Chapter 6

Application of Gradient Flows in Data Driven Inference of Passivity Properties

6.1 Introduction

In this chapter, we address the problem of computing passivity indices—quantitative measures of a system's passivity—due to their significance in controller design. In practical scenarios, the explicit input-output relationship of a system is often unavailable, and only input-output data from simulations or experiments can be accessed. We focus on determining the passivity indices of a discrete-time LTI system using optimal input-output samples tailored to the specific property being assessed. Here, passivity indices like IFP and OFP are formulated as an objective function of an optimization problem. To compute these indices, we introduce prescribed-time GFs, which has the property of guaranteed convergence, within a prior chosen time. Notably, the proposed method is data-driven and does not require knowledge of the system's mathematical model, as it relies solely on gradients computed from data. Furthermore, we leverage the feedback interconnection properties of passive systems to ensure both asymptotic and finite-time stability of the discrete-time LTI system, using only the computed passivity indices. Specifically, stabilization is achieved by employing input feedforward passive and output feedback passive systems as controllers, again without requiring the explicit model of the system.

The structure of this chapter is outlined as follows. Section 6.2 is dedicated to notations and preliminaries. In section 6.3, the main results of this chapter are presented by introducing modified gradient flows to find passivity indices with simulation results. Section 6.4 focuses on the design of controllers based on the calculated passivity indices calculated in section 6.3. Finally, Section 6.6 concludes the chapter.

6.2 Preliminaries and Problem Formulation

6.2.1 Discrete Time Dissipative Systems

Consider the following discrete-time dissipative system

$$z(k+1) = g(z(k), u(k)), \quad y(k) = h(z(k)), \tag{6.1}$$

where $z(k) \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$ are the states, input and output respectively, $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^m$.

Definition 6.1 *(see Definition* 2.3 *[52], [31]): The system* (6.1) *is said to be dissipative if there exists a supply rate function $r(u, y)$, positive definite storage function $V(z)$, and a positive definite dissipation rate function ψ such that (s.t.): $V(g(z, u)) - V(z) \leq r(u, h(z)) - \psi(z, u, h(z)), \ \ \forall (z, u) \in \mathcal{Z} \times \mathcal{U}$. The system is passive when $r(u, y) = y^\top u$. In addition, the system is said to be input feedforward passive if $\psi(u) = vu^\top u$, and output feedback passive, if $\psi(y) = \delta y^\top y$, where $v, \delta > 0$.*

6.2.2 Finite-Time Stability of Discrete-Time Systems

Consider the following discrete-time system

$$\zeta(k+1) = g(\zeta(k)), \quad \zeta(0) = \zeta_0, \quad k \in \mathbb{Z}, \tag{6.2}$$

where, $k \in \mathbb{Z}$, $\zeta(k) \in \mathcal{D} \subseteq \mathbb{R}^n, 0 \in \mathcal{D}, g : \mathcal{D} \to \mathcal{D}, g(0) = 0$. The following lemma discusses the existence and uniqueness of the solution to (6.2).

Lemma 6.2 *(see [34]): Consider the dynamical system* (6.2)*. Then for every $\zeta_0 \in \mathcal{D}$, there exists $T^+_{\zeta_0} \in \mathbb{Z}_+$ such that (2) has a unique solution $\zeta : T^+_{\zeta_0} \to \mathbb{R}^n$. Moreover, if $\mathcal{G}(\cdot)$ is continuous, then the solution $w(k, \cdot)$ is continuous for each $k \in T^+_{\zeta_0}$. If, in addition, $\mathcal{G}(\cdot)$ is a homeomorphism of $\mathcal{D}$ onto $\mathbb{R}^n$, then the solution $\zeta : T^+_{\zeta_0} \to \mathbb{R}^n$ is unique in all $T_{\zeta_0} \subseteq \mathbb{Z}$ and $w(k, \cdot)$ is continuous for all $k \in T_{\zeta_0}$. Finally, if $\mathcal{D} = \mathbb{R}^n$ then $T_{\zeta_0} = \mathbb{Z}$.*

In the above lemma, the solution $w(k, \cdot)$ is continuous for each k means that for every fixed k the solution map $\zeta_0 \mapsto w(k, \zeta_0)$ is continuous with respect to the initial condition ζ_0.

Definition 6.3 *(see [34]): Consider the dynamics* (6.2), *and let the solution sequence be defined as $\zeta(k) = w(k, \zeta_0)$. The zero solution $\zeta(k) \equiv 0$ of* (6.2) *is finite-time stable (FTS) if there exists an open neighborhood of origin ($N \subset \mathcal{D}$) and a settling-time function ($\mathcal{K}_{st} : N \backslash \{0\} \to \mathbb{Z}$) such that*

(a) ***Lyapunov stability:*** *for each $\epsilon > 0$, there exists $\delta > 0$ such that $\forall\ \zeta \in \mathcal{B}_\delta(0) \backslash \{0\}$, where, $\mathcal{B}_\delta(0) \subset N, \exists\ w(k, \zeta) \in \mathcal{B}_\epsilon(0)\ \forall k \in \{0, \ldots, \mathcal{K}_{st} - 1\}$.*

(b) ***Finite-time convergence (FTC):*** *$\forall\ \zeta \in N \backslash \{0\}$ and $k > \mathcal{K}_{st}$, $w(k, \zeta)$ is in $N \cap \{0\}$.*

Lemma 6.4 *(see [34]) Consider the dynamics* (6.2). *Assume there exists a continuous function $\mathcal{V} : \mathcal{D} \to \mathbb{R}$, $m \in \mathbb{R}_{>0}$, and a neighbourhood of the origin ($\mathcal{N} \subseteq \mathcal{D}$) s.t.: (a) $\mathcal{V}(0) = 0$, (b) $\mathcal{V}(\zeta) > 0$, $\forall \zeta \in \mathcal{N} \backslash \{0\}$, (c) $\Delta\mathcal{V}(\zeta) \leq -min\{\mathcal{V}(\zeta), m\}$, $\forall \zeta \in \mathcal{N} \backslash \{0\}$. Then the zero solution is FTS. In addition, there exists a settling-time function $\mathcal{K}_{st}(\zeta_0) \leq \left\lceil \frac{\mathcal{V}(\zeta_0)}{m} \right\rceil$.*

6.2.3 Polyak- Lojasiewicz Inequality and Rayleigh Quotient

Definition 6.5 *(see Theorem 4 [47] and Section 2 [9]) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable and suppose a minimizer x^* exists. We say f satisfies the Polyak-Lojasiewicz (PL) inequality if $\exists \mu > 0$ such that $\frac{1}{2}\|\nabla f(x)\|^2 \geq \mu(f(x) - f(x^*))$, $\forall x \in \mathbb{R}^n$. If this holds only in a neighborhood $B_\delta(x^*)$ of a minimizer x^*, we call it a local PL inequality.*

Lemma 6.6 *(see Proposition 4.6.2 [53]) Consider the problem of finding a minimizer of $f : \mathcal{S}^{n-1} \to \mathbb{R} : x \mapsto x^\top Ax$, the Rayleigh quotient on the sphere. Let A be $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_n$ with corresponding eigenvectors $v_1, \ldots, v_n$, Then $\pm v_1$ are the local and global minimizers of the Rayleigh quotient, and if λ_1 is simple, then they are the only minimizers.*

Proposition 6.7 *Let $A \in \mathbb{R}^{n \times n}$ be symmetric with eigenvalues $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$, $\varrho(x) = x^T Ax$ on $\|x\| = 1$, and v_1 be the corresponding eigenvector to λ_1. Then ϱ is strongly convex near v_1 with modulus $\mu = \lambda_2 - \lambda_1$.*

Proof 5 *On $\mathcal{S}^{n-1}$, the Riemannian Hessian of ϱ at v_1 satisfies [see Chapter 5 [54]]:* $\langle \mathrm{Hess}\,\varrho(v_1)[\eta], \eta\rangle = 2\eta^T(A - \lambda_1 I)\eta \geq 2(\lambda_2 - \lambda_1)\|\eta\|^2, \forall \eta \in T_{v_1}\mathcal{S}^{n-1}$*, where* $T_{v_1}\mathcal{S}^{n-1} = \{\eta \in \mathbb{R}^n : \eta^T v_1 = 0\}$ *(tangent vectors orthogonal to v_1). By continuity of the Hessian and strict positivity at v_1, $\exists \delta > 0$ such that $\forall x \in \mathcal{B}_\delta(v_1) \cap \mathcal{S}^{n-1}$ and $\xi \in T_x\mathcal{S}^{n-1}$:* $\langle \mathrm{Hess}\,\varrho(x)[\xi], \xi\rangle \geq (\lambda_2 - \lambda_1)\|\xi\|^2 > 0$. *This establishes local strong convexity of the Rayleigh quotient.*

6.2.4 Problem Statement

Consider the following single-input single-output discrete-time LTI system

$$z(k+1) = Az(k) + Bu(k),\ y(k) = Cz(k) + Du(k), \tag{6.3}$$

where, $z(k) \in \mathbb{R}^n, y(k) \in \mathbb{R},$ and $u(k) \in \mathbb{R}$ are the states, output and input respectively, $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times 1}, C \in \mathbb{R}^{1\times n}$, and $D \in \mathbb{R}$. We can write (6.3) as follows:

$$H_s: \quad y(k) = \sum_{\kappa=0}^{k} h_\kappa u(k-\kappa), \tag{6.4}$$

where h_κ denotes the impulse response sequence. For a given input sequence $u(k) = 0, 1, \ldots, N-1$, (6.4) can be written as follows in matrix notation

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(N-1) \end{bmatrix} = \begin{bmatrix} h_0 & 0 & 0 & \cdots & 0 \\ h_1 & h_0 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ h_{N-1} & h_{N-2} & \cdots & h_1 & h_0 \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix}, \tag{6.5}$$

which is denoted as $y = Hu$, where H is a lower triangular Toeplitz matrix.

Input Feedforward Passivity (IFP)

IFP notion for discrete-time LTI systems using the standard Euclidean inner product in matrix notation leads to $v \leq \frac{u^\top H u}{u^\top u},\ \forall\ \|u\| \neq 0$, where v is the IFP index. To determine v, we rewrite the above definition into an optimization problem

$$v = \min_{\|u\|\neq 0} \varrho(u) = \min_{\|u\|\neq 0} \frac{1}{2}\frac{u^\top (H + H^\top)u}{u^\top u}, \tag{6.6}$$

where $\varrho : \mathbb{R}^N \backslash \{0\} \to \mathbb{R}$ is the Rayleigh quotient (see [6]). Let $S^{N-1} = \{u \in \mathbb{R}^N | \|u\| = 1\}$ represent the unit sphere in $\mathbb{R}^N$. Since, $\varrho(lu) = \varrho(u),\ \forall\ l \in \mathbb{R}$, it suffices to consider ϱ on S^{N-1}.

Lemma 6.8 *(see [6]) Let λ_p is the smallest eigenvalue of a symmetric matrix $H_1 \in \mathbb{R}^{N\times N}$, where $H_1 = 0.5(H + H^\top)$. Then $\lambda_p = \min_{\|u\|=1} \varrho(u)$.*

Output Feedback Passivity (OFP)

OFP notion for discrete-time LTI systems in matrix notation is $u^\top H u \geq -s u^\top H^\top H u$, where s denotes the shortage of passivity. Assume that $h_0 \neq 0$, then H has full rank. To estimate s, we rewrite the above definition into an optimization problem

$$s = -\min_{\|u\|\neq 0} \varphi(u) = -\min_{\|u\|\neq 0} \frac{1}{2}\frac{u^\top(H + H^\top)u}{u^\top H^\top H u}, \tag{6.7}$$

where $\varphi : \mathbb{R}^N \backslash \{0\} \to \mathbb{R}$ is the Rayleigh quotient.

Remark 6.9 *Solution of the minimization problem* (6.7) *are the generalized eigenvectors and generalized eigenvalues of $\left(\frac{1}{2}(H + H^\top), H^\top H\right)$.*

Lemma 6.10 *(see [30]) The gradient vector field of ϱ and φ on S^{N-1} is $\nabla\varrho(u) = 2H_1 u - 2\varrho(u)u$, where $\nabla\varrho(u) \in \mathbb{R}^N$ and $\nabla\varphi(u) = ((H + H^\top)u - 2\varphi(u)H^\top H u)/(\|Hu\|^2)$, where $\nabla\varphi(u) \in \mathbb{R}^N$.*

6.3 Main Results

In this section, first, we propose gradient flow (GF) dynamics to find the optimal points of optimization problems (6.6) and (6.7) in a priori chosen time. Proposed GFs have advantages over iterative methods used in literature to calculate passivity indices. Here we need not worry about step-size calculation and can reach critical points in predefined time. Next, we use calculated shortage of passivity to design a controller which stabilizes a passivity-short system in finite time.

6.3.1 Prescribed-Time Stable Gradient Flow (PT-GF) to Calculate IFP Index (v)

Consider the following GF system whose equilibrium point is the critical point of (6.6)

$$\dot{u}(t) = \begin{cases} -c\phi(t, u(t)), & \text{for} \ \ t_0 \leq t < t_f, \\ -c\nabla\varrho(u(t)), & \text{for} \ \ t \geq t_f \end{cases} \tag{6.8}$$

where $u(t) \in D \subset \mathbb{R}^N$, t_0 is initial time, $\phi(t, u) := \frac{\nabla\varrho(u(t))}{t_f - t} \in \mathbb{R}^N$, and $c > 1$.

Proposition 6.11 *Consider the PT-GF system* (6.8)*. It leaves the sphere S^{N-1} invariant.*

Proof 6 *Since,* $\frac{d}{dt}(u^\top u) = \dot{u}^\top u + u^\top \dot{u}$*. Next, for* $t \in [t_0, t_f)$ *along the* (6.8) *we get,* $\frac{d}{dt}(u^\top u) = -2cpu^\top \nabla \varrho(u)$*, where* $p = (1/(t_f - t))$*. Next, using Lemma* (6.10)*, further we can write* $\frac{d}{dt}(u^\top u) = -4cpu^\top (H_1 - \varrho(u)I)u = -c_1(\varrho(u) - \varrho(u))\|u\|^2 = 0$*, where* $c_1 = 4cp$*. Next, for* $t \in [t_f, \infty)$*, proceeding in a similar way along the flow* (6.8) $\frac{d}{dt}(u^\top u) = -cu^\top (H_1 - \varrho(u)I)u = -c(\varrho(u) - \varrho(u))\|u\|^2 = 0$

Theorem 1 *Consider the PT-GF system* (6.8)*. Let there exist a continuously differentiable function* $V : [t_0, \infty) \times D \to \mathbb{R}$*, satisfying* $\gamma_1(u) \le V(t,u) \le \gamma_2(u)$ *,* $\forall u \in D \backslash \{0\}, \forall t \in [t_0, \infty)$*,* $\gamma_1(u)$ *and* $\gamma_2(u)$ *be continuous positive definite functions,* $V(t, 0) = 0$ *and for* $V \neq 0$*:* $\dot{V} < 0, \forall t \ge t_f$ *and* $\dot{V} \le \frac{-\vartheta V}{(t_f - t)}, \forall t \in [t_0, t_f), \vartheta > 1$ *for* (6.8)*, then the equilibrium point of dynamics* (6.8) *is prescribed-time stable. For almost all initial conditions* $u(0)$ *with* $\|u(0)\| = 1$*,* ϱ *converges to IFP index (v) as* $t \to t_f$*.*

Proof 7 *Since* $\varrho(u) : S^{N-1} \to \mathbb{R}$ *is a smooth function on a compact Riemannian manifold* (S^{N-1})*, and from Lemma 6.8,* λ_p *is an optimal point of* (6.6)*. Let us consider a Lyapunov function candidate as:* $V(u) = \frac{1}{2}(\varrho(u) - \lambda_p)^2$*. First, we carry out the analysis for* $t \in [t_0, t_f)$ *then* $\dot{V}(u) = (\varrho(u) - \lambda_p)\nabla(\varrho(u))^\top \dot{u} = -c(\varrho(u) - \lambda_p)\nabla(\varrho(u))^\top \phi(t, u) \le -(c(\varrho(u) - \lambda_p)\|\nabla \varrho(u)\|^2)/(t_f - t)$*. Next using Lojasiewicz's Inequality stated in Lemma 2.20, and an assumption that* $\varrho(u)$ *is gradient dominated,* $\|\nabla \varrho(u)\|^2 \ge 2\mu(\varrho(u) - \lambda_p)$*, where* $\mu > 0$*. Further, we can write* $\dot{V}(u) \le -(2c\mu(\varrho(u) - \lambda_p)^2)/(t_f - t) = -(4c\mu V)/(t_f - t)$*. Hence, the Lyapunov candidate satisfies all the conditions required for prescribed-time stability. Therefore, the trajectories of the gradient flow systems* (6.8) *converge to the minima of* $\varrho(u)$ *within a prior chosen predefined time. Next we carry out the analysis for* $t \in [t_f, \infty)$*, for PT-GF* (6.8)*, then* $\dot{V} = -c(\varrho(u) - \lambda_p)\|\nabla \varrho(u(t))\|^2 \le -4cV^2$*, which represents that the equilibrium point of* (6.8) *is asymptotically stable for* $t \ge t_f$*. Hence, the trajectory of* (6.8) *will remain at its equilibrium point. Hence, the equilibrium point of PT-GF system* (6.8) *is prescribed-time stable.*

Remark 6.12 (Regarding internal stability) *With* $c > \frac{1}{4\mu}$*, let* $\vartheta = 4c\mu > 1$*. Solving* $\dot{V} \le -\frac{\vartheta}{t_f - t} V$ *gives:*

$$V(t) \le V(t_0) \left(\frac{t_f - t}{t_f - t_0} \right)^\vartheta$$

Since $\vartheta > 1$, $V(t) \to 0$ as $t \to t_f^-$. Compactness of S^{N-1} and continuity of ϱ ensure $u(t)$ converges to the λ_p. (We may not have the knowledge of μ, so we choose c large enough to guarantee $c > \frac{1}{4\mu}$.)

6.3.2 Prescribed-Time Stable Gradient Flow (PT-GF) to Estimate Shortage of Passivity ($s > 0$)

Consider the following GF system whose equilibrium point is the optimal point of (6.7)

$$\dot{u}(t) = \begin{cases} -c_1\Phi(t, u(t)), & \text{for } \; t_0 \le t < t_f, \\ -c_1\nabla\varphi(u(t)), & \text{for } \; t \ge t_f \end{cases} \tag{6.9}$$

where $u(t) \in D \subset \mathbb{R}^N$, t_0 is initial time, $\Phi(t, u) := \frac{\nabla\varphi(u)}{t_f - t} \in \mathbb{R}^N$, and $c_1 > 1$. Again it can be verified, similar to Proposition 6.11, that PT-GF dynamics (6.9) leaves the unit sphere S^{N-1} invariant.

Since H is full rank, an inverse transformation always exists. Taking $y = Gu$ and $Q := (H^{-1})^\top (H + H^\top) H^{-1}$, we get the standard Rayleigh quotient and we can define $\varphi(u) = \frac{1}{2}\frac{y^\top Q y}{y^\top y}$, where eigenvalues of Q correspond to generalized eigenvalues of pair $\left(\frac{1}{2}(H + H^\top), H^\top H\right)$. Using this coordinate transform, we can use similar insights as in the Theorem 1 to prove the equilibrium point of (6.9) is prescribed-time stable.

6.3.3 Data-Driven Computation of $\nabla\varrho(u)$ and $\nabla\varphi(u)$

Data-Driven Computation of $\nabla\varrho(u)$

To implement the dynamics (6.8), we require the knowledge of $\nabla\varrho(u)$, and without the information of H, we can not calculate $\nabla\varrho(u)$ using Lemma 6.10. However, $\nabla\varrho(u)$ can be calculated using two measurements of the system (6.3) [30]. Since, $\nabla\varrho(u) = (H + H^\top)u - 2\varrho(u)u$. To calculate $\nabla\varrho(u)$ from $u \mapsto Hu$, define an involutory permutation matrix $P = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \ddots & & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{N\times N}$, with $P = P^{-1}$. As $PH^\top = HP$, we can write $H^\top u = PHPu$, which gives us $\nabla\varrho(u) = Hu + PHPu - (u^\top(Hu + PHPu)u)/(u^\top u)$. Let $y_1 = Hu$ be the output measurement for input u and $y_2 = HPu$ be the output for input Pu applied on the experimental setup of system (6.3). Then, $\nabla \varrho(u) = y_1 + Py_2 - (u^\top(y_1 + Py_2)u)/(u^\top u)$.

Data-Driven Computation of $\nabla\varphi(u)$

Similar to the discussion in subsection 6.3.3, we can not calculate $\nabla\varphi(u)$, using Lemma 6.10, without information about H. As $\nabla\varphi = ((H + H^\top)u - 2\varphi(u)H^\top Hu)/\|Hu\|^2$, here, we can write, $\nabla\varphi(u) = \frac{Hu+PHPu}{\|Hu\|^2} - \frac{u^\top(Hu+PHPu)}{\|Hu\|^4}(PH)^2u$. Let $\hat{y}_1$ be the output measurement for input u, $\hat{y}_2 = HPu$ be the output for input Pu, and $\hat{y}_3$ is the output for input $P\hat{y}_1$ applied on the experimental setup of system (6.3). Then, we can write, $\nabla\varphi(u) = \frac{\hat{y}_1+P\hat{y}_2}{\|\hat{y}_1\|^2} - \frac{u^\top(\hat{y}_1+P\hat{y}_2)}{\|\hat{y}_1\|^4}P\hat{y}_3$.

6.3.4 Application to a Coupled Tank System

In this segment, MATLAB/Simulink simulations are included to authenticate the effectiveness of the proposed data-driven approach to calculate passivity indices. We consider the coupled tank system as shown in the Figure 6.1 as our test bench. We consider the coupled tank in configuration#2, where the objective is to control the water level in that tank, which is not under direct influence of the control.

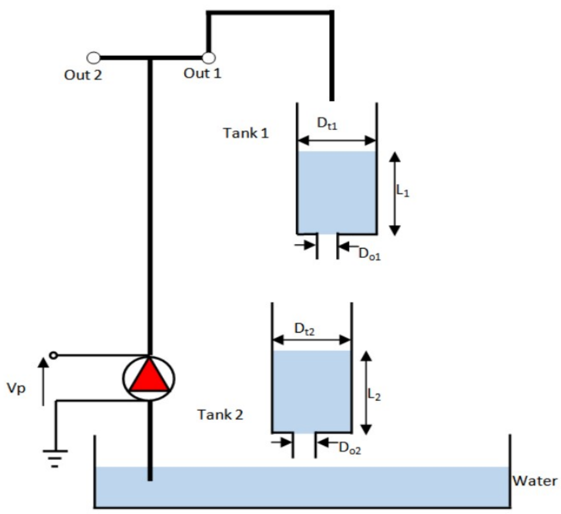

Figure 6.1: Schematic of coupled tank in configuration # 2

Here, we set $N = 200$ and take $u(0) = \frac{g}{\|g\|}$, where $g = [g(0), \cdots, g(N-1)]$, $g(i) = sin(2\pi i t_s), i = 0, \cdots, N-1$. Next, we use PT-GF (6.8) to calculate IFP index (v), where we take $\nabla\varrho(u)$ as given in subsection 6.3.3. Since, $\nabla\varrho(u) = y_1 + Py_2 - (u^\top(y_1+Py_2))/(u^\top u)$

is composed of y_1, y_2, u, $\nabla\varrho(u)$, is calculated using two sets of input-output data. These data are obtained using the coupled-tanks configuration #2 model in Simulink as a black box for simulation. Parameters in PT-GF were chosen as $t_f = 0.4$ seconds and $c = 1.2$. It is shown in Figure 6.2, that within $t_f \leq 0.4$ sec, we reach the optimal point of the optimization problem (6.6), which is $v = -2.68$. Figure 6.2 also shows that the proposed PT-GF gives better convergence results than classical GF: $\dot{u} = -c\nabla\varrho(u)$, $\forall t$, with the same gain $c = 1.2$, without much tuning of the parameters.

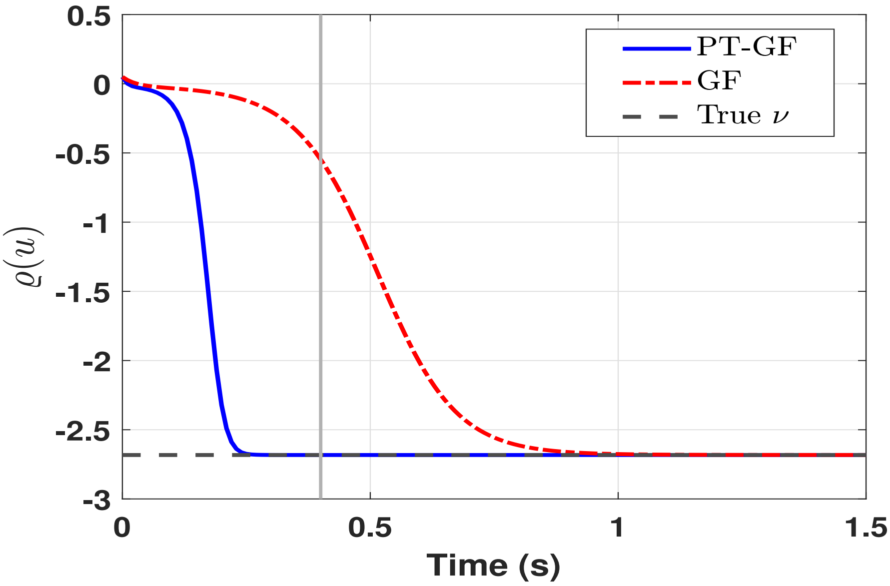

Figure 6.2: Comparison between computation of IFP index (ν) using proposed PT-GF (6.8) and GF

Verification of IFP index value

We calculated true IFP index values using the following linearized discrete-time mathematical model of the coupled tank system: $L(k+1) = A_d L(k) + B_d V_p(k)$*, where* $L := [L_1; L_2]$ *are the water levels in tanks (in cm),* V_p*: pump input voltage (u). State space matrices are:*

$$A_d = \begin{bmatrix} 1 - T_s \dfrac{a_1}{A_1}\sqrt{\dfrac{g}{2L_{10}}} & 0 \\ T_s \dfrac{a_1}{A_2}\sqrt{\dfrac{g}{2L_{10}}} & 1 - T_s \dfrac{a_2}{A_2}\sqrt{\dfrac{g}{2L_{20}}} \end{bmatrix}$$

$$B_d = \begin{bmatrix} T_s \dfrac{K_m}{A_1} \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 0.8,$$

where A_1, A_2*: Cross-sectional areas* (cm^2)*,* a_1, a_2*: Outflow orifice areas* (cm^2)*,* g*: Gravitational acceleration* ($\mathrm{cm/s}^2$)*,* L_{10}, L_{20}*: Operating levels (cm),* K_m*: Pump constant* ($(\mathrm{cm}^3/\mathrm{s})/\mathrm{V}$)*. Parameter values are:* $A_1 = A_2 = 15.5179\,\mathrm{cm}^2, a_1 = a_2 = 0.178\,139\,197\,65\,\mathrm{cm}^2, g = 980\,\mathrm{cm/s}^2, L_{10} = L_{20} = 15\,\mathrm{cm}, K_m = 4.6\,(\mathrm{cm}^3/\mathrm{s})/\mathrm{V}, T_s = 0.1\,\mathrm{s}$. *Using above given values, we calculated the lower triangular Toeplitz matrix* H*, and then* $H_1 = 0.5(H + H^\top)$*. Next, using Lemma 6.8, we can say that the minimum eigenvalue of* H_1 *is the true IFP index.*

Next, we use PT-GF (6.9) to calculate OFP index (s), where we take $\nabla\varphi(u)$ as given in subsection 6.3.3. Since, $\nabla\varphi(u) = \frac{\hat{y}_1 + P\hat{y}_2}{\|\hat{y}_1\|^2} - \frac{u^\top(\hat{y}_1 + P\hat{y}_2)}{\|\hat{y}_1\|^4} P\hat{y}_3$ is composed of $\hat{y}_1, \hat{y}_2, \hat{y}_3, u$, $\nabla\varphi(u)$, is calculated using three sets of input-output data. Parameters in PT-GF were chosen as $t_f = 0.4$ seconds and $c = 15$. It is shown in Figure 6.3, that within $t_f \leq 0.4$ sec, we reach the optimal point of the optimization problem (6.6), which is $v = -0.049$. Figure 6.3 also shows that the proposed PT-GF gives better convergence results than classical GF: $\dot{u} = -\nabla\varphi(u)$, $\forall t$, with the same gain $c = 15$, without much tuning of the parameters. Verification of OFP index can be done in a similar way as the discussion of verification of IFP index.

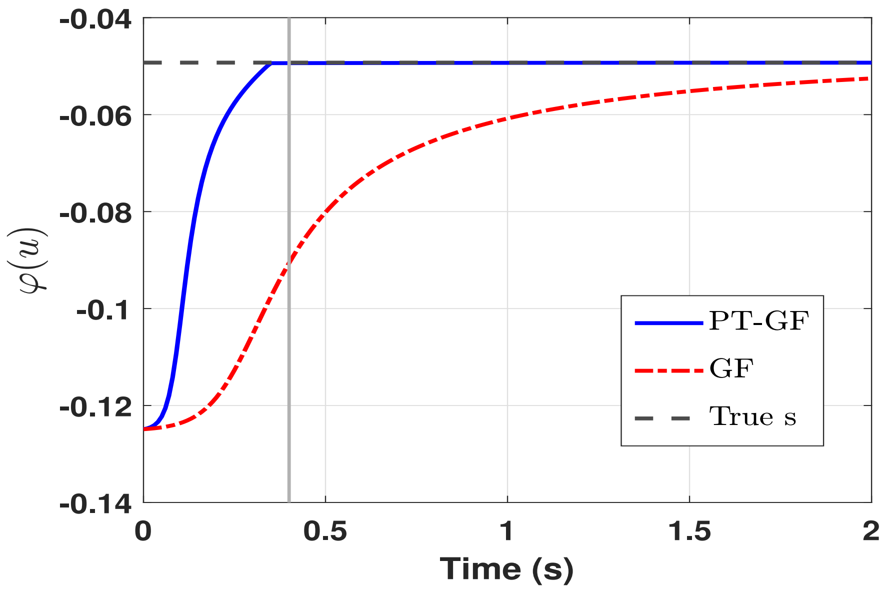

Figure 6.3: Comparison between computation of OFP index ($-s$) using proposed PT-GF (6.9) and GF

Remark 6.13 *As demonstrated in the preceding section, the shortage of passivity for a system was evaluated using the proposed gradient flow-based approach, which relies solely on the computation of* $\nabla\varphi(u)$. *The data-driven method for computing* $\nabla\varphi(u)$ *has been detailed in Subsection 6.3.3. Consequently, complete knowledge of the system dynamics is not required for estimating the passivity indices.*

6.4 Asymptotic and Finite-Time Stabilization Using Shortage of Passivity Property of System

In this section, we utilize the shortage of passivity property, which can be computed using the method discussed in subsection 6.3.2, to make the system output feedback passive and finite-time passive. We have given the definitions and examples of discrete finite time passive systems and discrete output feedback finite-time passive systems. Employing these properties and considering the assumption of zero state observability (ZSO), the origin of the close loop system is made asymptotically and finite time stable.

Definition 6.14 *The system* (6.3) *is said to be a Discrete Finite Time Passive (DFTP) if it is dissipative for supply rate* $r(u, y) = y(k)^\top u(k) - min\{V(z), m\}$, $\forall k \in \mathbb{Z}$, $m \in \mathbb{R}_+$. *System* (6.3) *is Discrete Output Feedback Finite Time Passive (DOF-FTP(*δ*)) if it is dissipative for supply rate* $r(u, y) = y(k)^\top u(k) - \delta y(k)^\top y(k) - min\{V(z), m\}$, $\forall k \in \mathbb{Z}$, $\delta, m \in \mathbb{R}_+$. *Here,* V *is a positive definite storage function and* $\delta = -s$, *where* s *denotes a shortage of passivity.*

Next, we show examples of DFTP and DOF-FTP properties discussed in Definition 6.14.

Example 6.15 *Consider the following input-output dynamics*

$$\begin{aligned} z_c(k+1) =& z_c(k) - min\{|z_c(k)|, m_c\} sign(z_c(k)) + u_c(k), \\ y_c(k) =& 2z_c(k+1), \end{aligned} \tag{6.10}$$

where $z_c, u_c, y_c \in \mathbb{R}$ *are the state, input and output respectively, and* $m_c \in \mathbb{R}_+$. *Let us consider a positive definite storage function* $V = z_c^2(k)$.

Further,

$$\begin{aligned}
\Delta V(z_c) &= z_c^2(k+1) - z_c^2(k) \\
&= [z_c(k) - min\{|z_c(k)|, m_c\} sign(z_c(k)) + u_c(k)]^2 - z_c^2(k) \\
&= [z_c(k) - min\{|z_c(k)|, m_c\} sign(z_c(k))]^2 - z_c^2(k) + 2u_c(k)(z_c(k) - min\{|z_c(k)|, m_c\} \\
&\quad sign(z_c(k)) + u_c^2(k) \\
&= -2|z_c(k)| min\{|z_c(k)|, m_c\} + [min\{|z_c(k)|, m_c\}]^2 + 2u_c(k)((z_c(k+1) - u_c(k)) + \\
&\quad u_c^2(k) \\
&= min\{|z_c(k)|, m_c\}[min\{|z_c(k)|, m_c\} - 2|z_c(k)|] + 2u_c(k) z_c(k+1) - u_c^2(k) \\
&\leq -[min\{z_c(k), m_c\}]^2 + 2u_c(k) z_c(k+1) - u_c^2(k) \\
&\leq -[min\{z_c(k), m_c\}]^2 + 2u_c(k) z_c(k+1) \\
&= -min\{V(k), m_c\} + u_c(k) y_c(k),
\end{aligned}$$

Therefore, from definition 6.14, we can say that the dynamics (6.10) *is DFTP from input* $u_c(k)$ *to output* $y_c(k)$ *with storage function* $V = z_c^2(k)$.

Example 6.16 *Consider the following input-output dynamics with virtual output*

$$\begin{aligned}
z_c(k+1) &= z_c(k) - min\{|z_c(k)|, m_c\} sign(z_c(k)) + u_c(k), \\
y_c(k) &= 2z_c(k) - 2min\{|z_c(k)|, m_c\} sign(z_c(k)) + u_c(k),
\end{aligned} \tag{6.11}$$

where $z_c, u_c, y_c \in \mathbb{R}$ *are the state, input and output respectively, and* $m_c \in \mathbb{R}_+$.
Assume a positive-definite storage function $V = z_c^2(k)$. *Further,*

$$\begin{aligned}
\Delta V(z_c) &= z_c^2(k+1) - z_c^2(k) \\
&= [z_c(k) - min\{|z_c(k)|, m_c\} sign(z_c(k)) + u_c(k)]^2 - z_c^2(k) \\
&= [z_c(k) - min\{|z_c(k)|, m_c\} sign(z_c(k))]^2 - z_c^2(k) + u_c^2(k) + 2u_c(k)[z_c(k) - \\
&\quad min\{|z_c(k)|, m_c\}] \\
&\leq -min\{|z_c(k)|^2, m_c^2\} + u_c^2(k) + 2u_c(k)[z_c(k) - min\{|z_c(k)|, m_c\} sign(z_c(k))] \\
&\leq -min\{|z_c(k)|^2, m_c^2\} + u_c(k) y_c(k),
\end{aligned}$$

We define a new input $u_n(k) := u_c(k) + s_n y_c(k),$. *Hence, we can further write,* $\Delta V(z_c) \leq -min\{|z_c(k)|^2, m_c^2\} + (u_n(k) - s_n y_c(k)) y_c(k) = -min\{|z_c(k)|^2, m_c^2\} + u_n(k) y_c(k) - s_n y_c^2(k)$. *Therefore, from definition 6.14, we can say that the dynamics* (6.11) *is DOF-FTP(s_n) from input* $u_n(k)$ *to output* $y_c(k)$ *with storage function* $V = z_c^2(k)$.

As we know, making a system output feedback passive (OFP) corresponds to adding some dissipative terms that help to stabilize a system. This intuitive way of stabilizing a system makes passivity-based control fascinating. The Next Lemma explains the OFP-based stabilization of passivity-short systems.

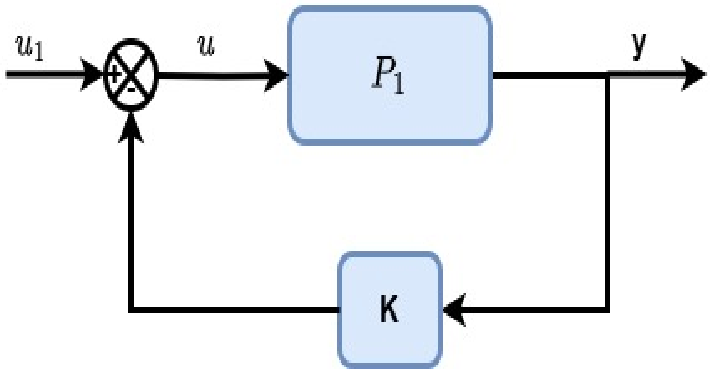

Figure 6.4: Output feedback interconnection of system P_1

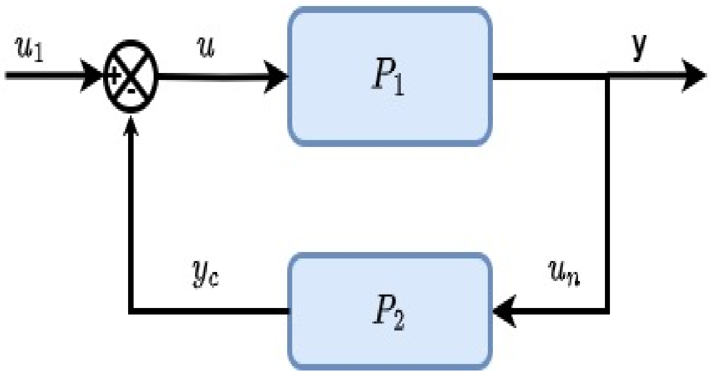

Figure 6.5: Feedback interconnection of system P_2 with P_1

Lemma 6.17 *Consider a passivity-short system P_1 with OFP index $(-s)$, if we add a feedback gain (K) as shown in Fig.6.4, then closed loop interconnected system becomes OFP(K_f) from u_1 to y, where, $K_f = K - s$. Choose K such that $K_f > 0$. Further, if in Fig. 6.4, we take $u_1 = 0$, and P_1 is ZSO, then the equilibrium point of the closed-loop system is asymptotically stable.*

Proof 8 *We consider a passivity-short system P_1 with OFP index $(-s)$, this property can be represented mathematically as $\Delta V(k) \leq y(k)^\top u(k) + sy(k)^\top y(k)$. As shown in Fig.6.4, $u(k) = u_1(k) - Ky(k)$, then $\Delta V(k) \leq y(k)^\top (u_1(k) - Ky(k)) + sy(k)^\top y(k) = y(k)^\top u_1(k) - K_f y(k)^\top y(k)$, where $K_f = K - s$ and let we chose K such that $K_f > 0$.*

Then closed loop interconnection shown in Fig. 6.4 is OFP(K_f) from u_1 to y, where $K_f > 0$.

Theorem 6.18 *Consider the feedback interconnection shown in Figure 6.5, where P_1 is given as strongly OFP(K_f) ($K_f \in \mathbb{R}_{>0}$ and large enough), with the assumption that P_1 is ZSO. P_2 is given as DFTP. If $u_1 = 0$, then the origin of the feedback interconnection is FTS.*

Proof 9 *Consider a storage function $V(k) = V_1(k) + V_2(k)$, where V_1 and V_2 are the storage functions of the systems P_1 and P_2 respectively. Using Lyapunov's direct method and Definitions 6.1 and 6.14 for OFP and DFTP, we have:*

$$\Delta V \leq y(k)^\top u(k) - K_f y(k)^\top y(k) + y_c(k)^\top u_n(k) - \min\{V_2(k), m_{c2}\},$$

where $m_{c2} > 0$ and $K_f >> 0$. From Fig. 6.5, $u = u_1 - y_c$ and $y = u_n$, so:

$$\begin{aligned}\Delta V &\leq y(k)^\top (u_1(k) - y_c(k)) - K_f y(k)^\top y(k) + y_c(k)^\top y(k) - \min\{V_2(k), m_{c2}\} \\ &= y(k)^\top u_1(k) - K_f y(k)^\top y(k) - \min\{V_2(k), m_{c2}\}.\end{aligned}$$

Since P_1 is ZSO and strongly OFP(K_f) with $K_f >> 0$, and let there exists a constant $m_{c1} > 0$ such that $K_f y(k)^\top y(k) \geq \min\{V_1(k), m_{c1}\}$. Therefore, we can write:

$$\begin{aligned}\Delta V &\leq y(k)^\top u_1(k) - K_f y(k)^\top y(k) - \min\{V_2(k), m_{c2}\} \\ &\leq y(k)^\top u_1(k) - \min\{V_1(k), m_{c1}\} - \min\{V_2(k), m_{c2}\} \\ &\leq y(k)^\top u_1(k) - \min\{V(k), m_c\},\end{aligned}$$

where $m_c = \min\{m_{c1}, m_{c2}\} > 0$.

Thus, by Definition 6.14, the closed-loop interconnection is DFTP from u_1 to y. When $u_1 = 0$,

$$\Delta V \leq -\min\{V(k), m_c\}.$$

By Lemma 6.4, the origin is FTS.

6.5 Simulation Results

As we now know the OFP index of the considered coupled tank system is $-s = -0.049$. First, we use Lemma 6.17 to asymptotically stabilize the origin of the considered coupled tank system. According to Lemma 6.17, we chose feedback gain $K = 0.1$, then $K_f = K - s = 0.05 > 0$. For simulation, we have taken initial conditions as $L(0) = [0 \quad 0]^\top$. We connect gain K in feedback as shown in Fig. 6.4, and it is shown in Fig. 6.6, that the system is asymptotically stable. Next to demonstrate the effectiveness of Theorem

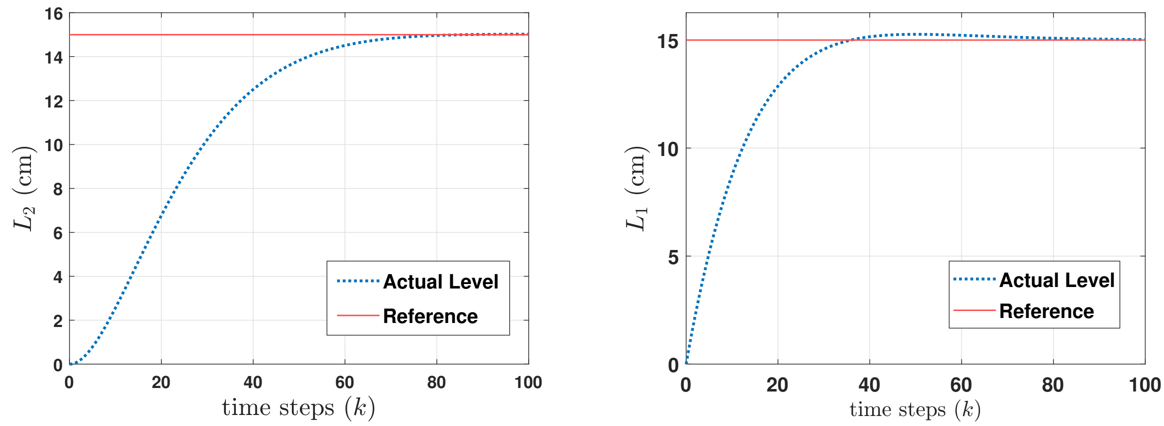

Figure 6.6: Evolution of levels of tanks after interconnecting feedback gain as shown in Fig.6.4.

6.18, we interconnect our OFP(K_f) dynamics with (6.10) as shown in Fig. 6.5. For simulation, we have taken initial conditions as $L(0) = [0 \quad 0]^\top$ and in dynamics (6.10) $z_c(0) = 1, m_c = 10$. It is shown in Fig. 6.7 that the coupled tank system is FTS with $\mathcal{K}_{st} \leq \left\lceil \frac{V_1(0)+V_2(z_c(0))}{m_c} \right\rceil$=23.

Remark 6.19 *While the mathematical model of coupled-tank system (as given in Subsection 6.3.4) was employed to obtain simulation results, the preceding subsection demonstrates that the controller design procedure is model-free.*

6.6 Conclusion

In this chapter, we discuss a PT-GFs approach to calculate IFP and OFP indices, as these system characteristics are formulated as optimization problems. We show the Lyapunov-based convergence analysis of the proposed GF dynamics. Additionally, we utilize the calculated indices to stabilize the equilibrium point of the discrete-time LTI system asymptotically and in finite time without requiring the complete knowledge of the mathematical model of the system. Here, we utilize the example of DFTP dynamics discussed in this chapter as a stabilizing controller in finite-time.

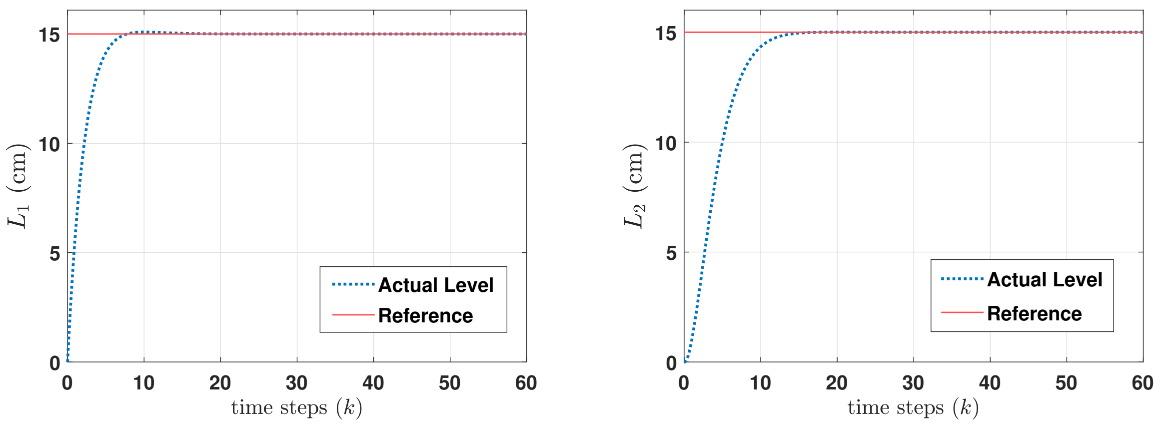

Figure 6.7: Evolution of levels of tanks after interconnecting with (6.10) as shown in Fig.6.5.