Chapter 4

Numerical study of variable order fractional physical model arising in chemical processes by using operational matrix and collocation methods

4.1 Introduction

Many researchers have confirmed that a wide range of critical dynamical problems display fractional order behaviour that may vary with space and/or time. This fact signifies that variable order fractional calculus is a genuine contender to deliver an impressive mathematical structure for describing complex dynamical problems. A fractional variable order operator is an extended form of the fractional constant order operator. Bounded functions are used to represent fractional integrals and derivatives by this operator. In recent years, fractional variable order operators have been found to be more accurate in depicting many scientific and engineering models as compared to fractional constant order operators [101–104]. The fractional variable order operator makes nonlocal properties of systems more visible and it has been used to explain plenty of real-world phenomena in mechanics, physics, signal processing and control [105–107]. It is very challenging to analyse equations described by fractional variable order derivatives analytically because of their high complexity. So, it is extremely important to present proper methods for obtaining their approximate solutions. As a result, several numerical approaches for fractional differential equations with variable order have developed in recent years. In [108] the authors have approximated the linear cable equation of variable order using Bernstein polynomials. Legendre wavelets are used as a numerical technique in [109] for a class of fractional differential equation of variable order. In [110], a finite difference technique is used to solve the fractional variable order telegraph equation. An accurate and efficient hybrid scheme based on the Bernoulli polynomial and radial basis function has been developed in [111] to solve 2D advection-reaction-diffusion equations with variable order. A highly accurate numerical method is proposed in [112] to deal with variable order reaction-diffusion and subdiffusion equations. In addition, numerous numerical techniques are present in the literature for solving variable order fractional differential equations, such as those given in [113–117].

P. Gray and S.K. Scott have firstly proposed the Gray-Scott model in [118]. The Gray-Scott model is one of the most widely used reaction-diffusion systems(RDS), which is a set of PDEs describing the reaction, diffusion, and transportation of constituents with the aid of a chemical reaction. RDS arise in studying many disciplines, such as population dynamics, epidemiology, chemistry, biology etc. [119–122].

In this chapter, the generalized Gray-Scott model obtained from the time-fractional Gray-Scott model by replacing the time-fractional derivatives with fractional variable order derivatives is studied, which is given as follows

$$\begin{aligned} \frac{\partial^{\alpha_1(x,t)}u(x,t)}{\partial t^{\alpha_1(x,t)}} &= d_1\frac{\partial^2 u(x,t)}{\partial x^2} - u(x,t)v^2(x,t) + A(1-u(x,t)) + f_1(x,t), \\ \frac{\partial^{\alpha_2(x,t)}v(x,t)}{\partial t^{\alpha_2(x,t)}} &= d_2\frac{\partial^2 v(x,t)}{\partial x^2} + u(x,t)v^2(x,t) + Bv(x,t) + f_2(x,t), \end{aligned} \tag{4.1}$$

subject to the initial conditions (ICs)

$$u(x,0) = u_0(x), \quad v(x,0) = \tilde{u}_0(x), \tag{4.2}$$

and boundary conditions (BCs)

$$\begin{aligned} u(0,t) &= u_1(t), \quad v(0,t) = \tilde{u}_1(t), \\ u(L,t) &= u_2(t), \quad v(L,t) = \tilde{u}_2(t), \end{aligned} \tag{4.3}$$

where $\alpha_1(x,t)$ and $\alpha_2(x,t)$ denote the variable order derivatives which depend on space and time such that $0 < \alpha_1(x,t), \alpha_2(x,t) < 1$. The concentration of two chemical substances are described by unknown variables $u(x,t)$ and $v(x,t)$. d_1 and d_2 are the diffusion coefficients, and constants A and B are positive. $f_1(x,t)$ and $f_2(x,t)$ are the source terms.

Solving the Gray-Scott equations typically involves numerical methods such as the finite difference method, finite element method, and many other techniques. Manaa and Rasheed [123] have used the successive approximation method and finite difference method to solve the Gray-Scott model. Owolabi and Patidar [124] investigated numerical simulations of a coupled one-dimensional Gray-Scott model using a combination of higher-order exponential time differencing Runge-Kutta and a higher-order symmetric finite difference scheme. Korkmaz et al. [125] obtained the numerical solutions of the Gray-Scott model with the help of third-fourth order implicit Rosenbrock and exponential B-spline collocation methods. Yadav and Jiwari [126] employed the Galerkin finite element technique to obtain approximation of solutions using Lagrange basis functions. Aljhani et al. [127] utilize the fractional homotopy analysis transformation method to obtain approximate solutions for the time fractional Gray-Scott model. Onarcan et al. [128] investigated numerical solutions of the model via the trigonometric quintic B-spline finite element collocation method. Han et al. [129] introduced a high-precision numerical approach to solving the space fractional Gray-Scott model.

In this chapter, the main purpose is to develop a highly efficient and effective technique, namely a combination of shifted Vieta–Lucas collocation and the operational matrix method to find the approximate solution of the fractional variable order Gray-Scott model. We choose this extremely accurate technique because of the higher order convergence, even for small grid approximations. The shifted Vieta–Lucas collocation method is one of the most efficient ways of dealing with non-linear variable order coupled partial differential equations. It is very important to choose a collocation point for efficiency and convergence of the numerical scheme because it transforms the concerned model into a nonlinear algebraic system of equations that are then solved by Newton's iterative method. The spatial domain and temporal domain should be finite in order to use this proposed numerical technique because the method is based on the operational matrix and order of matrix is a finite natural number.

4.2 Vieta–Lucas polynomials

The Vieta–Lucas polynomials $\mathbb{VL}_m(y)$ of degree $m \in \mathbb{N}_0$ in the variable y defined on the interval $[-2, 2]$ is given by [130]

$$\mathbb{VL}_m(y) = 2\cos(m\phi), \quad \phi = cos^{-1}\left(\frac{y}{2}\right), \quad \phi \in [0, \pi].$$

The Vieta–Lucas polynomials $\mathbb{VL}_m(y)$ can be created from the following recurrence relation

$$\mathbb{VL}_m(y) = y\mathbb{VL}_{m-1}(y) - \mathbb{VL}_{m-2}(y), \quad m = 2, 3, ...,$$

with the initial values

$$\mathbb{VL}_0(y) = 2, \quad \mathbb{VL}_1(y) = y.$$

The explicit power series form of $\mathbb{VL}_m(y)$ is represented as

$$\mathbb{VL}_m(y) = \sum_{j=0}^{\lceil \frac{m}{2} \rceil} (-1)^j \frac{m\Gamma(m-j)}{\Gamma(j+1)\Gamma(m+1-2j)} y^{m-2j}, \quad m = 2, 3, ..., \tag{4.4}$$

where $\lceil \frac{m}{2} \rceil$ is the integer part of $\frac{m}{2}$.

These polynomials $\mathbb{VL}_m(y)$ are also orthogonal to [-2,2] w.r.t. the weight function $\frac{1}{\sqrt{4-y^2}}$ as

$$\langle \mathbb{VL}_l(y), \mathbb{VL}_m(y) \rangle = \int_{-2}^{2} \frac{1}{\sqrt{4-y^2}} \mathbb{VL}_l(y)\mathbb{VL}_m(y)dy = \begin{cases} 0, & l \neq m \neq 0, \\ 2\pi, & l = m \neq 0, \\ 4\pi, & l = m = 0. \end{cases} \tag{4.5}$$

4.2.1 Shifted Vieta–Lucas polynomials

The shifted Vieta–Lucas polynomials $\mathbb{VL}_m^*(y)$ of degree m in y on $[0, 1]$ are given by

$$\mathbb{VL}_m^*(y) = \mathbb{VL}_m(4y - 2).$$

Also, polynomials $\mathbb{VL}_m^*(y)$ are obtained by using the recurrence formula

$$\mathbb{VL}_{m+1}^*(y) = (4y - 2)\mathbb{VL}_m^*(y) - \mathbb{VL}_{m-1}^*(y), \quad m = 1, 2, ...,$$

with the initial values

$$\mathbb{VL}_0^*(y) = 2, \quad \mathbb{VL}_1^*(y) = 4y - 2.$$

Moreover, one can generate the explicit analytical form of $\mathbb{VL}_m^*(y)$ by using the following formula

$$\mathbb{VL}_m^*(y) = 2m \sum_{j=0}^{m} (-1)^j \frac{4^{m-j} \Gamma(2m-j)}{\Gamma(j+1)\Gamma(2m-2j+1)} y^{m-j}, \quad m = 2, 3, ..., \tag{4.6}$$

where the polynomials $\mathbb{VL}_m^*(y)$ are orthogonal w.r.t. the following inner product

$$\langle \mathbb{VL}_l^*(y), \mathbb{VL}_m^*(y) \rangle = \int_0^1 w(y) \mathbb{VL}_l^*(y) \mathbb{VL}_m^*(y) dy = \begin{cases} 0, & l \neq m \neq 0, \\ 2\pi, & l = m \neq 0, \\ 4\pi, & l = m = 0, \end{cases} \tag{4.7}$$

where $w(y) = \frac{1}{\sqrt{y-y^2}}$ is the weight function.

4.3 Function approximation

Let the function $u(y) \in L^2[0,1]$, then the function $u(y)$ can be written as a power series of the shifted Vieta–Lucas polynomials as

$$u(y) = \sum_{j=0}^{\infty} b_j \mathbb{VL}_j^*(y), \tag{4.8}$$

where the coefficients b_j are unknown.

In general, the series in equation (4.8) can be approximated by a finite sum of $(m+1)$ terms of shifted Vieta–Lucas polynomials as

$$u_m(y) = \sum_{j=0}^{m} b_j \mathbb{VL}_j^*(y) = B^T \Omega(y), \tag{4.9}$$

where

$$B^T = [b_0, b_1, ..., b_m], \quad \Omega(y) = [\mathbb{VL}_0^*(y), \mathbb{VL}_1^*(y), ..., \mathbb{VL}_m^*(y)]^T, \tag{4.10}$$

and $b_j, j = 0, 1, ..., m$ are unknown coefficients which can be obtained by the following expression

$$b_j = \frac{1}{\delta_j \pi} \int_{-2}^{2} \frac{u\left(\frac{y+2}{4}\right) \mathbb{VL}_j(y)}{\sqrt{4-y^2}} dy, \tag{4.11}$$

or,

$$b_j = \frac{1}{\delta_j \pi} \int_{0}^{1} \frac{u(y)\mathbb{VL}_j^*(y)}{\sqrt{y-y^2}} dy, \tag{4.12}$$

where

$$\delta_j = \begin{cases} 4, & j = 0, \\ 2, & j = 1, 2, ..., m. \end{cases} \tag{4.13}$$

Suppose an arbitrary function $u(y,t) \in L^2[0,1] \times [0,1]$, it can be expanded in terms of the Vieta–Lucas polynomials as

$$u(y,t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} u_{jk} \mathbb{VL}_j^*(y) \mathbb{VL}_k^*(t), \tag{4.14}$$

where

$$u_{jk} = \frac{1}{\delta_j \delta_k \pi^2} \int_0^1 \int_0^1 u(y,t)\mathbb{VL}_j^*(y)\mathbb{VL}_k^*(t) w(y) w(t) dy dt, \quad j,k = 0, 1, 2, ... \tag{4.15}$$

The truncated series of equation(4.14), can be considered as

$$u(y,t) \approx \sum_{j=0}^{m} \sum_{k=0}^{n} u_{jk} \mathbb{VL}_j^*(y) \mathbb{VL}_k^*(t), = \Omega^T(y) U \Omega(t) \tag{4.16}$$

where

$$\Omega(y) = [\mathbb{VL}_0^*(y), \mathbb{VL}_1^*(y), ..., \mathbb{VL}_m^*(y)]^T,$$
$$\Omega(t) = [\mathbb{VL}_0^*(t), \mathbb{VL}_1^*(t), ..., \mathbb{VL}_n^*(t)]^T, U = (u_{jk})_{j,k=0}^{m,n}. \tag{4.17}$$

Theorem 1. As a result of shifting the Lucas vector $\Omega(y)$, we get its first order derivative as

$$\frac{d\Omega(y)}{dy} = P^{(1)}\Omega(y), \tag{4.18}$$

where $P^{(1)} = (p_{ij})_{(m+1)\times(m+1)}$ is an operational matrix of the derivatives, such that

$$p_{ij} = \begin{cases} 4i, \quad for\ j = i-k, & \begin{cases} k = 1, 3, ..., m, & m\ odd, \\ k = 1, 3, ..., m-1, & m\ even, \end{cases} \\ 0, \quad otherwise. \end{cases} \tag{4.19}$$

The n^{th} order derivatives of vector $\Omega(y)$ is given as

$$\frac{d^n\Omega(y)}{dy^n} = (P^{(1)})^n\Omega(y) = P^{(n)}\Omega(y),$$

where n is a natural number.

Proof. The proof is not given as it is straightforward.

4.4 Generating operational matrix of variable order fractional derivative

Theorem 2. Let $\Omega(y)$ be the shifted Vieta–Lucas vector defined in equation (4.10) and $\beta(y,t) > 0$, then

$$D_y^{\beta(y,t)}(\Omega(y)) \simeq P^{(\beta(y,t))}\Omega(y), \tag{4.20}$$

where $P^{(\beta(y,t))}$ is the $(m+1)\times(m+1)$ operational matrix of the Caputo variable derivative of order $\beta(y,t)$ and it is defined by

$$P^{(\beta(y,t))} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{l=0}^{j-q}\Theta_{q,0,l} & \sum_{l=0}^{j-q}\Theta_{q,1,l} & \cdots & \sum_{l=0}^{j-q}\Theta_{q,m,l} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{l=0}^{j-q}\Theta_{r,0,l} & \sum_{l=0}^{j-q}\Theta_{r,1,l} & \cdots & \sum_{l=0}^{j-q}\Theta_{r,m,l} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{l=0}^{j-q}\Theta_{m,0,l} & \sum_{l=0}^{j-q}\Theta_{m,1,l} & \cdots & \sum_{l=0}^{j-q}\Theta_{m,m,l} \end{pmatrix}$$

where

$$\Theta_{j,k,l} = \begin{cases} \frac{y^{-\beta(y,t)}(-1)^l 4^{j-l}\ j\ \Gamma(2j-l)\ \Gamma(j-l+\frac{1}{2})}{\sqrt{\pi}\ \Gamma(l+1)\ \Gamma(2j-2l+1)(\Gamma(j-l-\beta(y,t)+1))}, & k=0, \\ \frac{y^{-\beta(y,t)}(-1)^l 4^{j-l}\ 2j\ \Gamma(2j-l)\ \Gamma(j-l+1)}{\sqrt{\pi}\ \Gamma(l+1)\ \Gamma(2j-2l+1)(\Gamma(j-l-\beta(y,t)+1))} & \\ \times \sum_{p=0}^{k} \frac{(-1)^p 4^{k-p}\ k\ \Gamma(2k-p)\ \Gamma(j+k-p-l+\frac{1}{2})}{\Gamma(p+1)\ \Gamma(2k-2p+1)\ \Gamma(j+k-p-l+1}, & k=1,2,...m. \end{cases}$$

Proof. Using the Caputo derivative in equation (4.6), we get

$$D_y^{\beta(y,t)}(\mathbb{VL}_j^*(y)) = D_y^{\beta(y,t)}\left(\sum_{l=0}^{j}(-1)^l\frac{4^{j-l}\ 2j\ \Gamma(2j-l)}{\Gamma(l+1)\Gamma(2j-2l+1)}y^{j-l}\right). \tag{4.21}$$

By linearity property of the Caputo derivative, we have

$$D_y^{\beta(y,t)}(\mathbb{VL}_j^*(y)) = \sum_{l=0}^{j}(-1)^l\frac{4^{j-l}\ 2j\ \Gamma(2j-l)}{\Gamma(l+1)\Gamma(2j-2l+1)}D_y^{\beta(y,t)}y^{j-l}. \tag{4.22}$$

Using equation (1.19), we get

$$D_y^{\beta(y,t)}(\mathbb{VL}_j^*(y)) = 0, \quad j = 0, 1, ..., q-1, \tag{4.23}$$

and

$$\begin{aligned} D_y^{\beta(y,t)}(\mathbb{VL}_j^*(y)) &= \sum_{l=0}^{j-q}(-1)^l\frac{4^{j-l}\ 2j\ \Gamma(2j-l)\ \Gamma(j-l+1)}{\Gamma(l+1)\Gamma(2j-2l+1)\ \Gamma(j-l-\beta(y,t)+1)}y^{j-l-\beta(y,t)} \\ &= y^{-\beta(y,t)}\sum_{l=0}^{j-q}(-1)^l\frac{4^{j-l}\ 2j\ \Gamma(2j-l)\ \Gamma(j-l+1)}{\Gamma(l+1)\Gamma(2j-2l+1)\ \Gamma(j-l-\beta(y,t)+1)}y^{j-l}, \quad j = q, ..., m. \end{aligned} \tag{4.24}$$

Now, approximate y^{j-l} by $(m+1)$ terms of the Vieta–Lucas series, we have

$$y^{j-l} \simeq \sum_{k=0}^{m} b_{lk}\mathbb{VL}_k^*(y), \tag{4.25}$$

where

$$b_{lk} = \frac{1}{\delta_k\pi}\int_0^1 \frac{y^{j-l}\mathbb{VL}_k^*(y)}{\sqrt{y-y^2}}dy. \tag{4.26}$$

Now, putting the value of $\mathbb{VL}_k^*(y)$ into equation (4.26) and using equation (4.13), we get

$$b_{lk} = \begin{cases} \frac{1}{4\pi}\int_0^1 \frac{2y^{j-l}}{\sqrt{y-y^2}}dy, & k=0, \\ \frac{1}{2\pi}\int_0^1 \frac{y^{j-l}}{\sqrt{y-y^2}}\sum_{p=0}^{k}\frac{(-1)^p 4^{k-p}\ 2k\ \Gamma(2k-p)}{\Gamma(p+1)\ \Gamma(2k-2p+1)}y^{k-p}dy, & k=1,2,...,m, \end{cases}$$

$$= \begin{cases} \frac{1}{2\pi}\int_0^1 \frac{y^{j-l}}{\sqrt{y-y^2}}dy, & k=0, \\ \frac{1}{\pi}\sum_{p=0}^{k}\frac{(-1)^p 4^{k-p}\ k\ \Gamma(2k-p)}{\Gamma(p+1)\ \Gamma(2k-2p+1)}\int_0^1 \frac{y^{j+k-p-l}}{\sqrt{y-y^2}}dy, & k=1,2,...,m, \end{cases}$$

$$= \begin{cases} \frac{1}{2\sqrt{\pi}}\frac{\Gamma(j-l+\frac{1}{2})}{\Gamma(j-l+1)}, & k=0, \\ \frac{1}{\sqrt{\pi}}\sum_{p=0}^{k}\frac{(-1)^p 4^{k-p}\ k\ \Gamma(2k-p)\ \Gamma(j+k-p-l+\frac{1}{2})}{\Gamma(p+1)\ \Gamma(2k-2p+1)\ \Gamma(j+k-p-l+1}, & k=1,2,...m. \end{cases} \tag{4.27}$$

Putting equations (4.25) and (4.27) into equation (4.24), we obtain

$$D_y^{\beta(y,t)}(\mathbb{VL}_j^*(y)) \simeq \sum_{k=0}^{m}\sum_{l=0}^{j-q}\Theta_{j,k,l}\mathbb{VL}_k^*(y), \qquad j=q,...,m, \tag{4.28}$$

where

$$\Theta_{j,k,l} = \begin{cases} \frac{y^{-\beta(y,t)}(-1)^l 4^{j-l}\ j\ \Gamma(2j-l)\ \Gamma(j-l+\frac{1}{2})}{\sqrt{\pi}\ \Gamma(l+1)\ \Gamma(2j-2l+1)(\Gamma(j-l-\beta(y,t)+1))}, & k=0, \\ \frac{y^{-\beta(y,t)}(-1)^l 4^{j-l}\ 2j\ \Gamma(2j-l)\ \Gamma(j-l+1)}{\sqrt{\pi}\ \Gamma(l+1)\ \Gamma(2j-2l+1)(\Gamma(j-l-\beta(y,t)+1))} & \\ \times\sum_{p=0}^{k}\frac{(-1)^p 4^{k-p}\ k\ \Gamma(2k-p)\ \Gamma(j+k-p-l+\frac{1}{2})}{\Gamma(p+1)\ \Gamma(2k-2p+1)\ \Gamma(j+k-p-l+1}, & k=1,2,...m. \end{cases}$$

Equation (4.28) can be expressed in vector form as follows

$$D_y^{\beta(y,t)}(\mathbb{VL}_j^*(y)) \simeq \left[\sum_{l=0}^{j-q}\Theta_{j,0,l}, \sum_{l=0}^{j-q}\Theta_{j,1,l}, ..., \sum_{l=0}^{j-q}\Theta_{j,m,l}\right]\Omega(y). \tag{4.29}$$

Also, the equation (4.23) can be written as

$$D_y^{\beta(y,t)}(\mathbb{VL}_j^*(y)) \simeq [0,0,...,0]\,\Omega(y), \quad j=0,1,...,q-1. \tag{4.30}$$

The combination of equations (4.29) and (4.30) prove the desired results.

4.5 Description of the presented technique

The spectral collocation approach paired with the shifted Vieta–Lucas operational matrix of derivatives is provided in this section to solve fractional variable order Gray-Scott models, as shown in equations (4.1)-(4.3). Let us approximate $u(x,t)$ and $v(x,t)$ by shifted Vieta–Lucas polynomials as

$$\begin{aligned} u(x,t) &\approx \Omega^T(x)U\Omega(t),\\ v(x,t) &\approx \Omega^T(x)V\Omega(t), \end{aligned} \tag{4.31}$$

where $U=[u_{jk}]$ and $V=[\tilde{u}_{jk}]$ are unknown $(m+1)\times(m+1)$ matrices and $\Omega(t)=[\mathbb{VL}_0^*(t),\mathbb{VL}_1^*(t),...,\mathbb{VL}_m^*(t)]^T$ is the column vector. Now applying the Caputo variable operator $D_t^{\beta(x,t)}$ on the equation (4.31) and using Theorem (2), we get

$$\begin{aligned} D_t^{\alpha_1(x,t)}u(x,t) = \frac{\partial^{\alpha_1(x,t)}u(x,t)}{\partial t^{\alpha_1(x,t)}} \approx & \frac{\partial^{\alpha_1(x,t)}}{\partial t^{\alpha_1(x,t)}}\left(\Omega^T(x)U\Omega(t)\right)\\ =&\Omega^T(x)U\frac{\partial^{\alpha_1(x,t)}\Omega(t)}{\partial t^{\alpha_1(x,t)}}\\ =&\Omega^T(x)UP^{(\alpha_1(x,t))}\Omega(t), \end{aligned} \tag{4.32}$$

similarly

$$D_t^{\alpha_2(x,t)} v(x,t) = \frac{\partial^{\alpha_2(x,t)} v(x,t)}{\partial t^{\alpha_2(x,t)}} \approx \Omega^T(x) V P^{(\alpha_2(x,t))} \Omega(t), \tag{4.33}$$

and

$$\begin{aligned} \frac{\partial^2 u(x,t)}{\partial x^2} &\approx \Omega^T(x)(P^2)^T U \Omega(t), \\ \frac{\partial^2 v(x,t)}{\partial x^2} &\approx \Omega^T(x)(P^2)^T V \Omega(t). \end{aligned} \tag{4.34}$$

Using equations (4.31)-(4.34), the residuals $R_1(x,t)$ and $R_2(x,t)$ for equation (4.1) can be written as

$$\begin{aligned} R_1(x,t) =& \Omega^T(x) U P^{(\alpha_1(x,t))} \Omega(t) - d_1 \Omega^T(x)(P^2)^T U \Omega(t) + \Omega^T(x) U \Omega(t) (\Omega^T(x) V \Omega(t))^2 \\ & - A(1 - \Omega^T(x) U \Omega(t)) - f_1(x,t), \end{aligned} \tag{4.35}$$

$$\begin{aligned} R_2(x,t) =& \Omega^T(x) V P^{(\alpha_2(x,t))} \Omega(t) - d_2 \Omega^T(x)(P^2)^T V \Omega(t) - \Omega^T(x) U \Omega(t) (\Omega^T(x) V \Omega(t))^2 \\ & + B \Omega^T(x) V \Omega(t) - f_2(x,t), \end{aligned} \tag{4.36}$$

now using equation (4.31), we approximate the initial and boundary conditions, we get

$$\Omega^T(x) U \Omega(0) = u_0(x) \quad \Omega^T(x) V \Omega(0) = \tilde{u}_0(x), \tag{4.37}$$

and

$$\begin{aligned} \Omega^T(0) U \Omega(t) = u_1(t), &\quad \Omega^T(0) V \Omega(t) = \tilde{u}_1(t), \\ \Omega^T(L) U \Omega(t) = u_2(t), &\quad \Omega^T(L) U \Omega(t) = \tilde{u}_2(t). \end{aligned} \tag{4.38}$$

Now collocating equations (4.35)-(4.38) at points $x_i = \frac{2i+1}{2m+2}$ for $i = 1, 2, ..., m-1$ and $t_j = \frac{2j+1}{2m+2}$ for $j = 1, 2, ..., m$, we obtain a system of algebraic equations. By solving this algebraic system, we get the unknown matrices U and V. The approximate solutions can be obtained by substituting U and V in equation (4.31).

4.6 Convergence analysis

Theorem 3. Suppose that $g_{nm}(x,t)$ is an approximation of a continuous function $g(x,t)$ in terms of shifted Vieta–Lucas polynomials. If the function $g(x,t)$ has fourth order derivatives that are bounded, then

$$\begin{aligned}
&\left|c_{0,0}\right| \le \frac{1}{4}N_{0,0}, \quad \left|c_{1,0}\right| \le \frac{1}{2}N_{0,0}, \quad \left|c_{0,1}\right| \le \frac{1}{2}N_{0,0}, \quad \left|c_{1,1}\right| \le N_{0,0},\\
&\left|c_{i,0}\right| \le \frac{1}{8i(i-1)}N_{2,0}, \quad \left|c_{i,1}\right| \le \frac{1}{8i(i-1)}N_{2,0}, \quad i \ge 2,\\
&\left|c_{0,j}\right| \le \frac{1}{8j(j-1)}N_{0,2}, \quad \left|c_{1,j}\right| \le \frac{1}{8j(j-1)}N_{0,2}, \quad j \ge 2,\\
&\left|c_{i,j}\right| \le \frac{N_{2,2}}{16ij(i-1)(j-1)}, \quad i,j \ge 2,
\end{aligned} \tag{4.39}$$

where

$$\begin{aligned}
N_{0,0} &= max\left\{\left|g(x,t)\right| : (x,t) \in ([0,1]\times[0,1])\right\},\\
N_{2,0} &= max\left\{\left|g_{xx}(x,t)\right| : (x,t) \in ([0,1]\times[0,1])\right\}\\
N_{0,2} &= max\left\{\left|g_{tt}(x,t)\right| : (x,t) \in ([0,1]\times[0,1])\right\},\\
N_{2,2} &= max\left\{\left|g_{xxtt}(x,t)\right| : (x,t) \in ([0,1]\times[0,1])\right\}.
\end{aligned}$$

Proof. Assume that $g(x,t)$ can be expressed as a series of Vieta–Lucas polynomials as

$$g_{nm}(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{m} c_{i,j}\mathbb{VL}_i^*(x)\mathbb{VL}_j^*(t),$$

where unknowns can be determined using the following relation

$$c_{i,j} = \frac{1}{\pi^2\delta_i\delta_j}\int_0^1\int_0^1 g(x,t)\mathbb{VL}_i^*(x)w(x)\mathbb{VL}_i^*(t)w(t)dxdt. \tag{4.40}$$

Substituting $x = \frac{2+2\cos(\theta)}{4}$ and $t = \frac{2+2\cos(\phi)}{4}$ in equation (4.40), we get

$$c_{i,j} = \frac{4}{\pi^2\delta_i\delta_j}\int_0^\pi\int_0^\pi g\Big(\frac{2+2\cos(\theta)}{4}, \frac{2+2\cos(\phi)}{4}\Big)\cos(i\theta)\cos(j\phi)d\theta d\phi. \tag{4.41}$$

From equation (4.41), we have

$$c_{0,0} = \frac{4}{\pi^2\delta_0\delta_0}\int_0^\pi\int_0^\pi g\Big(\frac{1+\cos(\theta)}{2}, \frac{1+\cos(\phi)}{2}\Big)d\theta d\phi. \tag{4.42}$$

As a results

$$\big|c_{0,0}\big| \leq \frac{1}{4}N_{0,0}, \tag{4.43}$$

where $N_{0,0} = max\left\{\big|g(x,t)\big| : (x,t) \in ([0,1]\times[0,1])\right\}$.

A similar argument can be used to determine $\big|c_{1,0}\big| = \big|c_{0,1}\big| \leq \frac{1}{2}N_{0,0}$ and $\big|c_{1,1}\big| \leq N_{0,0}$.

From equation (4.41), for $i \geq 2$, we obtain

$$c_{i,0} = \frac{1}{2\pi^2}\int_0^\pi\int_0^\pi g\Big(\frac{1+\cos(\theta)}{2}, \frac{1+\cos(\phi)}{2}\Big)\cos(i\theta)d\theta d\phi. \tag{4.44}$$

Applying the integration by parts twice w.r.t. θ, we get

$$c_{i,0} = \frac{1}{16\pi^2}\int_0^\pi\int_0^\pi g_{\theta\theta}\Big(\frac{1+\cos(\theta)}{2}, \frac{1+\cos(\phi)}{2}\Big)\sin(\theta)\eta_i(\theta)d\theta d\phi, \tag{4.45}$$

where

$$\eta_i(\theta) = \frac{\sin(i-1)\theta}{i-1} - \frac{\sin(i+1)\theta}{i+1}.$$

After solving equation (4.45) provides the relation

$$\left|c_{i,0}\right| \leq \frac{1}{8i(i-1)} N_{2,0}, \quad i \geq 2, \tag{4.46}$$

where $N_{2,0} = max\left\{\left|g_{xx}(x,t)\right| : (x,t) \in ([0,1] \times [0,1])\right\}$.

In the same way, we can achieve the following outcomes

$$\left|c_{i,1}\right| \leq \frac{1}{8i(i-1)} N_{2,0}, \quad i \geq 2, \tag{4.47}$$

$$\left|c_{0,j}\right| \leq \frac{1}{8j(j-1)} N_{0,2}, \quad j \geq 2, \tag{4.48}$$

$$\left|c_{1,j}\right| \leq \frac{1}{8j(j-1)} N_{0,2}, \quad j \geq 2, \tag{4.49}$$

where $N_{0,2} = max\left\{\left|g_{tt}(x,t)\right| : (x,t) \in ([0,1] \times [0,1])\right\}$.

From equation (4.41), for $i, j \geq 2$, we have

$$c_{i,j} = \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} g\Big(\frac{1+\cos(\theta)}{2}, \frac{1+\cos(\phi)}{2}\Big) \cos(i\theta)\cos(j\phi) d\theta d\phi. \tag{4.50}$$

Now, applying the integration by parts twice w.r.t. θ, we have

$$c_{i,j} = \frac{1}{8\pi^2} \int_0^{\pi} \int_0^{\pi} g_{\theta\theta}\Big(\frac{1+\cos(\theta)}{2}, \frac{1+\cos(\phi)}{2}\Big) \sin(\theta)\eta_i(\theta)\cos(j\phi) d\theta d\phi, \tag{4.51}$$

Again integrating equation (4.51) twice w.r.t. ϕ, we achieve

$$c_{i,j} = \frac{1}{64\pi^2} \int_0^{\pi} \int_0^{\pi} g_{\theta\theta\phi\phi}\Big(\frac{1+\cos(\theta)}{2}, \frac{1+\cos(\phi)}{2}\Big) \sin(\theta)\eta_i(\theta)\cos(\phi)\eta_j(\phi) d\theta d\phi, \tag{4.52}$$

where

$$\eta_j(\phi) = \frac{\sin(j-1)\phi}{j-1} - \frac{\sin(j+1)\phi}{j+1}.$$

Now, taking modulus on both the sides of equation (4.52), we have

$$\left|c_{i,j}\right| \leq \frac{N_{2,2}}{16\pi^2\delta_i\delta_j}\int_0^\pi\int_0^\pi \left|\eta_i(\theta)\eta_j(\phi)\right| d\theta d\phi. \tag{4.53}$$

After performing some mathematical calculations, we get following inequality

$$\left|c_{i,j}\right| \leq \frac{N_{2,2}}{16ij(i-1)(j-1)}, \qquad i,j \geq 2, \tag{4.54}$$

where $N_{2,2} = max\left\{\left|g_{xxtt}(x,t)\right| : (x,t) \in ([0,1]\times[0,1])\right\}.$

Thus, the desired result is obtained.

Theorem 4. If the fourth order partial derivative of a continuous function $g(x,t)$ defined on $[0,1]\times[0,1]$ is bounded, then the error bound will be

$$\sigma_n \leq \sqrt{\Psi(n,m)},$$

where

$$\sigma_n = \left(\int_0^1\int_0^1 \left[g(x,t) - \sum_{i=0}^{n}\sum_{j=0}^{m} c_{i,j}\mathbb{VL}_i^*(x)\mathbb{VL}_j^*(t)\right]^2 w(x)w(t)dxdt\right)^{\frac{1}{2}},$$

and

$$\Psi(n,m) = \frac{\pi^2}{16}\left[\frac{N_{0,2}^2}{m^3} + \frac{N_{2,0}^2}{n^3} + \frac{\pi^4 N_{2,2}^2}{1080n^3} + \frac{N_{2,2}^2}{1080m^3}\left(\pi^4 - 15F_3(n)\right)\right]$$

Proof.

$$
\begin{aligned}
\sigma_n^2 &= \int_0^1 \int_0^1 \Big[g(x,t) - \sum_{i=0}^{n}\sum_{j=0}^{m} c_{i,j} \mathbb{VL}_i^*(x)\mathbb{VL}_j^*(t) \Big]^2 w(x)w(t)dxdt, \\
&= \int_0^1 \int_0^1 \Big[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} c_{i,j} \mathbb{VL}_i^*(x)\mathbb{VL}_j^*(t) - \sum_{i=0}^{n}\sum_{j=0}^{m} c_{i,j} \mathbb{VL}_i^*(x)\mathbb{VL}_j^*(t) \Big]^2 w(x)w(t)dxdt, \\
&= \int_0^1 \int_0^1 \Big[\sum_{i=0}^{n}\sum_{j=m+1}^{\infty} c_{i,j} \mathbb{VL}_i^*(x)\mathbb{VL}_j^*(t) + \sum_{i=n+1}^{\infty}\sum_{j=0}^{\infty} c_{i,j} \mathbb{VL}_i^*(x)\mathbb{VL}_j^*(t) \Big]^2 w(x)w(t)dxdt, \\
&= \sum_{i=0}^{n}\sum_{j=m+1}^{\infty} c_{i,j}^2 \left(\int_0^1 \mathbb{VL}_i^{*2}(x)w(x)dx \right) \left(\int_0^1 \mathbb{VL}_j^{*2}(t)w(t)dt \right) \\
&\qquad + \sum_{i=n+1}^{\infty}\sum_{j=0}^{\infty} c_{i,j}^2 \left(\int_0^1 \mathbb{VL}_i^{*2}(x)w(x)dx \right) \left(\int_0^1 \mathbb{VL}_j^{*2}(t)w(t)dt \right).
\end{aligned}
$$

Using the orthogonality condition of the shifted Vieta–Lucas polynomials, we have

$$
\begin{aligned}
\sigma_n^2 &= 8\pi^2 \sum_{j=m+1}^{\infty} c_{0,j}^2 + 4\pi^2 \sum_{j=m+1}^{\infty} c_{1,j}^2 + 4\pi^2 \sum_{i=2}^{n}\sum_{j=m+1}^{\infty} c_{i,j}^2 + 8\pi^2 \sum_{i=n+1}^{\infty} c_{i,0}^2 \\
&\quad + 4\pi^2 \sum_{i=n+1}^{\infty} c_{i,1}^2 + 4\pi^2 \sum_{i=n+1}^{\infty}\sum_{j=2}^{\infty} c_{i,j}^2.
\end{aligned}
$$

Now, from Theorem 3, we get

$$
\begin{aligned}
\sigma_n^2 &\le \frac{3}{16}\pi^2 N_{0,2}^2 \sum_{j=m+1}^{\infty} \frac{1}{j^2(j-1)^2} + \frac{1}{64}\pi^2 N_{2,2}^2 \sum_{i=2}^{n}\sum_{j=m+1}^{\infty} \frac{1}{i^2 j^2 (i-1)^2 (j-1)^2} \\
&\quad + \frac{3}{16}\pi^2 N_{2,0}^2 \sum_{i=n+1}^{\infty} \frac{1}{i^2(i-1)^2} + \frac{1}{64}\pi^2 N_{2,2}^2 \sum_{i=n+1}^{\infty}\sum_{j=2}^{\infty} \frac{1}{i^2 j^2 (i-1)^2 (j-1)^2}.
\end{aligned}
$$

$$
\begin{aligned}
\sigma_n^2 &< \frac{3}{16}\pi^2 N_{0,2}^2 \sum_{j=m+1}^{\infty} \frac{1}{(j-1)^4} + \frac{1}{64}\pi^2 N_{2,2}^2 \sum_{i=2}^{n} \frac{1}{(i-1)^4} \times \sum_{j=m+1}^{\infty} \frac{1}{(j-1)^4} \\
&\quad + \frac{3}{16}\pi^2 N_{2,0}^2 \sum_{i=n+1}^{\infty} \frac{1}{(i-1)^4} + \frac{1}{64}\pi^2 N_{2,2}^2 \sum_{i=n+1}^{\infty} \frac{1}{(i-1)^4} \times \sum_{j=2}^{\infty} \frac{1}{(j-1)^4},
\end{aligned}
$$

$$\sigma_n^2 < \frac{3}{16}\pi^2 N_{0,2}^2 \int_{m+1}^{\infty} \frac{1}{(t-1)^4} dt + \frac{1}{64}\pi^2 N_{2,2}^2 \times \frac{(\pi^4 - 15F_3(n))}{90} \times \int_{m+1}^{\infty} \frac{1}{(t-1)^4} dt$$
$$+ \frac{3}{16}\pi^2 N_{2,0}^2 \int_{n+1}^{\infty} \frac{1}{(x-1)^4} dx + \frac{1}{64}\pi^2 N_{2,2}^2 \times \int_{n+1}^{\infty} \frac{1}{(x-1)^4} dx \times \frac{\pi^4}{90}.$$

After some mathematical calculation, we get

$$\sigma_n^2 < \frac{\pi^2}{16}\left[\frac{N_{0,2}^2}{m^3} + \frac{N_{2,0}^2}{n^3} + \frac{\pi^4 N_{2,2}^2}{1080n^3} + \frac{N_{2,2}^2}{1080m^3}\Big(\pi^4 - 15F_3(n)\Big)\right] \tag{4.55}$$

where $F_k(t)$ is the polygamma function defined by

$$F_k(t) = (-1)^{k+1} k! \sum_{i=0}^{\infty} \frac{1}{(t+i)^{k+1}}$$

Finally, square root of equation (4.55) gives the required result.

4.7 Numerical examples

This section contains three examples to validate the proposed method. The maximum absolute error (MAE) and the convergence order (CO) have been calculated for each example. Using MAE and CO, it is numerically analyzed the errors and convergence analysis. The convergence order of the method is defined by [131]

$$CO = \frac{log\left(\frac{\bar{e_1}}{\bar{e_2}}\right)}{log\left(\frac{Q_2}{Q_1}\right)}, \tag{4.56}$$

where $\bar{e_1}$ and $\bar{e_2}$ are the first and second MAE values, respectively. Moreover, $Q_j = (m_j + 1)(n_j + 1)$ for $j = 1,\ 2$ represents the number of the shifted Vieta–Lucas polynomials used in the first and second experiments, respectively.

Example 1. Let us consider the problem introduced in equation (4.1) on the domain $[0,1] \times [0,1]$ with $d_1 = d_2 = 1$, $A = 0.02$, $B = 0.05$ and the aid of following ICs and BCs as given by

$$\begin{aligned} &u(x,0) = 0, \qquad v(x,0) = 0, \\ &u(0,t) = 0, \qquad v(0,t) = t^2, \\ &u(1,t) = t^2 sin(1), \ v(1,t) = t^2 cos(1). \end{aligned} \tag{4.57}$$

The exact solution of this particular problem is $u(x,t) = t^2 sin(x)$ and $v(x,t) = t^2 cos(x)$ with suitable force functions $f_1(x,t)$ and $f_2(x,t)$.

We shall apply the above proposed method to solve Example 1. Here we examined MAE and CO through Table 4.1 for the order of approximation $m = 3, 5, 7, 9, 11$. From Table 4.1, it is clear that the maximum absolute error decreases as the degree of shifted Vieta–Lucas polynomial is increased, which shows the effectiveness and validation of the presented technique. Figure 4.1 and Figure 4.2 exhibit the approximate solution and the absolute error between the exact and approximate solutions for various x and t values with variable orders $\alpha_1(x,t) = 0.85 - 0.25e^{-xt}$, $\alpha_2(x,t) = 0.45 + 0.33sin(xt)$ and $m = 11$. From these Figures, it can be ensured that the approximate solutions $u(x,t)$ and $v(x,t)$ are very close to the exact solutions. The illustrated results verify that the proposed numerical method is highly efficient and accurate.

TABLE 4.1: MAE and CO of Example 1 for two different values of $\alpha_1(x,t)$ and $\alpha_2(x,t)$.

m	$\alpha_1(x,t)=0.85-0.25e^{-xt}, \alpha_2(x,t)=0.45+0.33sin(xt)$				$\alpha_1(x,t)=0.80-0.25e^{-2xt}, \alpha_2(x,t)=0.55+0.33sin(2xt)$			
	$u(x,t)$		$v(x,t)$		$u(x,t)$		$v(x,t)$	
	MAE	CO	MAE	CO	MAE	CO	MAE	CO
3	7.0158×10^{-4}	–	1.4213×10^{-3}	–	7.0373×10^{-4}	–	1.3871×10^{-3}	–
5	3.8245×10^{-6}	6.4271	7.6322×10^{-6}	6.4456	3.8306×10^{-6}	6.4289	7.4965×10^{-6}	6.4377
7	1.0736×10^{-8}	10.2119	2.0977×10^{-8}	10.2486	1.0751×10^{-8}	10.2123	2.0689×10^{-8}	10.2415
9	1.8152×10^{-11}	14.3015	3.5034×10^{-11}	14.329	1.8169×10^{-11}	14.3025	3.4621×10^{-11}	14.3246
11	2.5369×10^{-14}	18.0259	5.0626×10^{-14}	17.9343	3.3307×10^{-14}	17.2818	9.3481×10^{-14}	16.2199

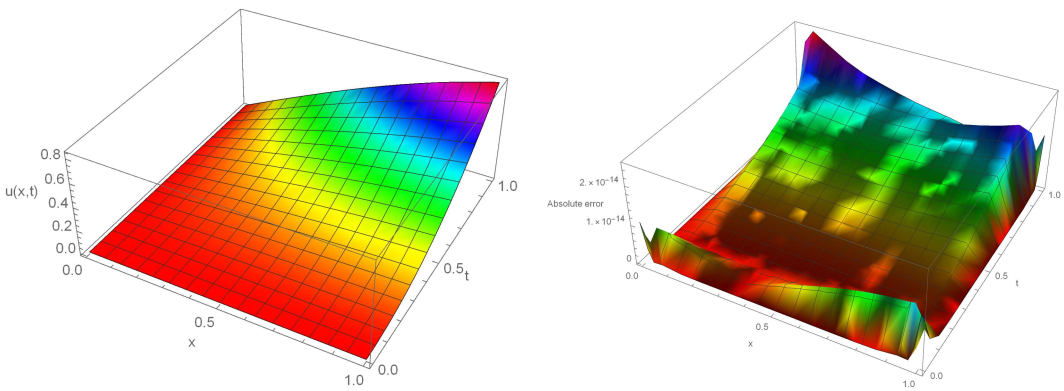

FIGURE 4.1: Plots of numerical result and absolute error of Example 1 for $u(x,t)$ with $\alpha_1(x,t)=0.85-0.25e^{-xt}$, $\alpha_2(x,t)=0.45+0.33sin(xt)$ and $m=11$.

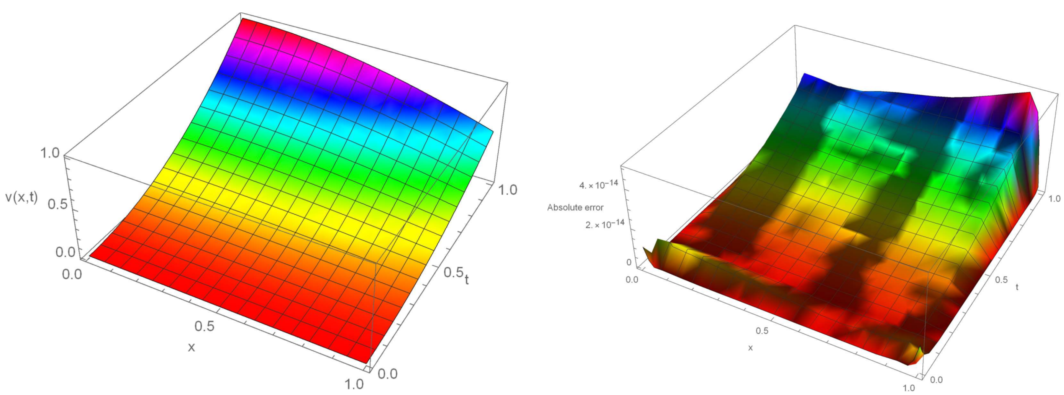

FIGURE 4.2: Plots of numerical result and absolute error of Example 1 for $v(x,t)$ with $\alpha_1(x,t)=0.85-0.25e^{-xt}$, $\alpha_2(x,t)=0.45+0.33sin(xt)$ and $m=11$.

Example 2. Consider the prescribed model in equation (4.1) on the domain $[0,1]\times$

$[0, 1]$ with $d_1 = 3$, $d_2 = 2$, $A = 0.06$, $B = 0.095$ and use of the following ICs and BCs as

$$\begin{aligned} &u(x,0) = 0, && v(x,0) = e^{-x}, \\ &u(0,t) = sin(t), && v(0,t) = cos(t), \\ &u(1,t) = e^{-1}sin(t), && v(1,t) = e^{-1}cos(t). \end{aligned} \tag{4.58}$$

The exact solution of this model is $u(x,t) = e^{-x}sin(t)$ and $v(x,t) = e^{-x}cos(t)$ with suitable force functions $f_1(x,t)$ and $f_2(x,t)$.

We shall apply the above proposed method to solve Example 2. Here MAE and CO have been found through Table 4.2 for the order of approximation $m = 3, 5, 7, 9, 11$. From Table 4.2, it is clear that the maximum absolute error decreases as the degree of shifted Vieta–Lucas polynomial is increased, which shows the effectiveness and validation of the presented technique. Figure 4.3 and Figure 4.4 exhibit the approximate solution and the absolute error between the exact and approximate solutions for various x and t values with variable orders $\alpha_1(x,t) = (cos(t)+e^{-2t}sin(x))/2$, $\alpha_2(x,t) = 0.7 + 0.25e^{-2t}sin(x)$ and $m = 11$. From these Figures, it can be ensured that the approximate solutions $u(x,t)$ and $v(x,t)$ are very close to the exact solutions. The illustrated results verify that the proposed numerical method is highly efficient and accurate.

TABLE 4.2: MAE and CO of Example 2 for two different values of $\alpha_1(x,t)$ and $\alpha_2(x,t)$.

m	$\alpha_1(x,t)=(cos(t)+e^{-2t}sin(x))/2, \alpha_2(x,t)=0.7+0.25e^{-2t}sin(x)$				$\alpha_1(x,t)=sin(t)+0.5e^{-2t}cos(x), \alpha_2(x,t)=0.25+0.65e^{-2t}cos(x)$			
	$u(x,t)$		$v(x,t)$		$u(x,t)$		$v(x,t)$	
	MAE	CO	MAE	CO	MAE	CO	MAE	CO
3	8.3610×10^{-4}	–	1.1619×10^{-3}	–	8.4928×10^{-4}	–	1.1337×10^{-3}	–
5	6.9860×10^{-6}	5.9004	6.8226×10^{-6}	6.3354	6.7522×10^{-6}	5.9617	7.1408×10^{-6}	6.2489
7	1.2382×10^{-8}	11.0111	1.5441×10^{-8}	10.5863	1.2491×10^{-8}	10.9368	1.5255×10^{-8}	10.6866
9	2.9499×10^{-11}	13.5331	2.7626×10^{-11}	14.1748	2.8598×10^{-11}	13.6222	2.8672×10^{-11}	14.0644
11	2.4647×10^{-14}	19.4367	5.6288×10^{-14}	16.992	3.9746×10^{-14}	18.0412	3.8858×10^{-14}	18.1102

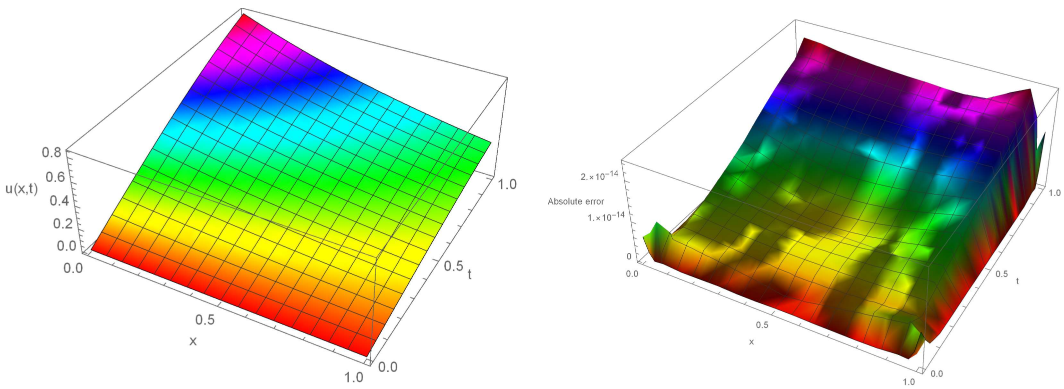

FIGURE 4.3: Plots of numerical result and absolute error of Example 2 for $u(x,t)$ with $\alpha_1(x,t)=(cos(t)+e^{-2t}sin(x))/2,\ \alpha_2(x,t)=0.7+0.25e^{-2t}sin(x)$ and $m=11$.

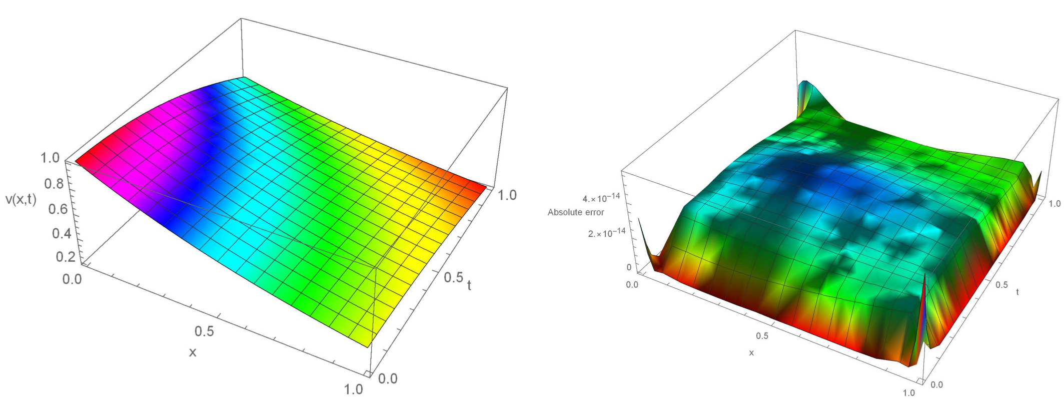

FIGURE 4.4: Plots of numerical result and absolute error of Example 2 for $v(x,t)$ with $\alpha_1(x,t)=(cos(t)+e^{-2t}sin(x))/2,\ \alpha_2(x,t)=0.7+0.25e^{-2t}sin(x)$ and $m=11$.

Example 3. Consider the prescribed model in equation (4.1) on the domain $[0, 1] \times [0, 1]$ with $d_1 = 2$, $d_2 = 1$, $A = 0.05$, $B = 0.07$ and use of the following ICs and BCs as

$$\begin{aligned} &u(x,0) = 0, && v(x,0) = sin(x), \\ &u(0,t) = sin(t), && v(0,t) = 0, \\ &u(1,t) = cos(1)sin(t), && v(1,t) = sin(1)cos(t). \end{aligned} \tag{4.59}$$

The exact solution of this model is $u(x,t) = cos(x)sin(t)$ and $v(x,t) = sin(x)cos(t)$ with suitable force functions $f_1(x,t)$ and $f_2(x,t)$.

The above proposed method is applied to solve Example 3. Here MAE and CO are examined through Table 4.3 for the order of approximation $m = 3, 5, 7, 9, 11$. From Table 4.3, it is clear that the maximum absolute error decreases with the increase in the degree of shifted Vieta–Lucas polynomial, which shows the effectiveness and validation of the presented technique. Figure 4.5 and Figure 4.6 exhibit the approximate solution and the absolute error between the exact and approximate solutions for various x and t values with variable orders $\alpha_1(x,t) = (1.35 + 0.25sin(xt))/2$, $\alpha_2(x,t) = (1 + 0.65e^{-2xt})/2$ and $m = 11$. From these Figures, it can be ensured that the approximate solutions $u(x,t)$ and $v(x,t)$ are very close to the exact solutions. The illustrated results verify that the proposed numerical scheme is highly efficient and accurate.

Table 4.3: MAE and CO of Example 3 for two different values of $\alpha_1(x,t)$ and $\alpha_2(x,t)$.

m	$\alpha_1(x,t) = (1.35 + 0.25sin(xt))/2, \alpha_2(x,t) = (1 + 0.65e^{-2xt})/2$				$\alpha_1(x,t) = 0.70 - 0.25sin(\pi xt), \alpha_2(x,t) = 0.45 + 0.35e^{-\sqrt{xt}}$			
	$u(x,t)$		$v(x,t)$		$u(x,t)$		$v(x,t)$	
	MAE	CO	MAE	CO	MAE	CO	MAE	CO
3	1.1457×10^{-3}	–	9.6441×10^{-4}	–	1.1611×10^{-3}	–	9.5588×10^{-4}	–
5	6.2341×10^{-6}	6.4293	4.7438×10^{-6}	6.5538	6.2709×10^{-6}	6.4385	4.6927×10^{-6}	6.5562
7	1.7265×10^{-8}	10.2354	1.2461×10^{-8}	10.3274	1.7344×10^{-8}	10.2377	1.2304×10^{-8}	10.3306
9	2.8975×10^{-11}	14.3181	2.0226×10^{-11}	14.393	2.9053×10^{-11}	14.3224	1.9968×10^{-11}	14.3933
11	3.9968×10^{-14}	18.0618	3.7970×10^{-14}	17.2166	4.3188×10^{-14}	17.8567	2.9671×10^{-14}	17.8578

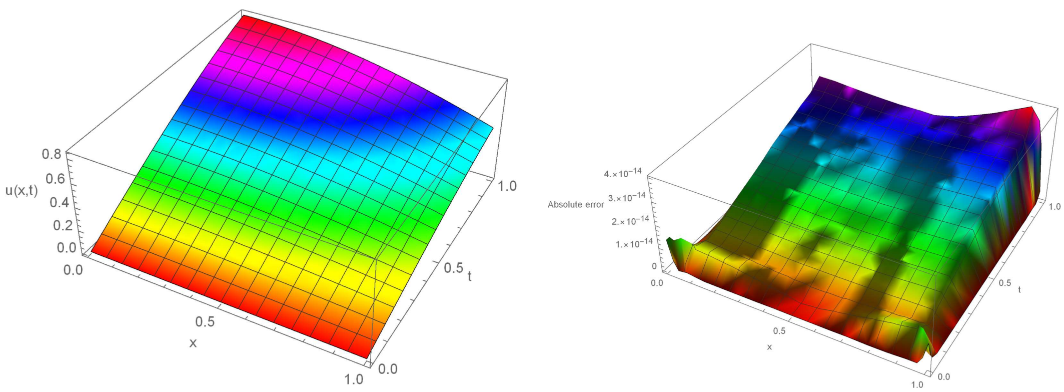

Figure 4.5: Plots of numerical result and absolute error of Example 3 for $u(x,t)$ with $\alpha_1(x,t) = (1.35 + 0.25sin(xt))/2,\ \alpha_2(x,t) = (1 + 0.65e^{-2xt})/2$ and $m = 11$.

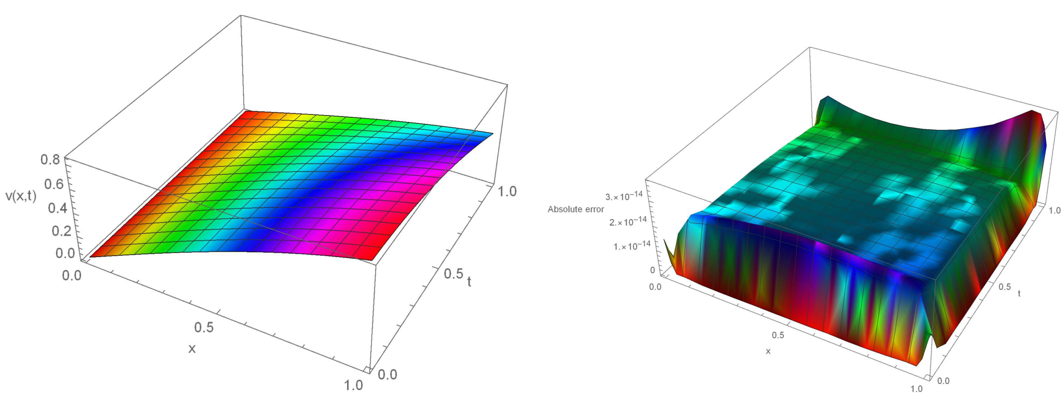

Figure 4.6: Plots of numerical result and absolute error of Example 3 for $v(x,t)$ with $\alpha_1(x,t) = (1.35 + 0.25sin(xt))/2,\ \alpha_2(x,t) = (1 + 0.65e^{-2xt})/2$ and $m = 11$.

4.8 Conclusion

In the present study, an efficient spectral collocation method based on shifted Vieta–Lucas polynomials is introduced to solve the generalized Gray-Scott model. A new operational matrix of fractional variable order derivatives is derived for the shifted Vieta–Lucas polynomials. While designing the technique, first the model has been approximated with the shifted Vieta–Lucas polynomial and then utilising the operational matrix the concerned model is converted into a nonlinear algebraic system of equations that can be solved numerically. The accuracy and efficiency of the presented method are validated by comparing the exact solutions and numerical solutions of three test problems through error analysis.

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