \right\} - k_j^I r_{\hat{j}}(t), \end{aligned} \quad (7.11)$$

where  $k = k^R + ik^I \in \mathbf{C}$ . In equation (7.11), the linear terms primarily handle proportional feedback, while the nonlinear terms account for complex interactions within the network, ensuring comprehensive stabilization.

In addition, networks (7.1) and (7.4) are globally  $\alpha$ -exponentially convergent to the ball

$$B_1 = \left\{ E(t) = q(t) + ir(t) \in \mathbf{C}^n : \sum_{\hat{j}=1}^n (|q_{\hat{j}}(t)| + |r_{\hat{j}}(t)|) \leq \frac{\beta}{YW} \right\}, \quad (7.12)$$

where

$$\begin{cases} W = \min \left\{ \min_{1 \leq \hat{j} \leq n} \{W_{\hat{j}}^1\}, \min_{1 \leq \hat{j} \leq n} \{W_{\hat{j}}^2\} \right\}, \\ Y = \min \left\{ \min_{1 \leq \hat{j} \leq n} \{A'_{\hat{j}}\}, \min_{1 \leq \hat{j} \leq n} \{B'_{\hat{j}}\} \right\}, \\ \beta = \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n (A'_{\hat{k}} |c_{\hat{k}\hat{j}}^R| + B'_{\hat{k}} |c_{\hat{k}\hat{j}}^I|) |H_{\hat{j}}^R(0, 0)| + \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n (A'_{\hat{k}} |c_{\hat{k}\hat{j}}^I| + B'_{\hat{k}} |c_{\hat{k}\hat{j}}^R|) |H_{\hat{j}}^I(0, 0)| + \sum_{\hat{j}=1}^n A'_{\hat{j}} (\hat{J}_{\hat{j}}^R(t) + \hat{I}'_{\hat{j}}^R(t)) + \sum_{\hat{j}=1}^n B'_{\hat{j}} (\hat{J}_{\hat{j}}^I(t) + \hat{I}'_{\hat{j}}^I(t)). \end{cases}$$

*Proof.* : From error dynamical system (7.8), let us design the following Lyapunov function as

$$V(t) = \sum_{\hat{j}=1}^n A'_{\hat{j}} |q_{\hat{j}}(t)| + \sum_{\hat{j}=1}^n B'_{\hat{j}} |r_{\hat{j}}(t)|.$$

Taking the fractional-order derivative of  $V(t)$ , and using Assumptions 7.2.1 and 7.2.2, we obtain

$$\begin{aligned} D^\alpha V(t) &= \sum_{\hat{j}=1}^n A'_{\hat{j}} \operatorname{sgn}(q_{\hat{j}}(t)) D^\alpha q_{\hat{j}}(t) + \sum_{\hat{j}=1}^n B'_{\hat{j}} \operatorname{sgn}(r_{\hat{j}}(t)) D^\alpha r_{\hat{j}}(t) \\ &= \sum_{\hat{j}=1}^n A'_{\hat{j}} \operatorname{sgn}(q_{\hat{j}}(t)) \left\{ -(d_{\hat{j}} + k_{\hat{j}}^R) q_{\hat{j}}(t) + \sum_{\hat{k}=1}^n c_{\hat{j}\hat{k}}^R H_{\hat{k}}^R(q_{\hat{k}}(t), r_{\hat{k}}(t)) \right. \end{aligned}$$

$$\begin{aligned}
& - \sum_{\hat{k}=1}^n c_{\hat{j}\hat{k}}^I H_{\hat{k}}^I(q_{\hat{k}}(t), r_{\hat{k}}(t)) + J_{\hat{j}}^R(t) - I_{\hat{j}}^{R}(t) \Big\} + \sum_{\hat{j}=1}^n B'_{\hat{j}} \text{sgn}(r_{\hat{j}}(t)) \\
& \times \left\{ - (d_{\hat{j}} + k_{\hat{j}}^I) r_{\hat{j}}(t) + \sum_{\hat{k}=1}^n c_{\hat{j}\hat{k}}^R H_{\hat{k}}^I(q_{\hat{k}}(t), r_{\hat{k}}(t)) + \sum_{\hat{k}=1}^n c_{\hat{j}\hat{k}}^I H_{\hat{k}}^R(q_{\hat{k}}(t), r_{\hat{k}}(t)) \right. \\
& \left. + J_{\hat{j}}^I(t) - I_{\hat{j}}^{I}(t) \right\} \\
\leq & - \sum_{\hat{j}=1}^n A'_{\hat{j}} (d_{\hat{j}} + k_{\hat{j}}^R) |q_{\hat{j}}(t)| + \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n A'_{\hat{j}} |c_{\hat{j}\hat{k}}^R| |H_{\hat{k}}^R(q_{\hat{k}}(t), r_{\hat{k}}(t))| \\
& - \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n A'_{\hat{j}} |c_{\hat{j}\hat{k}}^I| |H_{\hat{k}}^I(q_{\hat{k}}(t), r_{\hat{k}}(t))| - \sum_{\hat{j}=1}^n B'_{\hat{j}} (d_{\hat{j}} + k_{\hat{j}}^I) |r_{\hat{j}}(t)| \\
& + \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n B'_{\hat{j}} |c_{\hat{j}\hat{k}}^R| |H_{\hat{k}}^I(q_{\hat{k}}(t), r_{\hat{k}}(t))| + \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n B'_{\hat{j}} |c_{\hat{j}\hat{k}}^I| |H_{\hat{k}}^R(q_{\hat{k}}(t), r_{\hat{k}}(t))| \\
& + \sum_{\hat{j}=1}^n A'_{\hat{j}} |J_{\hat{j}}^R(t)| + \sum_{\hat{j}=1}^n A'_{\hat{j}} |I_{\hat{j}}^{R}(t)| + \sum_{\hat{j}=1}^n B'_{\hat{j}} |J_{\hat{j}}^I(t)| + \sum_{\hat{j}=1}^n B'_{\hat{j}} |I_{\hat{j}}^{I}(t)| \\
\leq & - \sum_{\hat{j}=1}^n A'_{\hat{j}} (d_{\hat{j}} + k_{\hat{j}}^R) |q_{\hat{j}}(t)| + \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n A'_{\hat{j}} |c_{\hat{j}\hat{k}}^R| \left( F_{\hat{k}}^{RR} |q_{\hat{k}}(t)| + F_{\hat{k}}^{RI} |r_{\hat{k}}(t)| \right. \\
& \left. + |H_{\hat{k}}^R(0, 0)| \right) + \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n A'_{\hat{j}} |c_{\hat{j}\hat{k}}^I| \left( F_{\hat{k}}^{IR} |q_{\hat{k}}(t)| + F_{\hat{k}}^{II} |r_{\hat{k}}(t)| + |H_{\hat{k}}^I(0, 0)| \right) \\
& - \sum_{\hat{j}=1}^n B'_{\hat{j}} (d_{\hat{j}} + k_{\hat{j}}^I) |r_{\hat{j}}(t)| + \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n B'_{\hat{j}} |c_{\hat{j}\hat{k}}^R| \left( F_{\hat{k}}^{IR} |q_{\hat{k}}(t)| + F_{\hat{k}}^{II} |r_{\hat{k}}(t)| \right. \\
& \left. + |H_{\hat{k}}^I(0, 0)| \right) + \sum_{\hat{j}=1}^n A'_{\hat{j}} \hat{J}_{\hat{j}}^R(t) + \sum_{\hat{j}=1}^n A'_{\hat{j}} \hat{I}_{\hat{j}}^{R}(t) + \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n B'_{\hat{j}} |c_{\hat{j}\hat{k}}^I| \left( F_{\hat{k}}^{RR} |q_{\hat{k}}(t)| \right. \\
& \left. + F_{\hat{k}}^{RI} |r_{\hat{k}}(t)| + |H_{\hat{k}}^R(0, 0)| \right) + \sum_{\hat{j}=1}^n B'_{\hat{j}} \hat{J}_{\hat{j}}^I(t) + \sum_{\hat{j}=1}^n B'_{\hat{j}} \hat{I}_{\hat{j}}^{I}(t) \\
\leq & - \sum_{\hat{j}=1}^n A'_{\hat{j}} \left\{ d_{\hat{j}} + k_{\hat{j}}^R - \frac{1}{A'_{\hat{j}}} \sum_{\hat{k}=1}^n \left( A'_{\hat{k}} |c_{\hat{k}\hat{j}}^R| + B'_{\hat{k}} |c_{\hat{k}\hat{j}}^I| \right) F_{\hat{j}}^{RR} - \frac{1}{A'_{\hat{j}}} \sum_{\hat{k}=1}^n \left( A'_{\hat{k}} |c_{\hat{k}\hat{j}}^I| \right. \right. \\
& \left. \left. + B'_{\hat{k}} |c_{\hat{k}\hat{j}}^R| \right) F_{\hat{j}}^{IR} \right\} |q_{\hat{j}}(t)| - \sum_{\hat{j}=1}^n B'_{\hat{j}} \left\{ d_{\hat{j}} + k_{\hat{j}}^I - \frac{1}{B'_{\hat{j}}} \sum_{\hat{k}=1}^n \left( A'_{\hat{k}} |c_{\hat{k}\hat{j}}^I| + B'_{\hat{k}} |c_{\hat{k}\hat{j}}^R| \right) F_{\hat{j}}^{II} \right. \\
& \left. - \frac{1}{B'_{\hat{j}}} \sum_{\hat{k}=1}^n \left( A'_{\hat{k}} |c_{\hat{k}\hat{j}}^R| + B'_{\hat{k}} |c_{\hat{k}\hat{j}}^I| \right) F_{\hat{j}}^{RI} \right\} |r_{\hat{j}}(t)| + \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n \left( A'_{\hat{k}} |c_{\hat{k}\hat{j}}^R| + B'_{\hat{k}} |c_{\hat{k}\hat{j}}^I| \right)
\end{aligned}$$

$$\begin{aligned}
 & \times |H_j^R(0,0)| + \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n \left( A'_k |c_{\hat{k}\hat{j}}^I| + B'_k |c_{\hat{k}\hat{j}}^R| \right) |H_j^I(0,0)| + \sum_{\hat{j}=1}^n A'_j \left( \hat{J}_j^R(t) + \hat{I}_j^R(t) \right) \\
 & + \sum_{\hat{j}=1}^n B'_j \left( \hat{J}_j^I(t) + \hat{I}_j^I(t) \right) \\
 & \leq -WV(t) + \beta.
 \end{aligned}$$

When  $V(t) \geq \frac{\beta}{W}$ , we can conclude from Lemma 7.2.1 that

$$V(t) - \frac{\beta}{W} \leq \left( V(0) - \frac{\beta}{W} \right) e^{-\frac{W}{\Gamma(\alpha+1)}t^\alpha}, \quad t \geq 0, \quad (7.13)$$

i.e.,

$$\|E(t)\| = \sum_{\hat{j}=1}^n \left( |q_{\hat{j}}(t)| + |r_{\hat{j}}(t)| \right) \leq \frac{\beta}{YW} + \frac{V(0)}{Y} e^{-\frac{W}{\Gamma(\alpha+1)}t^\alpha}.$$

From the definition 7.2.1, the NNs of error systems (7.8) will be globally Lagrange  $\alpha$ -exponentially stable and hence the Lagrange  $\alpha$ -exponential synchronization between FOCVNNs (7.1) and (7.4) is achieved. □

**Remark 7.3.1.** If  $\beta = 0$ , then Lemma 7.2.1 is reduced to the Lyapunov  $\alpha$ -exponential stability of fractional order system and thus the NNs of error systems (7.8) will be Lyapunov  $\alpha$ -exponential stable and in this case we will achieve the  $\alpha$ -exponential synchronization between FOCVNNs (7.1) and (7.4) in Lyapunov sense. If  $\alpha = 1$ ,  $W > 0$  and,  $\beta > 0$ , then Lemma 7.2.1 is reduced to the integer order Lagrange stability and in this case we will achieve the exponential synchronization between complex valued NNs in Lagrange sense.

Accordingly, we can obtain the following results.

**Corollary 7.3.1.** Let us suppose Assumptions 7.2.1 and 7.2.2 hold with  $|H_j^R(0, 0)| = 0$ ,  $|H_j^I(0, 0)| = 0$  and if there exist  $A'_j > 0$ ,  $B'_j > 0$  such that

$$W_j^1 = d_j + k_j^R - \frac{1}{A'_j} \sum_{\hat{k}=1}^n \left( A'_k |c_{\hat{k}\hat{j}}^R| + B'_k |c_{\hat{k}\hat{j}}^I| \right) F_j^{RR} - \frac{1}{A'_j} \sum_{\hat{k}=1}^n \left( A'_k |c_{\hat{k}\hat{j}}^I| + B'_k |c_{\hat{k}\hat{j}}^R| \right) F_j^{IR} > 0,$$

$$W_j^2 = d_j + k_j^I - \frac{1}{B'_j} \sum_{\hat{k}=1}^n \left( A'_k |c_{\hat{k}\hat{j}}^I| + B'_k |c_{\hat{k}\hat{j}}^R| \right) F_j^{II} - \frac{1}{B'_j} \sum_{\hat{k}=1}^n \left( A'_k |c_{\hat{k}\hat{j}}^R| + B'_k |c_{\hat{k}\hat{j}}^I| \right) F_j^{RI} > 0,$$

then the networks (7.1) and (7.4) are globally Lyapunov  $\alpha$ -exponential stable based on controllers given as

$$K_j^R(t) = (d_j - b_j) \eta_j(t) - \sum_{\hat{k}=1}^n \left\{ c_{\hat{j}\hat{k}}^R h_{\hat{k}}^R(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) - a_{\hat{j}\hat{k}}^R f_{\hat{k}}^R(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) \right\}$$

$$+ \sum_{\hat{k}=1}^n \left\{ c_{\hat{j}\hat{k}}^I h_{\hat{k}}^I(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) - a_{\hat{j}\hat{k}}^I f_{\hat{k}}^I(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) \right\} - J_j^R(t) + I_j^{IR}(t) - k_j^R q_j(t),$$

$$K_j^I(t) = (d_j - b_j) \nu_j(t) - \sum_{\hat{k}=1}^n \left\{ c_{\hat{j}\hat{k}}^R h_{\hat{k}}^I(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) - a_{\hat{j}\hat{k}}^R f_{\hat{k}}^I(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) \right\}$$

$$- \sum_{\hat{k}=1}^n \left\{ c_{\hat{j}\hat{k}}^I h_{\hat{k}}^R(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) - a_{\hat{j}\hat{k}}^I f_{\hat{k}}^R(\eta_{\hat{k}}(t), \nu_{\hat{k}}(t)) \right\} - J_j^I(t) + I_j^{II}(t) - k_j^I r_j(t),$$

where

$$\begin{cases} W = \min \left\{ \min_{1 \leq \hat{j} \leq n} \{W_j^1\}, \min_{1 \leq \hat{j} \leq n} \{W_j^2\} \right\}, \\ Y = \min \left\{ \min_{1 \leq \hat{j} \leq n} \{A'_j\}, \min_{1 \leq \hat{j} \leq n} \{B'_j\} \right\}. \end{cases}$$

*Proof.* : Let us define the following Lyapunov function based on the error dynamical system (7.8) as

$$V(t) = \sum_{\hat{j}=1}^n A'_j |q_{\hat{j}}(t)| + \sum_{\hat{j}=1}^n B'_j |r_{\hat{j}}(t)|.$$

Taking fractional-order derivative of  $V(t)$  and applying Assumptions 7.2.1 and 7.2.2, we get

$$\begin{aligned}
 D^\alpha V(t) &= \sum_{\hat{j}=1}^n A'_j \operatorname{sgn}(q_{\hat{j}}(t)) D^\alpha q_{\hat{j}}(t) + \sum_{\hat{j}=1}^n B'_j \operatorname{sgn}(r_{\hat{j}}(t)) D^\alpha r_{\hat{j}}(t) \\
 &= \sum_{\hat{j}=1}^n A'_j \operatorname{sgn}(q_{\hat{j}}(t)) \left\{ - (d_{\hat{j}} + k_{\hat{j}}^R) q_{\hat{j}}(t) + \sum_{\hat{k}=1}^n c_{\hat{j}\hat{k}}^R H_{\hat{k}}^R(q_{\hat{k}}(t), r_{\hat{k}}(t)) \right. \\
 &\quad \left. - \sum_{\hat{k}=1}^n c_{\hat{j}\hat{k}}^I H_{\hat{k}}^I(q_{\hat{k}}(t), r_{\hat{k}}(t)) \right\} + \sum_{\hat{j}=1}^n B'_j \operatorname{sgn}(r_{\hat{j}}(t)) \left\{ - (d_{\hat{j}} + k_{\hat{j}}^I) r_{\hat{j}}(t) \right. \\
 &\quad \left. + \sum_{\hat{k}=1}^n c_{\hat{j}\hat{k}}^R H_{\hat{k}}^I(q_{\hat{k}}(t), r_{\hat{k}}(t)) + \sum_{\hat{k}=1}^n c_{\hat{j}\hat{k}}^I H_{\hat{k}}^R(q_{\hat{k}}(t), r_{\hat{k}}(t)) \right\} \\
 &\leq - \sum_{\hat{j}=1}^n A'_j (d_{\hat{j}} + k_{\hat{j}}^R) |q_{\hat{j}}(t)| + \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n A'_j |c_{\hat{j}\hat{k}}^R| |H_{\hat{k}}^R(q_{\hat{k}}(t), r_{\hat{k}}(t))| \\
 &\quad - \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n A'_j |c_{\hat{j}\hat{k}}^I| |H_{\hat{k}}^I(q_{\hat{k}}(t), r_{\hat{k}}(t))| - \sum_{\hat{j}=1}^n B'_j (d_{\hat{j}} + k_{\hat{j}}^I) |r_{\hat{j}}(t)| \\
 &\quad + \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n B'_j |c_{\hat{j}\hat{k}}^R| |H_{\hat{k}}^I(q_{\hat{k}}(t), r_{\hat{k}}(t))| + \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n B'_j |c_{\hat{j}\hat{k}}^I| |H_{\hat{k}}^R(q_{\hat{k}}(t), r_{\hat{k}}(t))| \\
 &\leq - \sum_{\hat{j}=1}^n A'_j (d_{\hat{j}} + k_{\hat{j}}^R) |q_{\hat{j}}(t)| + \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n A'_j |c_{\hat{j}\hat{k}}^R| \left( F_{\hat{k}}^{RR} |q_{\hat{k}}(t)| + F_{\hat{k}}^{RI} |r_{\hat{k}}(t)| \right) \\
 &\quad + \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n A'_j |c_{\hat{j}\hat{k}}^I| \left( F_{\hat{k}}^{IR} |q_{\hat{k}}(t)| + F_{\hat{k}}^{II} |r_{\hat{k}}(t)| \right) - \sum_{\hat{j}=1}^n B'_j (d_{\hat{j}} + k_{\hat{j}}^I) |r_{\hat{j}}(t)| \\
 &\quad + \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n B'_j |c_{\hat{j}\hat{k}}^R| \left( F_{\hat{k}}^{IR} |q_{\hat{k}}(t)| + F_{\hat{k}}^{II} |r_{\hat{k}}(t)| \right) + \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n B'_j |c_{\hat{j}\hat{k}}^I| \left( F_{\hat{k}}^{RR} |q_{\hat{k}}(t)| \right. \\
 &\quad \left. + F_{\hat{k}}^{RI} |r_{\hat{k}}(t)| \right) \\
 &\leq - \sum_{\hat{j}=1}^n A'_j \left\{ d_{\hat{j}} + k_{\hat{j}}^R - \frac{1}{A'_j} \sum_{\hat{k}=1}^n \left( A'_{\hat{k}} |c_{\hat{k}\hat{j}}^R| + B'_{\hat{k}} |c_{\hat{k}\hat{j}}^I| \right) F_{\hat{j}}^{RR} - \frac{1}{A'_j} \sum_{\hat{k}=1}^n \left( A'_{\hat{k}} |c_{\hat{k}\hat{j}}^I| \right. \right. \\
 &\quad \left. \left. + B'_{\hat{k}} |c_{\hat{k}\hat{j}}^R| \right) F_{\hat{j}}^{IR} \right\} |q_{\hat{j}}(t)| - \sum_{\hat{j}=1}^n B'_j \left\{ d_{\hat{j}} + k_{\hat{j}}^I - \frac{1}{B'_j} \sum_{\hat{k}=1}^n \left( A'_{\hat{k}} |c_{\hat{k}\hat{j}}^I| + B'_{\hat{k}} |c_{\hat{k}\hat{j}}^R| \right) F_{\hat{j}}^{II} \right. \\
 &\quad \left. - \frac{1}{B'_j} \sum_{\hat{k}=1}^n \left( A'_{\hat{k}} |c_{\hat{k}\hat{j}}^R| + B'_{\hat{k}} |c_{\hat{k}\hat{j}}^I| \right) F_{\hat{j}}^{RI} \right\} |r_{\hat{j}}(t)| \\
 &\leq - WV(t).
 \end{aligned}$$

When  $\beta = 0$ , then Lemma 7.2.1 is converted into

$$V(t) \leq V(0)e^{-\frac{W}{\Gamma(\alpha+1)}t^\alpha}, \quad t \geq 0,$$

which shows that systems (7.1) and (7.4) are  $\alpha$ -exponentially synchronized in Lyapunov sense.  $\square$

**Remark 7.3.2.** If  $\beta = 0$  and  $\lim_{t \rightarrow \infty} V(t) = 0$ , then the NNs (7.1) and (7.4) are globally asymptotically synchronized. The proof of  $\lim_{t \rightarrow \infty} V(t) = 0$  is given in [127] by contradiction. Also [128] has investigated asymptotic stability of FOCVNNs with discrete and distributed delays based on adaptive controller, where the convergence rate was not estimated but in [106] the convergence rate of FOCVNNs is estimated.

**Remark 7.3.3.** In  $n$ -dimensional real space, the networks (7.1) and (7.4) are transformed into the fractional-order RVNNs as

$$D^\alpha \tilde{\omega}_{\hat{j}}(t) = -b_{\hat{j}} \tilde{\omega}_{\hat{j}}(t) + \sum_{\hat{k}=1}^n a_{\hat{j}\hat{k}} f_{\hat{k}}(\tilde{\omega}_{\hat{k}}(t)) + I'_{\hat{j}}(t), \quad (7.14)$$

$$D^\alpha \tilde{u}_{\hat{j}}(t) = -d_{\hat{j}} \tilde{u}_{\hat{j}}(t) + \sum_{\hat{k}=1}^n c_{\hat{j}\hat{k}} h_{\hat{k}}(\tilde{u}_{\hat{k}}(t)) + J_{\hat{j}}(t) + K_{\hat{j}}(t), \quad (7.15)$$

where  $b_{\hat{j}}, d_{\hat{j}} > 0$ ,  $\tilde{\omega}_{\hat{j}}(t), \tilde{u}_{\hat{j}}(t), c_{\hat{j}\hat{k}}, a_{\hat{j}\hat{k}}, f_{\hat{k}}(\tilde{\omega}_{\hat{k}}(t)), h_{\hat{k}}(\tilde{u}_{\hat{k}}(t)), I'_{\hat{j}}(t), J_{\hat{j}}(t) \in \mathbf{R}$  and  $K_{\hat{j}}(t)$  is a controller. Assumptions 7.2.3 and 7.2.4 are transformed into the following assumptions.

**Assumption 7.3.1.** There exists  $F > 0$ , where  $F = \text{diag}\{F_1, F_2, \dots, F_n\}$  is a diagonal matrix s.t.  $|H_{\hat{j}}(E) - H_{\hat{j}}(\tilde{E})| \leq F_{\hat{j}}|E - \tilde{E}|$ , for all  $E, \tilde{E}, H_{\hat{j}}(\cdot) \in \mathbf{R}$  with  $\hat{j} \in \mathbf{N}$ .

**Assumption 7.3.2.** For  $I'_{\hat{j}}(t), J_{\hat{j}}(t) \in \mathbf{R}, \hat{j} \in \mathbf{N}, \exists$  positive matrices  $\hat{I}' = (\hat{I}'_1, \hat{I}'_2, \dots, \hat{I}'_n)^T$  and  $\hat{J} = (\hat{J}_1, \hat{J}_2, \dots, \hat{J}_n)^T$  such that  $|I'_{\hat{j}}(t)| \leq \hat{I}'_{\hat{j}}, |J_{\hat{j}}(t)| \leq \hat{J}_{\hat{j}}$ .

The synchronization error is stated as  $E_{\hat{j}}(t) = \tilde{u}_{\hat{j}}(t) - \tilde{\omega}_{\hat{j}}(t)$ . We can derive the following error system from equations (7.14) and (7.15) as

$$D^\alpha E_{\hat{j}}(t) = -d_{\hat{j}}E_{\hat{j}}(t) + \sum_{\hat{k}=1}^n c_{\hat{j}\hat{k}}H_{\hat{k}}(E_{\hat{k}}(t)) - (d_{\hat{j}} - b_{\hat{j}})\tilde{\omega}_{\hat{j}}(t) + \sum_{\hat{k}=1}^n \left\{ c_{\hat{j}\hat{k}}h_{\hat{k}}(\tilde{\omega}_{\hat{k}}(t)) - a_{\hat{j}\hat{k}}f_{\hat{k}}(\tilde{\omega}_{\hat{k}}(t)) \right\} + J_{\hat{j}}(t) - I'_{\hat{j}}(t) + K_{\hat{j}}(t). \quad (7.16)$$

We get the following result from the derivation of Theorem 7.3.1.

**Corollary 7.3.2.** Let us assume Assumptions 7.3.1 and 7.3.2 hold and there exist  $\psi_{\hat{j}} > 0$  such that

$$W_{\hat{j}} = (d_{\hat{j}} + k_{\hat{j}}) - \sum_{\hat{k}=1}^n \frac{\psi_{\hat{k}}}{\psi_{\hat{j}}} |c_{\hat{j}\hat{k}}| F_{\hat{j}} > 0.$$

Then the systems (7.14) and (7.15) are globally Lagrange  $\alpha$ -exponentially synchronized under the controllers

$$K_{\hat{j}}(t) = (d_{\hat{j}} - b_{\hat{j}})\tilde{\omega}_{\hat{j}}(t) - \sum_{\hat{k}=1}^n \left\{ c_{\hat{j}\hat{k}}h_{\hat{k}}(\tilde{\omega}_{\hat{k}}(t)) - a_{\hat{j}\hat{k}}f_{\hat{k}}(\tilde{\omega}_{\hat{k}}(t)) \right\} - k_{\hat{j}}E_{\hat{j}}(t),$$

where  $k_{\hat{j}} > 0$ .

Moreover, networks (7.14) and (7.15) are globally  $\alpha$ -exponentially convergent to the ball

$$\tilde{B}_1 = \left\{ E(t) \in \mathbf{R}^n : \sum_{\hat{j}=1}^n |E_{\hat{j}}(t)| \leq \frac{\beta}{W \min_{1 \leq \hat{j} \leq n} \{\psi_{\hat{j}}\}} \right\}, \quad (7.17)$$

where  $W = \min_{1 \leq \hat{j} \leq n} \{W_{\hat{j}}\}$ , and  $\beta = \sum_{\hat{j}=1}^n \sum_{\hat{k}=1}^n \psi_{\hat{j}} |c_{\hat{j}\hat{k}}| |H_{\hat{j}}(0)| + \sum_{\hat{j}=1}^n \psi_{\hat{j}} (\hat{J}_{\hat{j}}(t) + \hat{I}'_{\hat{j}}(t))$ .

**Remark 7.3.4.** The Lyapunov  $\alpha$ -exponential synchronization between (7.14) and (7.15) is investigated in [129], while the Lagrange  $\alpha$ -exponential synchronization of

(7.14) and (7.15) is investigated here with a rate  $\lambda = \frac{W}{\Gamma(\alpha+1)}$ . Corollary 7.3.2 with  $\beta = 0$  becomes Theorem 3 of [129]. To put it another way, the Theorem 3 of [129] is a particular part of Corollary 7.3.2. As a result, the conclusions drawn here are that our proposed synchronization result is more general as compared to the results of [129].

**Remark 7.3.5.** In [130], the analysis of the exponential synchronization of two different chaotic nonlinear identical competitive NNs with time varying delays has been investigated by using the active control, in which the sufficient criteria depends on the delay parameter. By using Lyapunov stability theory, the authors have used the method which is based on feedback control gain matrix. Whereas in this chapter Lagrange  $\alpha$ -exponential synchronization and  $\alpha$ -exponential convergence of non-identical FOCVNN has been investigated by using Lagrange stability. Lyapunov stability is a specific case of Lagrange stability because unlike Lyapunov stability, Lagrange stability shows the stability of total system, not the stability of an equilibrium point.

## 7.4 Numerical simulation

In this section the following numerical example is taken to show the effectiveness of the theoretical results.

Assume that the drive and response systems of fractional-order non-identical CVNNs are as follows:

$$D^{0.98}\omega(t) = -B\omega(t) + Af(\omega(t)) + I'(t), \quad (7.18)$$

$$D^{0.98}u(t) = -Du(t) + Ch(u(t)) + J(t) + K(t), \quad (7.19)$$

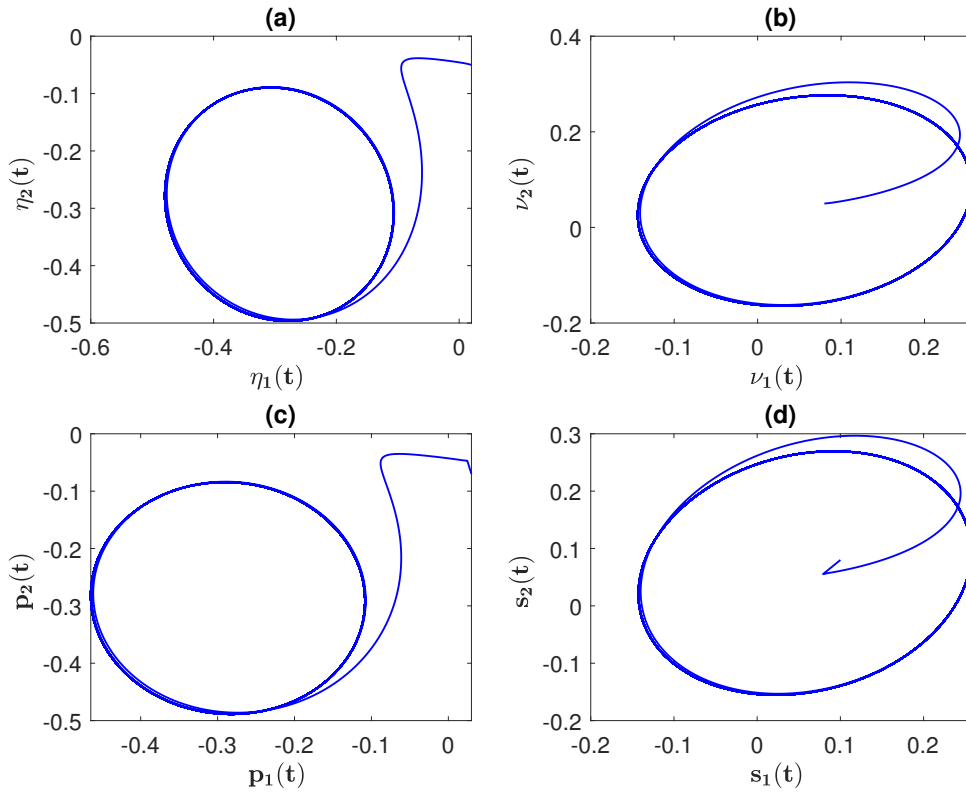


FIGURE 7.1: Plots of 2-D state trajectories of the drive and response systems.

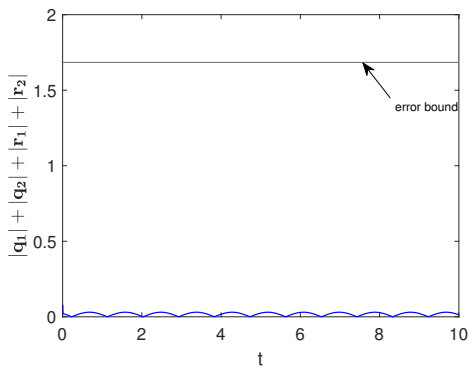


FIGURE 7.2: Plots of synchronization error of non-identical FOCVNNs in Lagrange sense.

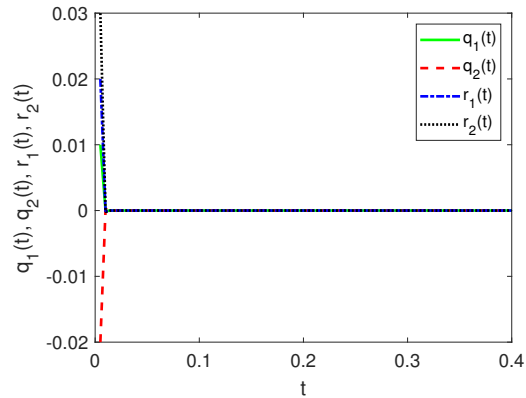


FIGURE 7.3: Plots of synchronization error of non-identical FOCVNNs in Lyapunov sense.

where  $B = D = \text{diag}\{9, 9\}$ ,  $A = \begin{bmatrix} 1+i & i \\ i & 1+i \end{bmatrix}$ ,  $C = \begin{bmatrix} 1+i & -i \\ -i & 1+i \end{bmatrix}$ ,

$$f_j(\omega_j(t)) = \frac{1-\exp(-\eta_j(t))}{1+\exp(-\eta_j(t))} + \frac{i}{1+\exp(-\nu_j(t))}, \quad h_j(u_j(t)) = \frac{1-\exp(-s_j(t))}{1+\exp(-s_j(t))} + \frac{i}{1+\exp(-p_j(t))}, \quad \text{for}$$

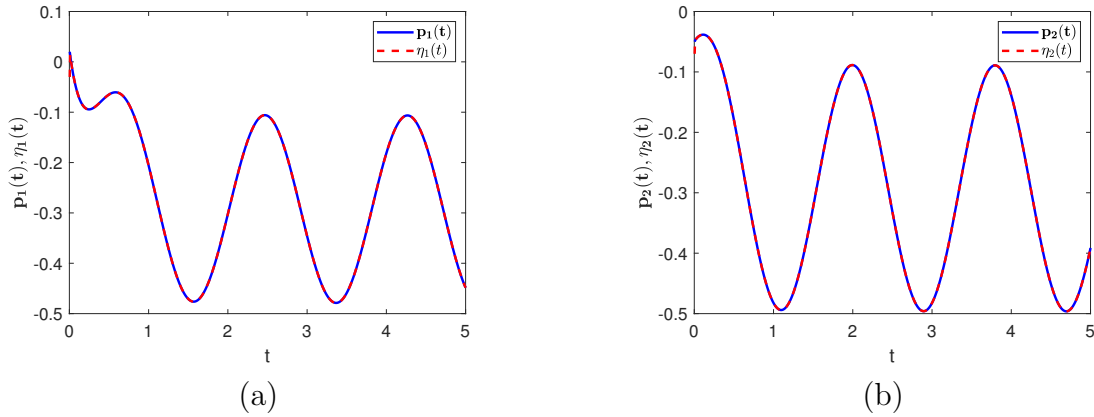


FIGURE 7.4: Plots of state trajectories of real components of the systems (7.18) and (7.19) with respect to time.

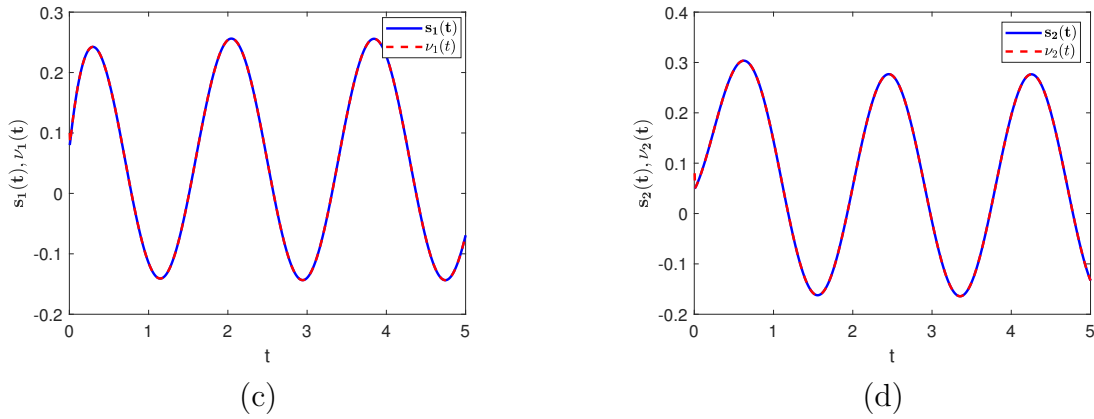


FIGURE 7.5: Plots of state trajectories of imaginary components of the systems (7.18) and (7.19) with respect to time.

$\hat{j} = 1, 2$ ,  $I_1'(t) = \sin(t) + \cos(t)i$ ,  $I_2'(t) = \cos(t) + \sin(t)i$ ,  $J_1(t) = \cos(t) + \sin(t)i$ ,  $J_2(t) = \sin(t) + \cos(t)i$  with initial conditions  $\omega_1(0) = 0.02 + 0.08i$ ,  $\omega_2(0) = -0.05 + 0.05i$ ,  $u_1(0) = 0.03 + 0.10i$ ,  $u_2(0) = -0.07 + 0.08i$  then the other parameters can be taken as  $F_j^{RR} = 0$ ,  $F_j^{RI} = 0.25$ ,  $F_j^{IR} = 0.5$ ,  $F_j^{II} = 0$ ,  $\hat{I}_j^R(t) = 1$ ,  $\hat{I}_j^I(t) = 1$ ,  $\hat{J}_j^R(t) = 1$ ,  $\hat{J}_j^I(t) = 1$ ,  $\hat{I}_j' = 1$ ,  $\hat{J}_j' = 1$  for ( $\hat{j} = 1, 2$ ). State curves of the drive and response systems in 2-D space are shown in Figure 7.1.

Case 1. Now choosing  $k_j(t) = -2.75 - 2.75i$  and  $A_j' = B_j' = 1$ , we can obtain  $W_j^1 = 4.75 > 0$ ,  $W_j^2 = 5.5 > 0$  ( $\hat{j} = 1, 2$ ),  $Y = 1$  and  $\beta = 8$  which satisfy all

the conditions of Theorem 7.3.1. Figure 7.2 clearly shows that the Lagrange  $\alpha$ -exponential synchronization between FOCVNNs (7.1) and (7.4) is achieved. Also networks (7.1) and (7.4) are globally  $\alpha$ -exponentially convergent to the ball  $B_1 = \left\{ E(t) = q(t) + ir(t) \in \mathbf{C}^n : \sum_{\hat{j}=1}^n \left( |q_{\hat{j}}(t)| + |r_{\hat{j}}(t)| \right) \leq \frac{8}{4.75} = 1.6842 \right\}$  with a rate 1.0083.

*Case 2.* Taking  $B = D = \text{diag}\{4, 4\}$  and the values of parameters as given in case 1, we get  $W_{\hat{j}}^1 = 1.0625 > 0$ ,  $W_{\hat{j}}^2 = 0.5 > 0$  ( $\hat{j} = 1, 2$ ) and  $\beta = 0$ , which meet all of the conditions of Corollary 7.3.1. Hence the systems (7.1) and (7.4) are  $\alpha$ -exponentially synchronized in Lyapunov sense which is shown in Figure 7.3. The trajectories of real and imaginary components of the systems (7.18) and (7.19) are shown through Figure 7.4 and Figure 7.5, respectively.

## 7.5 Conclusion

The present study focuses on globally achieving  $\alpha$ -exponential synchronization for FOCVNNs within the Lagrange framework. Effective criteria and inequalities have been explicitly derived for Lagrange  $\alpha$ -exponential synchronization, including establishing the framework for the  $\alpha$ -exponential convergence ball and estimating the exponential convergence rate, which is contingent upon the system's characteristics and differentiation order. Additionally, the study explores  $\alpha$ -exponential synchronization for FOCVNNs in the Lyapunov sense, investigating both the convergence ball and rate in  $n$ -dimensional real space. It is worth noting that our proposed results pertain to CVNNs. Hence, the outcomes of prior studies are encompassed as specific cases. This approach shows promise for application across various fractional-order complex-valued systems. The most intricate aspect of the research lies in selecting an appropriate activation function during the synchronization of FOCVNNs,

as this choice significantly impacts the synchronization process. While there exists literature on Lagrange  $\alpha$ -exponential stability for FOCVNNs, the investigation into Lagrange  $\alpha$ -exponential synchronization among non-identical FOCVNNs is unprecedented. Looking ahead, future research avenues include exploring Lagrange  $\alpha$ -exponential synchronization problems for FOCVNNs with time-variable and unbounded distributed delays.

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