Chapter 5

Semi-discrete scheme for the time-fractional Black-Scholes model governing European options

In this chapter, we proposed a semi-discrete scheme for time-factional Black-Scholes model (TFBSM) governing European option. The organization of this chapter is as follow: we discuss the descretization of Caputo time-fractional derivative by L-123 approximation in 5.2. The function approximtion by the operational matrix approach based on shifted Legendre polynomial (SLP) as basis function is discussed in Section 5.3. In section 5.4 we design the semi-discrete scheme for the TFBSM. The numerical example is provided in Sect. 5.5. Finally, the conclusion of the chapter is presented in section 5.6.

5.1 Introduction

In this section, we have provided model formulation of our proposed model and main contribution of this chapter.

5.1.1 The time-fractional Black-Scholes model

In this chapter, we consider the following time-fractional Black-Scholes model for pricing European option

$$\frac{\partial^{\alpha}\mathbb{C}(\mathbb{S},\zeta)}{\partial\zeta^{\alpha}}+\frac{1}{2}\sigma^{2}\mathbb{S}^{2}\frac{\partial^{2}\mathbb{C}(\mathbb{S},\zeta)}{\partial\mathbb{S}^{2}}+\mathrm{r}\mathbb{S}\frac{\partial\mathbb{C}(\mathbb{S},\zeta)}{\partial\mathbb{S}}-\mathrm{r}\mathbb{C}(\mathbb{S},\zeta)=0,(\mathbb{S},\zeta)\in(0,\infty)\times(0,T), \tag{5.1}$$

with the boundary conditions

$$\mathbb{C}(0,\zeta)=\psi_{1}(\zeta),\quad \mathbb{C}(\infty,\zeta)=\psi_{2}(\zeta), \tag{5.2}$$

and final (terminal) condition

$$\mathbb{C}(\mathbb{S},T)=\nu(\mathbb{S}). \tag{5.3}$$

Here, $0<\alpha<1$, ζ is the current time, $\mathbb{S}$ is the stock price, $\mathbb{C}$ is the European option price, T is the maturity date of contract, r is the risk free rate, and σ represents the volatility of the returns from the holding stock price $\mathbb{S}$. The fractional derivative in (5.1) is a modified right Riemann-Liouville derivative which is defined as

$$\frac{\partial^{\alpha}\mathbb{C}(\mathbb{S},\zeta)}{\partial\zeta^{\alpha}}=\begin{cases}\dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{d\zeta}\int_{\zeta}^{T}\dfrac{\mathbb{C}(\mathbb{S},\xi)-\mathbb{C}(\mathbb{S},T)}{(\xi-\zeta)^{\alpha}}d\xi, & 0<\alpha<1,\\ \dfrac{\partial\mathbb{C}(\mathbb{S},\zeta)}{\partial\zeta}, & \alpha=1.\end{cases} \tag{5.4}$$

Let $\zeta=T-\mathfrak{t}$, then

$$\frac{\partial^{\alpha}\mathbb{C}(\mathbb{S},\zeta)}{\partial\zeta^{\alpha}}=\frac{1}{\Gamma(1-\alpha)}\frac{d}{d\zeta}\int_{\zeta}^{T}\frac{\mathbb{C}(\mathbb{S},\xi)-\mathbb{C}(\mathbb{S},T)}{(\xi-\zeta)^{\alpha}}d\xi,$$

$$=\frac{1}{\Gamma(1-\alpha)}\frac{-d}{d\mathfrak{t}}\int_{T-\mathfrak{t}}^{T}\frac{\mathbb{C}(\mathbb{S},\xi)-\mathbb{C}(\mathbb{S},T)}{(\xi-(T-\mathfrak{t}))^{\alpha}}d\xi,$$

$$= \frac{-1}{\Gamma(1-\alpha)}\frac{d}{d\mathfrak{t}}\int_0^{\mathfrak{t}} \frac{\mathbb{C}(\mathbb{S}, T-\eta) - \mathbb{C}(\mathbb{S}, T)}{(\mathfrak{t}-\eta)^{\alpha}} d\eta. \tag{5.5}$$

Suppose, $\mathrm{x} = ln\mathbb{S}$ and $\mathfrak{U}(\mathrm{x}, \mathfrak{t}) = \mathbb{C}(e^{\mathrm{x}}, T-\mathfrak{t})$ then

$$ {}_0^{RL}\mathrm{D}_{\mathfrak{t}}^{\alpha}\mathfrak{U}(\mathrm{x}, \mathfrak{t}) = \frac{1}{2}\sigma^2\frac{\partial^2\mathfrak{U}(\mathrm{x}, \mathfrak{t})}{\partial S^2} + (\mathrm{r} - \frac{\sigma}{2})\frac{\partial\mathfrak{U}(\mathrm{x}, \mathfrak{t})}{\partial x} - \mathrm{r}\mathfrak{U}(\mathrm{x}, \mathfrak{t}), \quad (\mathrm{x}, \mathfrak{t}) \in R \times (0, T), \tag{5.6}$$

with boundary and initial conditions

$$\mathfrak{U}(-\infty, \mathfrak{t}) = \mathscr{G}_1(\mathfrak{t}), \ \mathbb{C}(\infty, \mathfrak{t}) = \mathscr{G}_2(\mathfrak{t}),$$

$$\mathfrak{U}(\mathrm{x}, 0) = \mathbb{H}(\mathrm{x}). \tag{5.7}$$

Where ${}_0^{RL}\mathrm{D}_{\mathfrak{t}}^{\alpha}\mathfrak{U}(\mathrm{x}, \mathfrak{t})$ is the modified left Riemann-Liouville derivatives of order α $(0 < \alpha < 1)$ which is defined as

$$ {}_0^{RL}\mathrm{D}_{\mathfrak{t}}^{\alpha}\mathfrak{U}(\mathrm{x}, \mathfrak{t}) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\mathfrak{t}}\int_0^{\mathfrak{t}} \frac{\mathfrak{U}(x, \eta) - \mathfrak{U}(\mathrm{x}, 0)}{(\mathfrak{t}-\eta)^{\alpha}} d\eta,$$

$$= \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\mathfrak{t}}\int_0^{\mathfrak{t}} \frac{\mathfrak{U}(\mathrm{x}, \eta)}{(\mathfrak{t}-\eta)^{\alpha}} d\eta - \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\mathfrak{t}}\int_0^{\mathfrak{t}} \frac{\mathfrak{U}(\mathrm{x}, 0)}{(\mathfrak{t}-\eta)^{\alpha}} d\eta$$

$$= \frac{\mathfrak{U}(\mathrm{x}, 0)\mathfrak{t}^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\mathfrak{t}}\int_0^{\mathfrak{t}} \frac{\mathfrak{U}'(\mathrm{x}, \eta)}{(\mathfrak{t}-\eta)^{\alpha}} d\eta - \frac{\mathfrak{U}(\mathrm{x}, 0)\mathfrak{t}^{-\alpha}}{\Gamma(1-\alpha)}$$

$$= \frac{1}{\Gamma(1-\alpha)}\frac{d}{d\mathfrak{t}}\int_0^{\mathfrak{t}} \frac{\mathfrak{U}'(x, \eta)}{(\mathfrak{t}-\eta)^{\alpha}} d\eta$$

$$= {}_0^{C}\mathrm{D}_{\mathfrak{t}}^{\alpha}\mathfrak{U}(\mathrm{x}, \mathfrak{t}). \tag{5.8}$$

Where ${}_0^{C}\mathrm{D}_{\mathfrak{t}}^{\alpha}\mathfrak{U}(\mathrm{x}, \mathfrak{t})$ is the Caputo derivative of order α. Now, we will truncate the infinite domain $\mathrm{R} \times (0, T)$ into a finite domain $(\mathrm{x}_1, \mathrm{x}_2) \times (0, T)$ to solve the TFBSM (5.6)-(5.7).

Thus, the TFBSM becomes

$$ {}_{0}^{C}\mathrm{D}_{\mathfrak{t}}^{\alpha}\mathfrak{U}(\mathrm{x},\mathfrak{t}) = \lambda_1 \frac{\partial^2 \mathfrak{U}(\mathrm{x},\mathfrak{t})}{\partial \mathrm{x}^2} + \lambda_2 \frac{\partial \mathfrak{U}(\mathrm{x},\mathfrak{t})}{\partial x} - \lambda_3 \mathfrak{U}(\mathrm{x},\mathfrak{t}) + \mathfrak{f}(\mathrm{x},\mathfrak{t}), \quad (\mathrm{x},\mathfrak{t}) \in (\mathrm{x}_1,\mathrm{x}_2)\times(0,T), \tag{5.9} $$

with boundary and initial conditions

$$ \begin{aligned} \mathfrak{U}(\mathrm{x}_1,\mathfrak{t}) = \mathscr{G}_1(\mathfrak{t}),\ \mathfrak{U}(\mathrm{x}_2,\mathfrak{t}) = \mathscr{G}_2(\mathfrak{t}) \\ \mathfrak{U}(\mathrm{x},0) = \mathbb{H}(\mathrm{x}), \end{aligned} \tag{5.10} $$

where $\lambda_1 = \frac{\sigma^2}{2} > 0,\ \lambda_2 = \mathrm{r} - \lambda_1,\ \ \lambda_3 = r > 0.$
For validation, we have added source function $\mathfrak{f}(\mathrm{x},\mathfrak{t})$.

5.1.2 Main contribution of this chapter

This chapter presents a computational method that combines the finite difference technique with the operational matrix approach to solve the time-fractional Black-Scholes model. The key contributions and advantages of this chapter are highlighted in the following points:

- We have applied the L-123 approximation for descretizing the Caputo time-fractional derivative and operational matrix approach based on shifted Legendre polynomials basis function in the spatial direction.
- The L-123 approximation first converts the TFBSM into a system of ordinary differential equations in space variables at each time level. Then, the operational matrix transforms the system of ODEs into a system of algebraic equations.
- The proposed numerical scheme is applied on one test example which yield highly accurate results with a higher order of convergence. A comparative study of the numerical results by our proposed scheme compared with the existing scheme given

by [190] has been discussed. It reflects that the proposed scheme is more accurate than earlier scheme.

5.2 Time discretization

In this section, we have approximate Caputo fractional derivative of proposed model by using the L-123 approximation.

5.2.1 L-123 approximation of the Caputo derivatives

The L-123 approximation of the Caputo time fractional derivative of order $\alpha \in (0,1)$ is given in [185]. Let $\{\mathsf{t_k} = \mathsf{k}\tau,\ \mathsf{k} = 0,1,2,...,N_\mathsf{t}\}$ with step length $\tau = T/N_\mathsf{t}$ where N_t denotes the temporal discretization parameter, then the L-123 approximation is defined as

$$ {}_0\mathbb{D}_\mathsf{t}^\alpha \mathfrak{U}(\mathsf{t_k}) = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{\mathsf{k}} \gamma_{\mathsf{k-j}}^{(\alpha)} \delta_t \mathfrak{U}_{\mathsf{j}-\frac{1}{2}}, \tag{5.11} $$

$$ = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left(\gamma_0^\alpha \mathfrak{U}(\mathsf{t}_k) - \sum_{j=1}^{k-1} (\gamma_{k-j-1}^\alpha - \gamma_{k-j}^\alpha) \mathfrak{U}(\mathsf{t}_j) - \gamma_{k-1}^\alpha \mathfrak{U}(\mathsf{t}_0) \right), \tag{5.12} $$

where for $k = 1, \quad \gamma_0^\alpha = 1$;

for $k = 2$, $\gamma_0^\alpha = a_0^\alpha + b_0^\alpha$, $\gamma_1^\alpha = a_1^\alpha - b_0^\alpha$;

for $k = 3$,

$$ \gamma_l^{(\alpha)} = \begin{cases} a_l^{(\alpha)} + b_l^{(\alpha)} + \beta_l^{(\alpha)}, & l = 0, \\ a_l^{(\alpha)} + b_l^{(\alpha)} - b_{j-1}^{(\alpha)} - 2\beta_{l-1}^{(\alpha)}, & l = 1, \\ a_l^{(\alpha)} - b_{l-1}^{(\alpha)} + \beta_{l-2}^{(\alpha)}, & l = 2. \end{cases} \tag{5.13} $$

Here, $\gamma_0^{(\alpha)} \in (1, 11/6)$, $\gamma_1^{(\alpha)} \in (-7/6, 1)$, and $\gamma_2^{(\alpha)} \in (0, 1)$.

For $k \geq 4$,

$$\gamma_l^{(\alpha)} = \begin{cases} a_l^{(\alpha)} + b_l^{(\alpha)} + \beta_l^{(\alpha)}, & l = 0, \\ a_l^{(\alpha)} + b_l^{(\alpha)} - b_{l-1}^{(\alpha)} + \beta_l^{(\alpha)} - 2\beta_{l-1}^{(\alpha)}, & l = 1, \\ a_l^{(\alpha)} + b_l^{(\alpha)} - b_{l-1}^{(\alpha)} + \beta_l^{(\alpha)} - 2\beta_{l-1}^{(\alpha)} + \beta_{l-2}^{(\alpha)}, & 2 \leq l \leq k-3, \\ a_l^{(\alpha)} + b_l^{(\alpha)} - b_{l-1}^{(\alpha)} - 2\beta_{l-1}^{(\alpha)} + \beta_{l-2}^{(\alpha)}, & l = k-2, \\ a_l^{(\alpha)} - b_{l-1}^{(\alpha)} + \beta_{l-2}^{(\alpha)}, & l = k-1. \end{cases} \tag{5.14}$$

where $a_{\mathrm{j}}^{(\alpha)}$ and $b_{\mathrm{j}}^{(\alpha)}$ are given below.

$$a_l^{(\alpha)} = (l+1)^{2-\alpha} - l^{2-\alpha}, \qquad 0 \leq l \leq k-1, \tag{5.15}$$

$$b_l^{(\alpha)} = \frac{1}{(2-\alpha)}\{(l+1)^{2-\alpha} - l^{2-\alpha}\} - [(l+1)^{1-\alpha} + l^{1-\alpha}]/2, \quad l \geq 0. \tag{5.16}$$

5.3 Basis function and operational matrices

5.3.1 Shifted Legendre polynomials

Let $\mathbb{P}_i(\mathrm{x})$ be the Legendre polynomial of order i which is defined over the interval $[-1, 1]$. It satisfies the recursive formula

$$(i+1)\mathbb{P}_{i+1}(\mathrm{x}) = (2i+1)\mathrm{x}\mathbb{P}_i(\mathrm{x}) - i\mathbb{P}_{i-1}(\mathrm{x}), \quad i = 1, 2, 3..., \tag{5.17}$$

where $\mathbb{P}_0(\mathrm{x}) = 1, \quad \mathbb{P}_1(\mathrm{x}) = \mathrm{x}, \quad \mathbb{P}_2(\mathrm{x}) = \frac{1}{2}(3\mathrm{x}^2 - 1),...$

The shifted Legendre polynomial defined over the domain $[0, 1]$ is given by

$$p_i(\mathrm{x}) = \mathbb{P}_i(2\mathrm{x} - 1), \quad i = 0, 1, 2, 3, ..., \tag{5.18}$$

The Legendre polynomial are orthogonal w.r.t. the weight function $w(\mathrm{x}) = 1$ such that

$$\int_0^1 w(\mathrm{x})\mathbb{P}_i(\mathrm{x})\mathbb{P}_j(\mathrm{x}) = \begin{cases} \frac{1}{2i+1}\delta_{ij} & for \quad i = j, \\ 0 & otherwise. \end{cases} \tag{5.19}$$

Therefore, the orthonormal shifted Legendre polynomial in $[0, 1]$ is given by

$$\phi_i(\mathrm{x}) = \sqrt{i + \frac{1}{2}}\mathbb{P}_i(2\mathrm{x} - 1), \quad i = 0, 1, 2, 3, ... \tag{5.20}$$

The set $\Phi(\mathrm{x}) = [\phi_0(\mathrm{x}), \phi_1(\mathrm{x}), ..., \phi_n(\mathrm{x})]$ form an orthonormal basis of $L_2[0, 1]$ for the polynomial of degree $\leq n$. The orthonormal nature of the SLP saves the computational time in unknown Legendre's coefficient calculation.

5.3.2 Function approximation

Let $\mathfrak{f}(\mathrm{x}) \in L_2[0, 1]$ and $\Phi(\mathrm{x}) = [\phi_0(\mathrm{x}), \phi_1(\mathrm{x}), ..., \phi_M(\mathrm{x})]^T$ be the orthonormal basis of $L_2[0, 1]$ then the approximation of the function $\mathfrak{f}(\mathrm{x})$ can be written as

$$\mathfrak{f}(\mathrm{x}) \approx \sum_{i=0}^{M} \mathfrak{f}_i \phi_i(\mathrm{x}) = \mathbb{F}^T \Phi(\mathrm{x}), \tag{5.21}$$

where $\mathbb{F} = [\mathfrak{f}_0, \mathfrak{f}_1, ..., \mathfrak{f}_M]^T$, and the unknown coefficients $\mathfrak{f}_i$ are calculated by

$$\mathfrak{f}_i =< \mathfrak{f}(\mathrm{x}), \phi_i(\mathrm{x}) >= \int_0^1 w(\mathrm{x})\mathfrak{f}(\mathrm{x})\phi_i(\mathrm{x})d\mathrm{x}. \tag{5.22}$$

5.3.3 The operational matrix

Let $\Phi(\mathrm{x})$ be the orthogonal basis of $L_2[0,1]$ defined in Sec 5.3.1, then

$$\int_{\mathrm{x}_1}^{\mathrm{x}} \Phi(\xi)d\xi = \begin{bmatrix} \int_{\mathrm{x}_1}^{\mathrm{x}} \phi_0(\xi)d\xi \\ \int_{\mathrm{x}_1}^{\mathrm{x}} \phi_1(\xi)d\xi \\ . \\ . \\ . \\ \int_{\mathrm{x}_1}^{\mathrm{x}} \phi_M(\xi)d\xi \end{bmatrix} = \begin{bmatrix} k_0(\mathrm{x}) \\ k_1(\mathrm{x}) \\ . \\ . \\ . \\ k_M(\mathrm{x}) \end{bmatrix}. \tag{5.23}$$

Now approximating $k_i(\mathrm{x})$ as mentioned in 5.21

$$\begin{bmatrix} k_0(\mathrm{x}) \\ k_1(\mathrm{x}) \\ . \\ . \\ . \\ k_M(\mathrm{x}) \end{bmatrix} \approx \begin{bmatrix} \sum_{j=0}^{M} k_{0j}\phi_j(\mathrm{x}) \\ \sum_{j=0}^{M} k_{1j}\phi_j(\mathrm{x}) \\ \vdots \\ \sum_{j=0}^{M} k_{Mj}\phi_j(\mathrm{x}) \end{bmatrix} = \begin{bmatrix} k_{00} & k_{01} & \dots & k_{0M} \\ k_{10} & k_{11} & \dots & k_{1M} \\ \vdots & \vdots & \ddots & \vdots \\ k_{M0} & k_{M1} & \dots & k_{MM} \end{bmatrix} \begin{bmatrix} \phi_0(\mathrm{x}) \\ \phi_1(\mathrm{x}) \\ \vdots \\ \phi_M(\mathrm{x}) \end{bmatrix} = \mathbb{K}\Phi(\mathrm{x}), \tag{5.24}$$

therefore,

$$\int_{\mathrm{x}_1}^{\mathrm{x}} \Phi(\xi)d\xi = \mathbb{K}\Phi(\mathrm{x}), \tag{5.25}$$

where,

$$\mathbb{K} = \begin{bmatrix} k_{00} & k_{01} & \dots & k_{0M} \\ k_{10} & k_{11} & \dots & k_{1M} \\ \vdots & \vdots & \ddots & \vdots \\ k_{M0} & k_{M1} & \dots & k_{MM} \end{bmatrix} \tag{5.26}$$

is called the operational matrix of integration. The order of $\mathbb{K}$ is $(M+1)$. The coefficient k_{ij} can be calculated by

$$k_{ij} =< k_i(\mathrm{x}), \phi_j(\mathrm{x}) >= \int_0^1 w(\mathrm{x}) k_i(\mathrm{x}) \phi_j(\mathrm{x}) d\mathrm{x}. \tag{5.27}$$

5.3.4 Approximation for spatial terms

Let the numerical solution of the time-fractional Black-Scholes model (5.9) at time level $\mathfrak{t}_k$ for $k = 1, 2, ..., N$ is denoted by $\mathfrak{U}^k(\mathrm{x}) = \mathfrak{U}(\mathrm{x}, \mathfrak{t}_k)$. Therefore, let us suppose

$$\frac{d^2\mathfrak{U}^k(\mathrm{x})}{d\mathrm{x}^2} \approx \sum_{i=0}^{M} a_i^k \phi_i(\mathrm{x}) = \mathbb{A}_k^T \Phi(\mathrm{x}), \quad k = 1, 2, ..., N, \quad i = 0, 1, ..., M. \tag{5.28}$$

Integrating (5.28) from x_1 to x, we get,

$$\int_{\mathrm{x}_1}^{\mathrm{x}} \frac{d^2\mathfrak{U}^k(\mathrm{x})}{d\mathrm{x}^2} d\mathrm{x} \approx \int_{\mathrm{x}_1}^{\mathrm{x}} \mathbb{A}_k^T \Phi(\mathrm{x}) d\mathrm{x},$$

$$\frac{d\mathfrak{U}^k(\mathrm{x})}{d\mathrm{x}} \approx \frac{d\mathfrak{U}^k(\mathrm{x}_1)}{d\mathrm{x}} + \mathbb{A}_k^T \int_{\mathrm{x}_1}^{\mathrm{x}} \Phi(\mathrm{x}) d\mathrm{x}, \tag{5.29}$$

again integrating (5.29) from x_1 to x, one can obtain

$$\mathfrak{U}^k(\mathrm{x}) = \mathfrak{U}^k(\mathrm{x}_1) + (\mathrm{x} - \mathrm{x}_1) \frac{d\mathfrak{U}^k(\mathrm{x}_1)}{d\mathrm{x}} + \mathbb{A}_k^T \int_{\mathrm{x}_1}^{\mathrm{x}} \int_{\mathrm{x}_1}^{\mathrm{x}} \Phi(\mathrm{x}) d\mathrm{x}. \tag{5.30}$$

Putting $\mathrm{x} = \mathrm{x}_2$ and usning the boundary condition $\mathfrak{U}(\mathrm{x}_2, \mathfrak{t}_k) = \phi_2(\mathfrak{t}_k)$ in (5.30), we get

$$\frac{d\mathfrak{U}^k(\mathrm{x}_1)}{d\mathrm{x}} \approx \frac{1}{\mathrm{x}_2 - \mathrm{x}_1}\Big(\mathscr{G}_2(\mathfrak{t}_k) - \mathscr{G}_1(\mathfrak{t}_k) - \mathbb{A}_k^T \int_{\mathrm{x}_1}^{\mathrm{x}_2}\int_{\mathrm{x}_1}^{\mathrm{x}} \Phi(\mathrm{x})d\mathrm{x}\Big) \tag{5.31}$$

Using (5.31) in (5.29), we get

$$\frac{d\mathfrak{U}^k(\mathrm{x}_1)}{d\mathrm{x}} \approx \frac{1}{\mathrm{x}_2 - \mathrm{x}_1}\Big(\mathscr{G}_2(\mathfrak{t}_k) - \mathscr{G}_1(\mathfrak{t}_k) - \mathbb{A}_k^T \int_{\mathrm{x}_1}^{\mathrm{x}_2}\int_{\mathrm{x}_1}^{\mathrm{x}} \Phi(\mathrm{x})d\mathrm{x}\Big) + \mathbb{A}_k^T \int_{\mathrm{x}_1}^{\mathrm{x}} \Phi(\mathrm{x})d\mathrm{x},$$

$$\approx (\mathbb{G}_k^T + \mathbb{A}_k^T K_{12})\Phi(\mathrm{x}), \tag{5.32}$$

where $\mathbb{K}_{12} = \mathbb{K}_1 - \mathbb{K}_2$. The calculation of $\mathbb{G}_k^T, \mathbb{K}_1$ and $\mathbb{K}_2$ is given in the remark (5.1) and (5.3). Now, using (5.31) in (5.30) and simplifying, we get

$$\mathfrak{U}^k(\mathrm{x}) \approx \mathscr{G}_1(\mathfrak{t}_k) + \frac{(\mathrm{x} - \mathrm{x}_1)}{(\mathrm{x}_2 - \mathrm{x}_1)}\left(\mathscr{G}_2(\mathfrak{t}_k) - \mathscr{G}_1(\mathfrak{t}_k) - \mathbb{A}_k^T \int_{\mathrm{x}_1}^{\mathrm{x}}\int_{\mathrm{x}_1}^{\mathrm{x}} \Phi(\mathrm{x})d\mathrm{x}\right) + \mathbb{A}_k^T \int_{\mathrm{x}_1}^{\mathrm{x}}\int_{\mathrm{x}_1}^{\mathrm{x}} \Phi(\mathrm{x})d\mathrm{x},$$

$$\approx (\mathbb{B}_k^T + \mathbb{A}_k^T \mathbb{K}_{13})\Phi(\mathrm{x}), \tag{5.33}$$

where, $\mathbb{K}_{13} = \mathbb{K}_1^2 - \mathbb{K}_3$. The calcultion of $\mathbb{B}_k^T, \mathbb{K}_1^2$ and $\mathbb{K}_3$ is given in the remark below.

Remark 5.1. Since $\mathscr{G}_1(\mathfrak{t}_k), \mathscr{G}_2(\mathfrak{t}_k)$ are the boundary condition at each time level $\mathfrak{t}_k$ are the function of x also therefore assuming

$$\frac{1}{(\mathrm{x}_2 - \mathrm{x}_1)}(\mathscr{G}_2(\mathfrak{t}_k) - \mathscr{G}_1(\mathfrak{t}_k)) = g(\mathrm{x}) \approx \mathbb{G}_k^T \Phi(\mathrm{x}). \tag{5.34}$$

$$\left(1 - \frac{(\mathrm{x} - \mathrm{x}_1)}{(\mathrm{x}_2 - \mathrm{x}_1)}\right)\mathscr{G}_1(\mathfrak{t}_k) + \frac{(\mathrm{x} - \mathrm{x}_1)}{(\mathrm{x}_2 - \mathrm{x}_1)}\mathscr{G}_2(\mathfrak{t}_k) = \lambda_2(\mathrm{x}) \approx \mathbb{B}_k^T \Phi(\mathrm{x}). \tag{5.35}$$

Remark 5.2. We can approximate the source function $\mathfrak{f}(\mathrm{x}, \mathfrak{t}_k) \approx \mathbb{F}_k^T \Phi(\mathrm{x})$ at each time level $\mathfrak{t}_k$ and initial condition $\mathfrak{U}(\mathrm{x}, 0) = \mathbb{H}(X) \approx \mathbb{H}^T \Phi(\mathrm{x})$.

Remark 5.3. The operational matrix of integration $\mathbb{K}_1, \mathbb{K}_2$ and $\mathbb{K}_3$ are calculated as

$$\int_{x_1}^{\mathrm{x}} \Phi(\mathrm{x}) d\mathrm{x} = k_1(\mathrm{x}) \approx \mathbb{K}_1 \Phi(\mathrm{x}), \tag{5.36}$$

$$\int_{\mathrm{x}_1}^{\mathrm{x}} \int_{\mathrm{x}_1}^{\mathrm{x}} \Phi(\mathrm{x}) d\mathrm{x} = k_{12}(\mathrm{x}) \approx \mathbb{K}_1^2 \Phi(\mathrm{x}), \tag{5.37}$$

$$\frac{1}{(\mathrm{x}_2 - \mathrm{x}_1)} \int_{\mathrm{x}_1}^{\mathrm{x}_2} \int_{\mathrm{x}_1}^{\mathrm{x}} \Phi(\mathrm{x}) d\mathrm{x} = k_2(\mathrm{x}) \approx \mathbb{K}_2 \Phi(\mathrm{x}), \tag{5.38}$$

$$\frac{\mathrm{x} - \mathrm{x}_1}{(\mathrm{x}_2 - \mathrm{x}_1)} \int_{\mathrm{x}_1}^{\mathrm{x}_2} \int_{\mathrm{x}_1}^{\mathrm{x}} \Phi(\mathrm{x}) d\mathrm{x} = k_3(\mathrm{x}) \approx \mathbb{K}_3 \Phi(\mathrm{x}). \tag{5.39}$$

5.4 Derivation of numerical scheme for TFBSM

In this section, we design the numerical algorithm by combining the finite difference method with operational matrix approach for the TFBSM (5.9) – (5.10). The TFBSM at each time level $\mathfrak{t}_k$ can be written as

$${}_0^C\mathrm{D}_{\mathfrak{t}}^{\alpha} \mathfrak{U}(\mathrm{x}, \mathfrak{t}) = \lambda_1 \frac{\partial^2 \mathfrak{U}(\mathrm{x}, \mathfrak{t})}{\partial \mathrm{x}^2} + \lambda_2 \frac{\partial \mathfrak{U}(\mathrm{x}, \mathfrak{t})}{\partial \mathrm{x}} - \lambda_3 \mathfrak{U}(\mathrm{x}, \mathfrak{t}) + \mathfrak{f}(\mathrm{x}, \mathfrak{t}), \tag{5.40}$$

with boundary and initial conditions

$$\mathfrak{U}(\mathrm{x}_1, \mathfrak{t}_k) = \mathscr{G}_1(\mathfrak{t}_k),\ \mathfrak{U}(\mathrm{x}_2, \mathfrak{t}_k) = \mathscr{G}_2(\mathfrak{t}_k), \tag{5.41}$$

$$\mathfrak{U}(\mathrm{x}, 0) = \mathbb{H}(\mathrm{x}). \tag{5.42}$$

Applying L-123 approximation for Caputo deivative at time $\mathtt{t}_k$, $(k \geq 1)$ in (5.40) and re-arranging the terms, we have

$$\lambda_1 \frac{d^2\mathfrak{U}^k(\mathrm{x})}{d\mathrm{x}^2} + \lambda_2 \frac{d\mathfrak{U}^k(\mathrm{x})}{d\mathrm{x}} - \lambda_3 \mathfrak{U}^k(\mathrm{x}) = {}_0^C\mathrm{D}_{\mathfrak{t}}^{\alpha}\mathfrak{U}^k(\mathrm{x}) - \mathfrak{f}^k(\mathrm{x}). \tag{5.43}$$

Since from (5.12)

$${}_0^C\mathrm{D}_{\mathfrak{t}}^{\alpha}\mathfrak{U}^k(\mathrm{x}) = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\Big[\gamma_0^{\alpha}\mathfrak{U}^k(\mathrm{x}) + \sum_{j=1}^{k-1}\big(\gamma_{k-j}^{\alpha} - \gamma_{k-j-1}^{\alpha}\big)\mathfrak{U}^k(\mathrm{x}) - \gamma_{k-1}^{\alpha}\mathfrak{U}^o(\mathrm{x})\Big]. \tag{5.44}$$

Therefore, (5.43) becomes

$$\begin{aligned}\lambda_1 \frac{d^2\mathfrak{U}^k(\mathrm{x})}{d\mathrm{x}^2} + \lambda_2 \frac{d\mathfrak{U}^k(\mathrm{x})}{d\mathrm{x}} - \lambda_3 \mathfrak{U}^k(\mathrm{x}) &= \mathtt{d}\gamma_0^{\alpha}\mathfrak{U}^k(\mathrm{x}) + \mathtt{d}\left(\sum_{j=1}^{k-1}\big(\gamma_{k-j}^{\alpha} - \gamma_{k-j-1}^{\alpha}\big)\mathfrak{U}^k(\mathrm{x})\right) \\ &\quad - \mathtt{d}\gamma_{k-1}^{\alpha}\mathfrak{U}^o - \mathfrak{f}^k(\mathrm{x}),\end{aligned} \tag{5.45}$$

where $\mathtt{d} = \dfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}$.

$$\lambda_1 \frac{d^2\mathfrak{U}^k(\mathrm{x})}{d\mathrm{x}^2} + \lambda_2 \frac{d\mathfrak{U}^k(\mathrm{x})}{d\mathrm{x}} - \big(\lambda_3 + \mathtt{d}\gamma_0^{\alpha}\big)\mathfrak{U}^k(\mathrm{x}) = \mathtt{d}\left(\sum_{j=1}^{k-1}\big(\gamma_{k-j-1}^{\alpha} - \gamma_{k-j}^{\alpha}\big)\mathfrak{U}^k(\mathrm{x}) - \gamma_{k-1}^{\alpha}\mathfrak{U}^o\right) - \mathfrak{f}^k(\mathrm{x}), \tag{5.46}$$

Now, using the approximation as discussed in Section (5.3.4) in (5.46), we get

$$\begin{aligned}&\lambda_1\mathbb{A}_k^T\Phi(\mathrm{x}) + \lambda_2(\mathbb{G}_k^T + \mathbb{A}_k^T\mathbb{K}_{12})\Phi(\mathrm{x}) - (\lambda_3 + \mathtt{d}\gamma_0^{\alpha})(\mathbb{B}_k^T + \mathbb{A}_k^T\mathbb{K}_{13})\Phi(\mathrm{x}) = \\ &\mathtt{d}\sum_{j=1}^{k-1}\big(\gamma_{k-j}^{\alpha} - \gamma_{k-j-1}^{\alpha}\big)(\mathbb{B}_k^T + \mathbb{A}_k^T\mathbb{K}_{13})\Phi(\mathrm{x}) - \mathtt{d}\gamma_{k-1}^{\alpha}\mathbb{H}^T\Phi(\mathrm{x}) - \mathbb{F}_k^T\Phi(\mathrm{x}).\end{aligned} \tag{5.47}$$

Comparing the coefficients of unknowns a_{ij} from both sides, we get (M+1) system of equations at each time level $\mathtt{t}_k$, $(k \geq 1)$

$$\lambda_1 \mathbb{A}_k^T + \lambda_2 \mathbb{A}_k^T \mathbb{K}_{12} - (\lambda_3 + \mathtt{d}\gamma_0^\alpha) \mathbb{A}_k^T \mathbb{K}_{13} = \mathtt{d} \sum_{j=1}^{k-1} \big(\gamma_{k-j-1}^\alpha - \gamma_{k-j}^\alpha\big)(\mathbb{B}_k^T + \mathbb{A}_k^T \mathbb{K}_{13}) \\ -\mathtt{d}\gamma_{k-1}^\alpha \mathbb{H}^T - \mathbb{F}_k^T - \lambda_2 \mathbb{G}_k^T + (\lambda_3 + \mathtt{d}\gamma_0^\alpha)\mathbb{B}_k^T. \quad (5.48)$$

Therefore, the matrix form of the numerical scheme is

$$\left(\lambda_1 \mathbb{I} + \lambda_2 \mathbb{K}_{12}^T - (\lambda_3 + \mathtt{d}\gamma_0^\alpha)\mathbb{K}_{13}^T\right)\mathbb{A}_k = \mathtt{d} \sum_{j=1}^{k-1} \left(\gamma_{k-j-1}^\alpha - \gamma_{k-j}^\alpha\right)(\mathbb{B}_k + \mathbb{A}_j \mathbb{K}_{13}^T) \\ -\mathtt{d}\gamma_{k-1}^\alpha \mathbb{H} - \mathbb{F}_k - \lambda_2 \mathbb{G}_k + (\lambda_3 + \mathtt{d}\gamma_0^\alpha)\mathbb{B}_k. \quad (5.49)$$

Solving the system of algebraic equation (5.49), we get $\mathbb{A}_k$ whose elements are unknown Legendre's coefficients a_{ij}. Putting the values of $\mathbb{A}_k$, $\mathbb{B}_k$, $\mathbb{K}_{13}$ in $\mathfrak{U}^k(\mathrm{x}) \approx (\mathbb{B}_k^T + \mathbb{A}_k^T \mathbb{K}_{13})\Phi(\mathrm{x})$ to get the approximate solution at each time level $\mathtt{t}_k$. The advantage of using a meshfree operational matrix approach is that we get the solution at any point in the spatial domain.

The algorithm for solving by the proposed numerical scheme (5.49) is given below.

Algorithm 4: To evaluate the numerical solution of TFBSM (5.9) − (5.10).

Input: The constants $\lambda_1, \lambda_2, \lambda_3, \mathbb{H}(\mathrm{x}), \mathscr{G}_1(\mathfrak{t}_k), \mathscr{G}_2(\mathfrak{t}_k), \mathfrak{f}(\mathrm{x}), \alpha \in (0, 1)$.

Output: The approximate solutions at each time level

$\mathfrak{U}^k(\mathrm{x}) \approx (\mathbb{B}_k^T + \mathbb{A}_k^T \mathbb{K}_{13})\Phi(\mathrm{x})$.

for Numerical solution of TFBSM by the semi-discrete scheme **do**

Step-1.1 Discritize the rectangular domain in uniform step size

$\mathfrak{t}_k = k\Delta\mathfrak{t},\ k = 0, 1, \ldots N$.

Step-1.2 Approximate the Caputo fractional derivative by L-123 approximation, we transform the TFBSM in a function of space variable 'x' at each time level $\mathfrak{t}_k$.

step-1.3 Generating basis function $\phi_i(\mathrm{x}); i = 0, ..., M$ by using shifted Legendre polynomial as given in section 5.3.1.

Step-1.4 By using the boundary conditions $\mathfrak{U}(\mathrm{x}_1, \mathfrak{t}_k) = \mathscr{G}_1(\mathfrak{t}_k)$ and $\mathfrak{U}(\mathrm{x}_2, \mathfrak{t}_k) = \mathscr{G}_2(\mathfrak{t}_k)$, compute the operational vector values of $\mathbb{G}_k^T$ and $\mathbb{B}_k^T$ as defined in remark 5.1.

Step-1.5 With the help of source term $\mathfrak{f}(\mathrm{x}, \mathfrak{t}_k)$, compute the operational vector $\mathbb{F}_K^T$ and by using initial condition $\mathfrak{U}(\mathrm{x}, 0)$, compute $\mathbb{H}^T$ as given in remark 5.2.

Step-1.6 Compute the operational matrix of integration $\mathbb{K}_1, \mathbb{K}_1^2, \mathbb{K}_2, \mathbb{K}_3$ of order $M + 1$ as defined in Remark 5.3.

Step-1.7 Compute $\mathbb{K}_{12} = \mathbb{K}_1 - \mathbb{K}_2$ and $\mathbb{K}_{13} = \mathbb{K}_1^2 - \mathbb{K}_3$.

Step-1.8 Assuming $\mathfrak{U}_{\mathrm{xx}}^k(\mathrm{x}) \approx \mathbb{A}^T\phi(\mathrm{x})$ we get the $\mathfrak{U}_{\mathrm{x}}^k(\mathrm{x}) \approx ((\mathbb{G}_k^T + \mathbb{A}_k^T\mathbb{K}_{13})\Phi(\mathrm{x}))$ and $\mathfrak{U}^k(\mathrm{x}) \approx (\mathbb{B}_k^T + \mathbb{A}_k^T\mathbb{K}_{13}\Phi(\mathrm{x}))\ \ i = 1, ...N - 1$.

Step-1.9 Using step-1.8 in step-1.1, we get the matrix form of the numerical scheme at each time level k given in (5.49).

Step-1.10 Solving the system of equations (5.49) we get the values of unknown vector $\mathbb{A}_k$ at each time level k.

Step-1.11 Put the values of $\mathbb{A}_k, \mathbb{B}_k, \mathbb{K}_{13}$ in $\mathfrak{U}^k(\mathrm{x}) \approx (\mathbb{B}_k^T + \mathbb{A}_k^T\mathbb{K}_{13})\Phi(\mathrm{x})$ to get the approximate solution at each time level $\mathfrak{t}_k$.

end

5.5 Numerical results and discussion

In this section, we give a numerical example of the TFBSM for the European options to validate the error efficiency of the proposed numerical algorithm.

5.5.1 Numerical examples for the time-fractional Black-Scholes model

Example 5.1. Consider the following TFBSM [132] with homogeneous boundary conditions

$$\begin{cases} {}_0^C\mathrm{D}_\mathfrak{t}^\alpha \mathfrak{U}(\mathrm{x},\mathfrak{t}) = \lambda_1 \dfrac{\partial^2 \mathfrak{U}(\mathrm{x},\mathfrak{t})}{\partial \mathrm{x}^2} + \lambda_2 \dfrac{\partial \mathfrak{U}(\mathrm{x},\mathfrak{t})}{\partial \mathrm{x}} - \lambda_3 \mathfrak{U}(\mathrm{x},\mathfrak{t}) + \mathfrak{f}(\mathrm{x},\mathfrak{t}) \in (0,1)\times(0,1) \\ \text{Initial condition} : \mathfrak{U}(\mathrm{x},0) = \mathrm{x}^4 + 1, \quad \mathrm{x} \in (0,1), \\ \text{Boundary condition} : \mathfrak{U}(0,\mathfrak{t}) = \mathfrak{t}^3 + 1, \quad \mathfrak{U}(1,\mathfrak{t}) = 2(\mathfrak{t}^3+1), \quad \mathfrak{t} \in (0,1), \end{cases} \tag{5.50}$$

with $\alpha \in (0,1)$ with the source function

$$\mathfrak{f}(\mathrm{x},\mathfrak{t}) = \frac{6\mathfrak{t}^{3-\alpha}}{\Gamma(4-\alpha)}(1+\mathrm{x}^4) - (\mathfrak{t}^3+1)[12\lambda_1\mathrm{x}^2 + 4\lambda_2\mathrm{x}^3 - \lambda_3(\mathrm{x}^4+1)]. \tag{5.51}$$

The exact solution of Ex. 5.1 is

$$\mathfrak{U}(\mathrm{x},\mathfrak{t}) = (1+\mathrm{x}^4)(\mathfrak{t}^3+1). \tag{5.52}$$

Here, M=5 SLP basis function, and the parameters values are $\mathrm{r} = 0.05$, $\sigma = \sqrt{2}$, $\lambda_1 = 1$, $\lambda_2 = -0.5$, $\lambda_3 = 0.5$.

TABLE 5.1: L_2 error and order for Ex. 5.1 different values of α using M=5 SLP basis.

τ	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 0.9$	
	L_2 error	order	L_2 error	order	L_2 error	order
1/10	5.0898e-07	0	3.3198e-06	0	7.0955e-06	0
1/20	3.0120e-08	4.0788	2.2790e-07	3.8646	5.8691e-07	3.5957
1/40	1.8458e-09	4.0284	1.6998e-08	3.7449	6.2184e-08	3.2385
1/80	1.1496e-10	4.0050	1.3332e-09	3.6724	6.9780e-09	3.1556
1/160	7.2254e-13	3.9919	1.0823e-10	3.6227	8.0118e-10	3.1226
1/320	4.6456e-13	3.9592	8.9966e-12	3.5886	9.2839e-11	3.1093

TABLE 5.2: L_∞ error and order for Ex. 5.1 different values of α using M=5 SLP basis.

τ	$\alpha = 0.1$		$\alpha = 0.5$		$\alpha = 0.9$	
	L_∞ error	order	L_∞ error	order	L_∞ error	order
1/10	6.9249e-07	0	4.5958e-06	0	9.9586e-06	0
1/20	4.0993e-08	4.0279	3.1632e-07	3.8608	8.2455e-07	3.5943
1/40	2.5131e-09	4.0279	2.3655e-08	3.7412	8.7428e-08	3.2374
1/80	1.5658e-10	4.0045	1.8595e-09	3.6691	9.8192e-09	3.1550
1/160	9.8446e-12	3.9914	1.5125e-10	3.6199	1.1272e-09	3.1223
1/320	6.3283e-13	3.9594	1.2592e-11	3.5863	1.3063e-10	3.1092

Table 5.3: Comparision of L_∞ error for Ex. 5.1 different values of α using M=5 SLP basis.

τ	$\alpha = 0.2$		$\alpha = 0.5$		$\alpha = 0.8$	
	L_∞ error	L_∞ error [190]	L_∞ error	L_∞ error [190]	L_∞ error	L_∞ error [190]
1/10	1.4326e-06	5.3144e-06	4.5958e-06	3.0317e-05	9.0352e-06	1.2014e-04
1/20	8.6065e-08	8.1141e-07	3.1632e-07	5.4650e-06	8.2775e-07	2.6248e-05
1/40	5.3873e-09	1.2242e-07	2.3655e-08	9.7909e-07	8.1051e-08	5.7259e-06
1/80	3.4427e-10	1.8200e-08	1.8595e-09	1.7459e-07	8.3990e-09	1.2477e-06
1/160	2.2280e-11	2.5779e-09	1.5125e-10	3.0941e-08	8.9384e-10	2.7161e-07
1/320	1.4651e-12	-	1.2592e-11	-	9.6292e-11	-

- Based on both Tables 5.1-5.2, we conclude that the L_2 and L_∞-error are almost the same with the SLP basis function. Furthermore, the temporal convergence rate in the L_2 and L_∞ norm is observed to be of order $(4-\alpha)$, which aligns well with the theoretical expectations.

- Table 5.3 presents a comparison of the L_∞ error obtained using the proposed algorithm and the scheme described in [190]. The findings clearly show that the proposed algorithm achieves higher error efficiency relative to the existing scheme.

5.6 Conclusion

We have presented a new semi-discrete scheme for time fractional Black-Scholes model with a combination of finite difference and operational matrix approach. The Caputo derivative of order α, where $1 < \alpha < 2$, has been approximated by L-123 approximation, and the space derivatives have been approximated by operational matrix approach based on shifted Legendre polynomial. The algorithm is observed to be highly accurate and showing a higher order of convergence, $(4-\alpha)$ in time for all alpha $0 < \alpha < 1$. The comparative study of the numerical results has shown that the proposed algorithm performs better in terms of error efficiency and convergence order. Therefore, this semi-discrete approach is found to be an effective numerical approach for solving other fractional and variable-order mathematical models.

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