

4 Hypersurface-Homogeneous Cosmological Models with Matter and Dark energy

4.1 Introduction

Recent measurements from some type Ia supernovae (SNe) at intermediate and high redshifts (Perlmutter et al.(1999), Riess et al.(1998)) indicate that the bulk of the energy in the universe is repulsive and appears like a quintessence component, that is, an unknown form of energy, called dark energy (in addition to the ordinary CDM matter) probably of primordial origin (Turner (2000)). Together with the observations of CMB anisotropies (Bernardis et al.(2000)) such results seem to have provided an important piece of information connecting an early inflationary stage with the astronomical observations. This state of affairs has stipulated interest in more general models containing an extra component describing the dark energy, and simultaneously accounting for the present accelerated stage of the universe. A

The contents of this chapter have been published in *Advanced Studies in Theoretical Physics* 6(24), 1189-1198 (2012), *Journal of Dynamical System and Geometric Theories* 12(2), 93-101 (2014) and *Canadian Journal of Physics*, 10.1139/cjp-2015-0040 (2015).

possible list of old and new candidates for quintessence includes a time varying cosmological term Λ (Overduin and Cooperstock (1998), Weinberg (1989)). The basic reason is the widespread belief that the early universe evolved through some phase transitions, thereby yielding a vacuum energy density. The Λ term has been interpreted in terms of a Higg's scalar field by Bergmann (1968). Drietein (1974) has suggested that the mass of the Higg's boson is connected with Λ , being a function of temperature and is related to the process of broken symmetries; and therefore it could be a function of time in a spatially homogeneous expanding universe (Weinberg (1967)). By taking into account the conservation of energy momentum tensor of matter and vacuum taken together, several cosmologists have evolved the idea of a decreasing vacuum energy and hence a time varying cosmological term Λ with cosmic expansions in the framework of Einstein's theory.

The cosmological models of cosmic fluid with viscosity play significant role in the study of evolution of universe at its early stages. In the early stages of cosmic expansion, it have been argued for a long time that the dissipative process may well account for the high degree of isotropy we observe today. Dissipative effects including both bulk and shear viscosities, play very important role in the early evolution of the universe. Bulk viscosity is associated with Grand Unified Theory (GUT) phase transition and string creation. Johri and Sudarshan (1988) have investigated the effect of bulk viscosity on the evolution of Friedman model and found that the presence of tiny time-dependent component of bulk viscosity would play a crucial role in driving the present day universe into a steady state. Padmanabhan and Chitre (1987) have shown that the presence of bulk viscosity leads to inflationary

like solutions in general relativity. The effect of bulk viscosity on the cosmological evolution has been investigated by a number of authors viz. Misner (1967, 1968), Banerjee and Sanyal (1986), Banerjee et al. (1986), Bali and Jain (1991), Saha (2005), Pradhan and Pandey (2006), Sahni and Starobinsky (2000), Padmanabhan (2003), Bali and Pradhan (2007), Bali and Kumawat (2008), Verma and Ram (2010) and Pradhan and Kumar (2009) etc.

Pradhan et al. (2011) have studied DE models by considering the special law of variation of Hubble's parameter that yields a constant value of the deceleration parameter. Some authors (Akarsu and Kilinc (2010a, 2010b), Yadav et al.(2011), Yadav and Yadav (2011), Kumar and Yadav (2011), Adhav et al.(2011), Yadav and Saha (2012) and Saha and Yadav (2012) have studied DE models with variable EoS parameter. Recently, Adhav et al. (2013) have studied anisotropic and spatially homogeneous LRS Bianchi type-I, Bianchi type-V, Bianchi type-III, Bianchi type VI_0 and Kantowski-Sachs space times with variable EoS parameter in general relativity, which correspond to early decelerating and late time accelerating anisotropic cosmological models with dynamical EoS parameter.

In this chapter, we first study a hypersurface-homogeneous models of the universe with bulk viscosity and time varying cosmological term in Sec.(4.2). In Sec.(4.3), we present hypersurface-homogeneous space-time filled with perfect fluid and isotropic EoS parameter and obtain exact solutions of field equations for three different values of the parameter K in the metric. In Sec.(4.4), we investigate hypersurface homogeneous cosmological models in the presence of anisotropic fluid with dynamical EoS parameter in the framework of Lyra geometry.

The general metric for a hypersurface-homogeneous space-time given by

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)[dy^2 + \Sigma^2(y, K)dz^2] \quad (4.1)$$

where $\Sigma(y, K) = \sin y, y, \sinh y$ respectively for $K = 1, 0, -1$. Stewart and Ellis (1968) have obtained some general solutions of Einstein's field equations for a perfect fluid distribution satisfying the barotropic equation of state. Hajj-Boutros (1985) proposed a method to construct exact solutions of the field equations for the metric (4.1) in the presence of perfect fluids and obtained some solutions not satisfying the barotropic equation of state. Verma and Shri Ram (2010) have presented some hypersurface homogeneous bulk viscous cosmological models with time-dependent cosmological term. Further, Shri Ram and Verma (2010) have investigated bulk viscous fluid hypersurface-homogeneous models with time varying gravitational constant and cosmological term. Singh and Beesham (2011) have investigated some hypersurface-homogeneous cosmological models in the presence of a dynamically anisotropic EoS parameter and a dynamical energy density. Katore et al. (2012) have investigated accelerating hypersurface-homogeneous cosmological models in Barber's second self-creation theory.

4.2 Hypersurface-Homogeneous Bulk Viscous Fluid Models with Decaying Cosmological Term

In this section, we present hypersurface-homogeneous cosmological models with bulk viscous fluid and decaying cosmological term.

4.2.1 Field Equations

The energy momentum tensor T_{ij} for a bulk viscous fluid is given by

$$T_{ij} = (\rho + \bar{p})u_i u_j - \bar{p}g_{ij} \quad (4.2)$$

where

$$\bar{p} = p - \xi u_{;i}^i. \quad (4.3)$$

Here ρ , p , \bar{p} and ξ are respectively energy density of matter, isotropic pressure, effective pressure and bulk viscosity coefficient. In comoving coordinates, Einstein's field equations

$$R_{ij} - \frac{1}{2}Rg_{ij} = -T_{ij} + \Lambda g_{ij}. \quad (4.4)$$

with time varying Λ in the presence of bulk viscous fluid for the metric (4.1), lead to the following set of equations:

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{K}{B^2} = -\bar{p} + \Lambda, \quad (4.5)$$

$$\frac{\ddot{B}}{B} + \frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB} = -\bar{p} + \Lambda, \quad (4.6)$$

$$\frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} + \frac{K}{B^2} = \rho + \Lambda. \quad (4.7)$$

The kinematical quantities that are of cosmological interest have the following expressions:

$$V = a^3 = AB^2, \quad (4.8)$$

$$H_x = \frac{\dot{A}}{A}, \quad H_y = H_z = \frac{\dot{B}}{B}, \quad (4.9)$$

$$H = \frac{1}{3} \left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B} \right), \quad (4.10)$$

$$\theta = \frac{\dot{A}}{B} + \frac{2\dot{B}}{B}, \quad (4.11)$$

$$\sigma^2 = \frac{1}{2}\sigma^{ij}\sigma_{ij} = \frac{1}{3}\left(\frac{\dot{A}}{B} - \frac{\dot{B}}{B}\right)^2. \quad (4.12)$$

4.2.2 Solutions of Field Equations

We first show that the system of highly non-linear differential equations (4.5)-(4.7) are solvable for any arbitrary cosmic scale function. From Eqs.(4.5) and (4.6), we obtain

$$\frac{\ddot{B}}{B} - \frac{\ddot{A}}{A} + \frac{\dot{B}^2}{B^2} - \frac{\dot{A}\dot{B}}{AB} + \frac{K}{B^2} = 0 \quad (4.13)$$

which, on integration, gives

$$-B^2\dot{A} + AB\dot{B} = -K \left[\int A dt + c_1 \right] \quad (4.14)$$

where c_1 is an arbitrary constant. Eq.(4.14) can be written in the form

$$\frac{d}{dt}(B^2) - \frac{2\dot{A}}{A}(B^2) = F(t) \quad (4.15)$$

where

$$F(t) = -\frac{2K}{A} \left[\int A dt + c_1 \right]. \quad (4.16)$$

The linear differential equation (4.15) admits the general solution as given by

$$B^2 = A^2 \left[\int \frac{F(t)}{A^2} dt + c_2 \right] \quad (4.17)$$

where c_2 is integration constant. Thus, the solution of Einstein's field equations reduces to the integration in Eq.(4.17) if $A(t)$ is known as an explicit function of time.

Here we obtain solutions of the field equations by choosing

$$A = t^n \quad (4.18)$$

where n is a positive real number. Performing integration in Eq.(4.17) , we obtain

$$B^2 = \frac{Kt^2}{n^2 - 1} + \frac{2Kc_1t^{1-n}}{3n - 1} + c_2t^{2n+1}. \quad (4.19)$$

If we choose $c_1 = c_2 = 0$, then the scale function B has the particular solution given by

$$B^2 = \frac{Kt^2}{n^2 - 1}, n \neq 1. \quad (4.20)$$

Hence, the hypersurface-homogeneous space-time (4.1) corresponding to Eqs. (4.18) and (4.20) can be written in the form becomes

$$ds^2 = dt^2 - t^{2n}dx^2 - \frac{Kt^2}{n^2 - 1} \left[dy^2 + \sum^2(y, K)dz^2 \right]. \quad (4.21)$$

The metric (4.21) is well defined for $n \neq 1$. For the metric(4.21), the expressions for effective pressure \bar{p} and energy density ρ , as calculated from Eqs.(4.5) and (4.7), are given by

$$\bar{p} = \Lambda - \frac{n^2}{t^2}, \quad (4.22)$$

$$\rho = -\Lambda + \frac{n(n+2)}{t^2}. \quad (4.23)$$

It is clear that we can determine explicitly physical parameters ρ , p , Λ if the bulk viscosity coefficient ξ is specified. In most of the applications, the bulk viscosity coefficient is assumed as a simple power function of the energy density (Weinberg (1968)):

$$\xi(t) = \xi_0\rho^\beta \quad (4.24)$$

where ξ_0 and β are constants. Murphy (1973) has assumed $\beta=1$, which corresponds to a radiating fluid. We assume $\xi = \xi_0\rho$. We also assume that the fluid obeys the barotropic equation of state

$$p = \omega\rho, \quad 0 \leq \omega \leq 1. \quad (4.25)$$

Then, from Eqs.(4.22), (4.24) and (4.25), we have

$$\rho(\omega - \xi_0\theta) = \Lambda - \frac{n^2}{t^2} \quad (4.26)$$

where the expansion scalar θ is given by

$$\theta = \frac{n+2}{t}. \quad (4.27)$$

Combining Eqs.(4.23) and (4.26), we get

$$\rho = \frac{2n}{t[(1+\omega)t - (n+2)\xi_0]}. \quad (4.28)$$

Substituting Eq.(4.28) in Eq.(4.23), we obtain

$$\Lambda = \frac{[n^2 + (n+2)\omega]t - n(n+2)^2\xi_0}{t^2[(1+\omega)t - (n+2)\xi_0]}. \quad (4.29)$$

From Eqs. (4.28) and (4.29), we observe that the bulk viscosity contributes significantly to the expressions of ρ , p and Λ . The bulk viscosity coefficient $\xi(t)$ has the expression given by

$$\xi = \frac{2n\xi_0}{t[(1+\omega)t - (n+2)\xi_0]}. \quad (4.30)$$

We now look at cosmological models for $K = 1$ and -1 in the following subsections.

4.2.2.1 Model I

When $K = 1$, the metric of our solution (4.21) reduces to

$$ds^2 = dt^2 - t^{2n}dx^2 - \frac{t^2}{n^2 - 1}(dy^2 + \sin^2 y dz^2). \quad (4.31)$$

This metric is well defined for $n > 1$.

For the model (4.31), the expressions for the kinematical parameters spatial

volume, scalar expansion, shear scalar and deceleration parameter are given by

$$V = \frac{t^{n+2}}{n^2 - 1}, \quad (4.32)$$

$$\theta = \frac{n + 2}{t}, \quad (4.33)$$

$$\sigma^2 = \frac{(n - 1)^2}{3t^2}, \quad (4.34)$$

$$q = \frac{1 - n}{n + 2}. \quad (4.35)$$

The deceleration parameter is negative and therefore model (4.31) represents an inflationary accelerating model which is consistent with present day universe.

We observe that the spatial volume is zero at $t = 0$, and it increases with cosmic time. This means that the model starts expanding with a big-bang at $t = 0$. The physical and kinematical parameters ρ , p , θ and σ diverge at this initial singularity. The physical and kinematical parameters are well defined and are decreasing functions for $0 < t < \infty$, and ultimately tend to zero for large time. The cosmological term $\Lambda(t)$ is infinite at the beginning and gradually decreases as time increases and ultimately becomes zero at late times. The bulk viscosity coefficient, being infinite at the initial epoch $t = 0$, is a decreasing function of time and dies out as $t \rightarrow \infty$. Since $\frac{\sigma}{\theta} = \frac{n-1}{\sqrt{3}(n+2)}$, the anisotropy in the universe is maintained throughout the passage of time.

4.2.2.2 Model II

When $K = -1$, the line-element (4.21) becomes

$$ds^2 = dt^2 - t^{2n} dx^2 - \frac{t^2}{1 - n^2} [dy^2 + \sinh^2 y dz^2] \quad (4.36)$$

where $-1 < n < 1$.

The deceleration parameter q is positive since $n < 1$. Therefore, the model (4.36) decelerates in the standard way. It deserves mention that the decelerating models are also consistent with recent CMB observations made by WMAP, as well as with the high redshift supernovae Ia data including SN 1997 ff at $Z=1.755$ (Vishwakarma (2003)) The other physical properties of the model are similar to that of model I.

4.2.3 Conclusions

We have investigated hypersurface-homogeneous cosmological models in the presence of bulk viscous fluid and time-dependent cosmological term Λ with the assumptions that (i) the cosmic fluid satisfies the barotropic equation of state and (ii) the bulk viscosity coefficient ξ is directly proportional to energy density ρ . For the specific choice of one of the scale functions we have presented two classes of exact solutions of Einstein's field equations for $K = 1$ and $K = -1$ which represent expanding, shearing and accelerating/decelerating models of the universe respectively. The bulk viscosity contributes significantly to the expressions of energy density, pressure and cosmological constant. The cosmological evolution of the models is expansionary with the scale factors monotonically increasing functions of time. The universe starts expanding with a big-bang singularity at $t = 0$. The parameters p , ρ and ξ starts of with extremely large values, which continued to decrease with the expansion of the universe and ultimately tend to zero. The cosmological term Λ , being initially large, decreases with the increase of time and

approaches to zero for large time. This nature of decaying Λ is supported by recent results from the observations of type Ia supernova explosion (SNe Ia).

4.3 Hypersurface-Homogeneous Cosmological Models with Matter and Dark Energy

In this section, we present hypersurface-homogeneous cosmological models with matter and dark energy.

4.3.1 Field Equations

In comoving coordinates, Einstein field equations (2.6) for the metric (4.1) in the presence of a perfect fluid satisfying the equation of state parameter ($p = \omega\rho$), lead to the following set of equations:

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{K}{B^2} = -\omega\rho, \quad (4.37)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} = -\omega\rho, \quad (4.38)$$

$$\frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} + \frac{K}{B^2} = \rho \quad (4.39)$$

where EoS parameter ω is not necessarily a constant.

4.3.2 Solutions of Field Equations

We now have a set of three highly non-linear equations in four unknowns A , B , ρ and ω . In order to solve them completely, we need an extra condition either on

physical ground or for mathematical convenience. We assume that

$$B = A^n \tag{4.40}$$

where n is a constant. This is a physically plausible condition as discussed by Thorne (1967), Bali and Jain (1998), Pradhan and Singh (2004) etc.

Combining Eqs.(4.8) and (4.40) , we obtain

$$A = V^{\frac{1}{2n+1}}, \quad B = V^{\frac{n}{2n+1}}. \tag{4.41}$$

From Eqs.(4.37) and (4.38), we get

$$\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \frac{\dot{B}^2}{B^2} - \frac{K}{B^2} = 0. \tag{4.42}$$

Inserting A and B from Eq.(4.41) into Eq.(4.42), we find equation for defining V as

$$\ddot{V} = \frac{K(2n+1)}{1-n} V^{\frac{1}{2n+1}}. \tag{4.43}$$

Eq.(4.43) has the general solution in the quadrature form as

$$\int \frac{dV}{\sqrt{V^{\frac{2(n+1)}{2n+1}} + C}} = \left(\sqrt{\frac{K}{1-n^2}} \right) (2n+1)t \tag{4.44}$$

where C is an integration constant. Eq.(4.44) imposes some restrictions on the choice of n . Thus, we see that V plays central role here in defining all physical quantities. In what follows we find V from Eq.(4.44) for different values of K and $C = 0$.

One can not solve Eq.(4.44) in general. So, in order to solve the problem completely, we have to choose either C or n in such a manner that Eq.(4.44) be integrable. Obviously the easiest way is to choose $C = 0$ in (4.44). We now consider the following cases of physical interest and obtain corresponding cosmological solutions.

4.3.2.1 Model I

When $K = 0$ and $C = 0$, Eq.(4.44) gives $\ddot{V} = 0$, which has the general solution

$$V = c_1 t + c_2 \quad (4.45)$$

where c_1 and c_2 are integration constants. From Eqs.(4.41) and (4.45), we obtain

$$A = (c_1 t + c_2)^{\frac{1}{2n+1}}, \quad (4.46)$$

$$B = (c_1 t + c_2)^{\frac{n}{2n+1}}. \quad (4.47)$$

The metric of the solutions can be written in the form

$$ds^2 = dT^2 - T^{\frac{2}{2n+1}} dx^2 - T^{\frac{2n}{2n+1}} (dy^2 + y^2 dz^2) \quad (4.48)$$

where $T = c_1 t + c_2$. The energy density and EoS parameter are given by

$$\rho = \frac{n(n+2)}{(2n+1)^2} \frac{1}{T^2}, \quad (4.49)$$

$$\omega = 1. \quad (4.50)$$

Thus, the metric (4.48) represents a hypersurface-homogeneous space time in the presence of stiff matter ($\rho = p$). From theoretical perspective the present model is important in relativistic cosmology for the description of the very early stages of the universe (Zel'dovich (1972)).

The expansion scalar, shear scalar and Hubble parameter are found to

$$\theta = \frac{1}{T}, \quad (4.51)$$

$$\sigma^2 = \frac{1}{3} \left(\frac{1-n}{2n+1} \right)^2 \frac{1}{T^2}, \quad (4.52)$$

$$H = \frac{3}{T}. \quad (4.53)$$

The deceleration parameter q is zero, which shows the coasting cosmology i.e. universe inflates with marginal inflation.

The model (4.48) has singularity at $T = 0$. At this epoch the energy density, expansion scalar, shear scalar and Hubble parameter all are infinite. These physical parameters are decreasing functions of time and ultimately tend to zero as $T \rightarrow \infty$. The model would essentially give an empty space time for large T .

4.3.2.2 Model II

When $K = 1$ and $C = 0$, Eq.(4.44) can be integrated to give

$$V = \left[\frac{n(t+k)}{\sqrt{1-n^2}} \right]^{\frac{2n+1}{n}}, \quad -1 < n < 1 \quad (4.54)$$

where k is an integrating constant. Then from Eqs.(4.41) and (4.54), we find the solution for the scale factors of the form

$$A = \left[\frac{n(t+k)}{\sqrt{1-n^2}} \right]^{\frac{1}{n}}, \quad (4.55)$$

$$B = \frac{n(t+k)}{\sqrt{1-n^2}}. \quad (4.56)$$

Hence, the metric of our solutions can be written in the form

$$ds^2 = dT^2 - \left(\frac{nT}{\sqrt{1-n^2}} \right)^{\frac{2}{n}} dx^2 - \left(\frac{nT}{\sqrt{1-n^2}} \right)^2 (dy^2 + \sin^2 y dz^2) \quad (4.57)$$

where again $T = t + k$.

The energy density, EoS parameter and deceleration parameter are given by

$$\rho = \frac{1+2n}{n^2 T^2}, \quad (4.58)$$

$$\omega = -\frac{1}{2n+1}, \quad (4.59)$$

$$q = -\frac{1+n}{2n+1}. \quad (4.60)$$

For the energy density to be positive and ω, q to be negative, we must have $-\frac{1}{2} < n < 1$. We restrict ourselves to $-\frac{1}{2} < n < 1$. The expansion scalar θ , shear scalar σ and Hubble parameter are found to be

$$\theta = \frac{1+2n}{nT}, \quad (4.61)$$

$$\sigma^2 = \frac{1}{3} \left(\frac{1-n}{nT} \right)^2, \quad (4.62)$$

$$H = \frac{1+2n}{3nT}. \quad (4.63)$$

We observe that this model has a singularity at $T = 0$. Since $\frac{\sigma}{\theta}$ tends to a constant as $T \rightarrow \infty$, the anisotropy in the model is maintained through the passage of time.

Thus, the model (4.57) represents a hypersurface-homogeneous cosmological model in presence of dark energy. The negative EoS parameter may be attributed to the current accelerating expansion of the universe.

4.3.2.3 Model III

When $K = -1, C = 0$, Eq.(4.44) can be integrated to give

$$V = \left[\frac{n(t+l)}{\sqrt{n^2-1}} \right]^{\frac{2n+1}{n}} \quad (4.64)$$

where l is constant of integration. For physically relativistic model we take $n > 1$.

Combining Eqs.(4.41) and (4.64), we obtain

$$A = \left[\frac{n(t+l)}{\sqrt{n^2-1}} \right]^{\frac{1}{n}}, \quad (4.65)$$

$$B = \frac{n(t+l)}{\sqrt{n^2-1}}. \quad (4.66)$$

The metric of the solutions can be written in the form

$$ds^2 = dT^2 - \left(\frac{nT}{\sqrt{n^2-1}}\right)^{\frac{2}{n}} dx^2 - \left(\frac{nT}{\sqrt{n^2-1}}\right)^2 (dy^2 + \sinh^2 y dz^2). \quad (4.67)$$

For the model (4.67), the energy density, EoS parameter and deceleration parameter are obtained as

$$\rho = \frac{1+2n}{n^2 T^2}, \quad (4.68)$$

$$\omega = -\frac{1}{2n+1}, \quad (4.69)$$

$$q = \frac{1-n}{2n+1}. \quad (4.70)$$

If $n > 1$, ρ is positive and ω , q are negative. The expansion scalar, shear scalar and Hubble parameter has the values given by Eqs. (4.61) to (4.63) with $n > 1$. Since the deceleration parameter is negative, the present solution represents an accelerating expanding model filled with DE. We also observe that the anisotropy is maintained for all time.

4.3.3 Conclusions

We have investigated hypersurface-homogeneous cosmological models filled with stiff fluid and DE. The model filled with stiff matter is important in relativistic cosmology for the description of very early stages of the universe. For other two models, the EoS parameter ω is negative constant with certain restrictions over n , which may be attributed to the current accelerated expansion of the universe. The anisotropy in the models are maintained throughout the evolution of the universe. The models obtained in this chapter are of considerable interest and may be useful

for describing the early evolution and studying the present accelerating model of the universe.

4.4 Hypersurface-Homogeneous Cosmological Models with Dynamical Equation of State Parameter in Lyra Geometry

4.4.1 Field Equations

The energy-momentum tensor of a fluid can be written in anisotropic diagonal form

$$T_j^i = \text{diag}[T_1^1, T_2^2, T_3^3, T_4^4,]. \quad (4.71)$$

The simplest generalization of the EoS parameter of a perfect fluid may be to determine the EoS parameter separately on each spatial axis while preserving the diagonal form of the energy momentum tensor in a consistent way with the considered metric. We parameterize it as follows:

$$\begin{aligned} T_j^i &= \text{diag}[-p_x, -p_y, -p_z, \rho] \\ &= \text{diag}[-\omega_x, -\omega_y, -\omega_z, 1]\rho \\ &= \text{diag}[-\omega, -(\omega + \delta), -(\omega + \delta), 1]\rho \end{aligned} \quad (4.72)$$

where ρ is the energy density of the fluid, p_x , p_y and p_z are the pressures and ω_x , ω_y and ω_z are the directional EoS parameters of the fluid. Parameterizing the deviation from isotropy by setting $\omega_x = \omega$ and then introducing skewness parameter δ that is the deviation from ω on the y and z axes respectively. The ω and δ are not

necessarily constants and can be a functions of the cosmic time t .

In a comoving coordinate system, the field equations (3.7) for the metric (4.1) with the help of equation Eq.(4.72) yield the following set of equations:

$$\frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} + \frac{K}{B^2} - \frac{3}{4}\beta^2 = \rho, \quad (4.73)$$

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{K}{B^2} + \frac{3}{4}\beta^2 = -\omega\rho, \quad (4.74)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{3}{4}\beta^2 = -(\omega + \delta)\rho. \quad (4.75)$$

Using Bianchi identities to (3.7) and assuming that the matter field is conserved separately, we obtain

$$\dot{\beta} + \beta \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = 0. \quad (4.76)$$

4.4.2 Solutions of Field Equations

Using Eqs.(4.9) and (4.10) in Eq.(1.41), the anisotropy parameter of expansion takes the form

$$A_m = \frac{2}{9H^2}(H_x - H_y)^2. \quad (4.77)$$

Subtracting Eq.(4.74) from Eq.(4.75) and integrating, we obtain

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{\lambda}{a^3} + \frac{1}{a^3} \int \left(\frac{K}{B^2} - \delta\rho \right) a^3 dt \quad (4.78)$$

where λ is a constant of integration and the term δ arises due to the possible intrinsic anisotropy of the fluid. Using Eq.(4.78) in Eq.(4.77), we get

$$A_m = \frac{2}{9H^2} \left[\lambda + \int \left(\frac{K}{B^2} - \delta\rho \right) a^3 dt \right]^2 a^{-6}. \quad (4.79)$$

The integral term in Eq.(4.79) vanishes for

$$\delta = \frac{K}{\rho B^2}. \quad (4.80)$$

The anisotropy parameter A_m then assumes the form

$$A_m = \frac{2}{9} \frac{\lambda^2}{H^2} a^{-6}. \quad (4.81)$$

Hence, the field equations (4.73)- (4.75) reduce to

$$\frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} - \frac{3}{4}\beta^2 = \rho - \frac{K}{B^2} = (1 - \delta)\rho, \quad (4.82)$$

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{K}{B^2} + \frac{3}{4}\beta^2 = -\omega\rho, \quad (4.83)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{K}{B^2} + \frac{3}{4}\beta^2 = -\omega\rho. \quad (4.84)$$

Singh and Beesham (2011) have presented solutions of Eqs.(4.82)- (4.84) in general relativity by assuming power-law and exponential form of the average scale factor a . We obtain exact solutions of Eqs.(4.82)- (4.84) by applying a special law of variation of the Hubble's parameter which leads to a negative constant deceleration parameter.

For an accelerating model of the universe, we take constant as negative. Integration of Eq.(1.34) leads the solution for the average scale factor given as

$$a = (c_1 t + c_2)^{\frac{1}{1+q}} \quad (4.85)$$

where c_1 and c_2 are integration constants. Equation (4.85) implies that the condition of accelerated expansion is $1 + q > 0$.

From Eqs.(4.80), (4.83) and (4.84), we obtain, after integration

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = \frac{\lambda}{a^3} = \lambda(c_1 t + c_2)^{\frac{-3}{1+q}}. \quad (4.86)$$

Also, from Eqs.(4.8) and (4.85), we obtain

$$\frac{\dot{A}}{A} + \frac{2\dot{B}}{B} = \frac{3c_1}{(1+q)}(c_1 t + c_2)^{-1}. \quad (4.87)$$

Solving Eqs.(4.86)-(4.87), we obtain the solutions of the scale factors A and B as

$$A = (c_1 t + c_2)^{\frac{1}{1+q}} \exp \left\{ \frac{2\lambda(1+q)}{3c_1(q-2)} (c_1 t + c_2)^{\frac{q-2}{1+q}} \right\}, \quad (4.88)$$

$$B = (c_1 t + c_2)^{\frac{1}{1+q}} \exp \left\{ \frac{-\lambda(1+q)}{3c_1(q-2)} (c_1 t + c_2)^{\frac{q-2}{1+q}} \right\}. \quad (4.89)$$

The solution of Eq.(4.76) is given by

$$\beta = l(c_1 t + c_2)^{\frac{-3}{1+q}} \quad (4.90)$$

where l is an arbitrary constant.

From Eq.(4.82), we can obtain the energy density of the fluid as

$$\rho = \left[\frac{3c_1^2}{(1+q)^2(c_1 t + c_2)^2} + \frac{K}{(c_1 t + c_2)^{\frac{2}{1+q}}} \exp \left\{ \frac{2\lambda(1+q)}{3c_1(q-2)} (c_1 t + c_2)^{\frac{q-2}{1+q}} \right\} - \frac{\lambda^2}{3(c_1 t + c_2)^{\frac{6}{1+q}}} - \frac{3l^2}{4(c_1 t + c_2)^{\frac{6}{1+q}}} \right]. \quad (4.91)$$

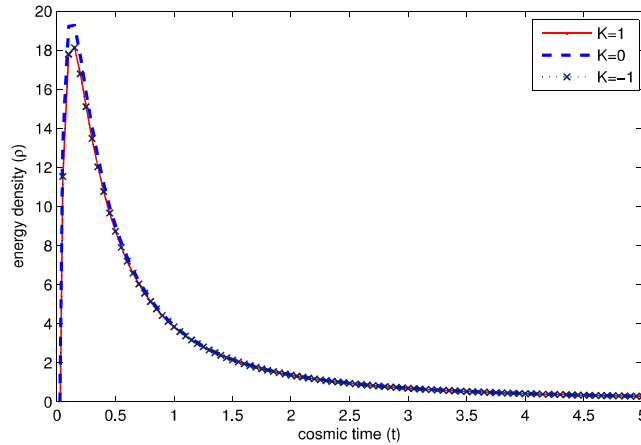


Figure 4.1: Variation of energy density ρ with time t for $\lambda = 1.02$, $l = 1.12$, $c_1 = 1.5$, $c_2 = .56$ and $q = -0.4$

Using Eqs.(4.89), (4.90) and (4.91) in Eq.(4.83) the expression for EoS parameter

can be obtained as

$$\omega = - \left[\frac{3c_1^2 - 2c_1^2(1+q)}{(1+q)^2(c_1t+c_2)^2} + \frac{K}{(c_1t+c_2)^{\frac{2}{1+q}}} \exp \left\{ \frac{2\lambda(1+q)}{3c_1(q-2)} (c_1t+c_2)^{\frac{q-2}{1+q}} \right\} + \frac{4\lambda^2 + 9l^2}{12(c_1t+c_2)^{\frac{6}{1+q}}} \right] \times \left[\frac{K}{(c_1t+c_2)^{\frac{2}{1+q}}} \exp \left\{ \frac{2\lambda(1+q)}{3c_1(q-2)} (c_1t+c_2)^{\frac{q-2}{1+q}} \right\} + \frac{3c_1^2}{(1+q)^2(c_1t+c_2)^2} - \frac{\lambda^2}{3(c_1t+c_2)^{\frac{6}{1+q}}} - \frac{3l^2}{4(c_1t+c_2)^{\frac{6}{1+q}}} \right]^{-1}. \quad (4.92)$$

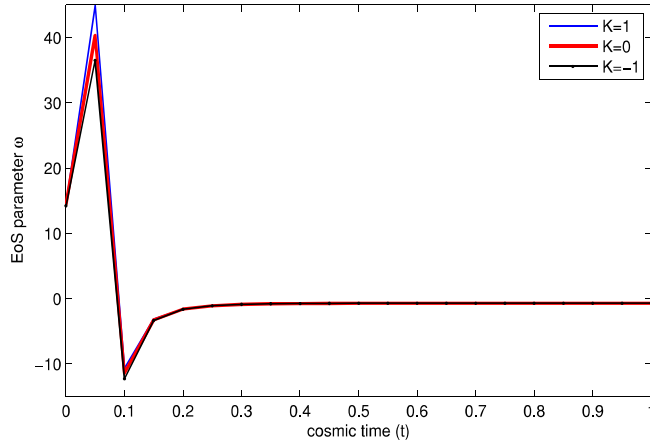


Figure 4.2: Variation of EoS parameter with time t for $\lambda = 1.02$, $l = 1.12$, $c_1 = 1.5$, $c_2 = .56$ and $q = -0.4$

Using Eqs.(4.89) and (4.91) into Eq.(4.80), the skewness parameter δ is obtained as

$$\delta = K \left[(c_1t+c_2)^{\frac{-2}{1+q}} \exp \left\{ \frac{2\lambda(1+q)}{3c_1(q-2)} (c_1t+c_2)^{\frac{q-2}{1+q}} \right\} \right] \left[\frac{3c_1^2}{(1+q)^2(c_1t+c_2)^2} - \frac{\lambda^2}{3(c_1t+c_2)^{\frac{6}{1+q}}} - \frac{3l^2}{4(c_1t+c_2)^{\frac{6}{1+q}}} + \frac{K}{(c_1t+c_2)^{\frac{2}{1+q}}} \exp \left\{ \frac{2\lambda(1+q)}{3c_1(q-2)} (c_1t+c_2)^{\frac{q-2}{1+q}} \right\} \right]^{-1} \quad (4.93)$$

The directional Hubble parameter on the x, y and z axes are given by

$$H_x = \frac{1}{(1+q)(c_1t+c_2)} + \frac{2\lambda}{3(c_1t+c_2)^{\frac{3}{1+q}}}, \quad (4.94)$$

$$H_y = H_z = \frac{1}{(1+q)(c_1t+c_2)} - \frac{\lambda}{3(c_1t+c_2)^{\frac{3}{1+q}}}. \quad (4.95)$$

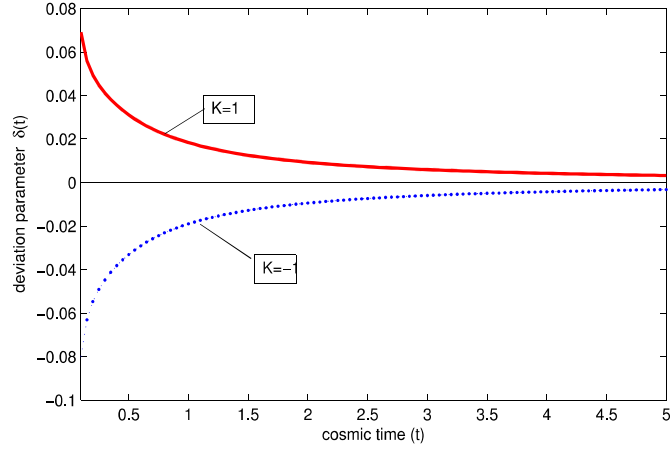


Figure 4.3: Variation of $\delta(t)$ with time t for $\lambda = 1.02$, $l = 1.12$, $c_1 = 1.5$, $c_2 = .56$ and $q = -0.4$

The mean Hubble parameter H is given by

$$H = \frac{1}{(1+q)(c_1 t + c_2)}. \quad (4.96)$$

The expressions for anisotropy parameter, expansion scalar and shear scalar are given as follows:

$$A_m = \frac{2\lambda^2(1+q)^2}{9(c_1 t + c_2)^{\frac{4-2q}{1+q}}}, \quad (4.97)$$

$$\theta = \frac{3c_1}{(1+q)(c_1 t + c_2)}, \quad (4.98)$$

$$\sigma^2 = \lambda^2(c_1 t + c_2)^{\frac{-6}{1+q}}. \quad (4.99)$$

We find that the volume V of the universe is zero at $t = -\frac{c_2}{c_1}$, and it takes infinitely large value as $t \rightarrow \infty$. The expansion scalar is infinite at $t = -\frac{c_2}{c_1}$ and decreases with increase in cosmic time. Thus, the universe starts evolving with zero volume at early time with an infinite rate of expansion and the expansion rate slows down for later times of the universe. We observe that the scale factors $A(t)$ and $B(t)$ are zero at $t = -\frac{c_2}{c_1}$, which implies that model has initial point singularity at $t = -\frac{c_2}{c_1}$, whereas these diverges for later time of the universe.

The dynamical parameters H_x , H_y , H_z and H are infinite at $t = -\frac{c_2}{c_1}$ and converges to zero at $t \rightarrow \infty$. It is also observed that the energy density first increases and attains a maximum value and decreases with time and then tends to zero at $t \rightarrow \infty$. The behavior of energy density is shown in fig.(4.1) with appropriate choice of integration constants and other physical parameters. The anisotropy parameter of expansion is infinite for earlier times of the universe and decreases with time and ultimately becomes zero as $t \rightarrow \infty$. This shows that anisotropy of expansion is not supported by the anisotropy of the fluid. Thus, the space approaches to isotropy in this model because $A_m \rightarrow 0$, $V \rightarrow \infty$ and $\rho \geq 0$ as $t \rightarrow \infty$.

The variation of EoS parameter ω with cosmic time t is shown in fig.(4.2) as a representative case with appropriate choice of integration constants and other physical parameters. From fig.(4.2) we observe that in early stages, the EoS parameter was positive (matter dominated universe) and while at late time it is evolving negative value (i.e. at the present time). Thus, the early matter dominated phase later on converted to dark energy phase. Since the deceleration parameter is negative for all time, the universe is accelerated expanding, which shows the dominance of DE. A graph of $\delta(t)$ is shown in fig.(4.3) for $K = \pm 1$, which shows that the fluid is highly anisotropic at early stage of evolution of universe and ultimately becomes null for large time. Therefore, the anisotropic fluid also isotropizes for the late time of the universe.

4.4.3 Conclusions

We have obtained the solutions of hypersurface homogeneous cosmological model in the presence of anisotropic fluid with dynamical EoS parameter in Lyra manifold. Reducing the anisotropy parameter of the expansion to a simple form, the exact solutions of the field equations have been obtained by assuming the special law of variation of mean Hubble parameter which leads to a negative constant value of deceleration parameter. The corresponding cosmological model represents accelerated expanding universe having singularity at $t = -\frac{c_2}{c_1}$. All the physical and kinematical parameter are well defined for finite time. The isotropy condition $\frac{\sigma}{\theta} \rightarrow 0$ as $t \rightarrow \infty$ of the expansion of the universe is also satisfied. Here, we observe that even in the presence of anisotropic fluid, universe approaches to isotropy for large time. Anisotropy of the fluid is also isotropizes at later time. The EoS parameter exhibits a transition from positive value to negative value which shows that anisotropic fluid was matter dominated at early time but at late time it corresponds to dark energy. We have found that $\beta \rightarrow 0$ as $t \rightarrow \infty$, so the concept of Lyra geometry will not survive for very long time.