INTRODUCTION

The concept of ring theory was initiated in the 1870s by Richard Dedekind and was further developed by mathematicians such as Hilbert, Fraenkel, and Noether. The concept of a ring is the first generalization of the Dedekind domain that emerged in number theory and since then, ring theory has found widespread applications in various fields such as algebraic geometry, number theory, algebraic graph theory, and coding theory.

There are two primary approaches to understanding the structure of rings. The first approach involves examining the inner conditions of a ring by studying its left and right ideals. The second approach is to investigate the outer conditions of a ring by studying the modules over them. In this thesis, the structure of rings is studied through the second approach, where the focus is on the properties of modules over the rings.

R. Baer introduced the concept of injectivity [3] in 1940, but mathematicians were attracted towards the study of injective modules only after Eckmann and Schopf [12] published their paper "Uber Injektive moduln." in 1953, where they proved the existence of injective hull of a module.

The concept of projectivity is dual to the concept of injectivity, although it originated simultaneously. However, the work on projective modules started only after the publication of the book "Homological Algebra" by Carten and Eilenberg [8] in 1956.

Several generalizations of projectivity have been made so far. A generalization of projectivity can be obtained either by specialization of R-modules or by restricting the R-homomorphism. For instance, quasi-projectivity by Y. Miyashita [31] and

M-projectivity by G. Azumaya [2], are due to specialization of R-modules whereas pseudo projectivity is obtained by restricting the R-homomorphism.

In 1958, I. Kaplansky [22] proved the following:

1. If R is local ring then any projective R-module is free.

2. If R is a commutative semi-hereditary ring, then any projective module is a direct sum of R-modules isomorphic to finitely generated ideals.

3. If R is a regular ring, then any projective module is a direct sum of principal ideals.

In 1959, P. M. Cohn [9] defined a submodule P of module M is pure iff $0 \longrightarrow E \oplus P \longrightarrow E \oplus M$ is exact for all modules E. In 1960, Maranda [29] extended a definition of pure submodules to pure projective modules. A module Q is called pure projective iff $Hom(Q, M) \longrightarrow Hom(Q, M/P) \longrightarrow 0$ is exact whenever P is pure in M. Later, D. J. Fieldhouse [15] proved that pure projective modules are direct summands of direct sums of finitely presented modules.

In 1960, H. Bass [5] introduced the concept of the projective cover of a module, dual to the injective envelope of a module and he proved that the projective cover of the module does not always exist but whenever it exists it is unique. Later on Pandeya et al., [37] discussed the concept of direct projective covers of modules and found its equivalence with projective covers.

S. Eilenberg [13]characterized that a category of modules is perfect if every module in this category has a projective cover.

B. Banaschewski [4] in 1964, proved the following results on projective modules:

1. A module is projective if and only if it is the direct sum of principal indecomposable modules.

2. If A and B are projective R-modules and J is an ideal of R some power of which annihilates A and B both then $A/JA \cong B/JB \implies A \cong B$

Later on, Y. Miyashita [31] generalized the concept of projective modules to quasi-projective modules. A detailed study quasi-projective was made by Wu and Jans [45] in 1967. They proved that if a module has a projective cover then it must have a quasi-projective cover. The converse of this theorem is false.

In 1970, A. Koehler [23] proved that if every module has a quasi-projective cover then every module has a projective cover and she characterized semi-simple rings in terms of quasi-projective module.

In 1970, J.S Golan [16] proved that if P is a projective module and $f : P \longrightarrow M$ is an epimorphism then M is projective if and only if $P \bigoplus M$ is quasi-projective, and M has projective cover if and only if $P \bigoplus M$ has a quasi-projective cover.

The notion of quasi-projective modules was generalized to pseudo-projective modules by Tiwary and Pandeya [40]. By defining the concept of a pseudo-projective cover of a module, they proved that if a module has a projective cover then it has a pseudo-projective cover. But the converse is not true.

In 1971, Roger Ware [42] studied the relationship between certain projective modules and their endomorphism rings. In [42] Ware described the projective modules whose endomorphism rings are regular, local, semi-perfect, or perfect.

In 1976, Nicholson [34] introduced the concept of direct-projective modules. He defined an R-module M is direct-projective if for any direct summand N of M with projection $p_N : M \longrightarrow N$ and any epimorphism $g : M \longrightarrow N$, there exists an $f \in End_R(M)$ such that $g \circ f = p_N$. In 1991, Wisbauer [43] defined an R-module M as direct-projective if for every direct summand X of M every epimorphism $M \longrightarrow X$ splits.

In 2016, Ibrahim et al. [18] introduced and studied the notion of simple-direct-projective modules. An R-module M is called simple-direct-projective if, whenever A and B are submodules of M with B simple and $M/A \cong B \leq^{\oplus} M$, then $A \leq^{\oplus} M$.

The following implications are true
Projective Module ⇒ Quasi-Projective Module ⇒ Direct Projective Module ⇒ Simple Direct Projective Module
But the converse of the above implications need not be true (see [24], [32]).

In 1993, W. Xue [46] characterized semisimple and hereditary rings in terms of direct-projective and direct-injective modules. In 2003, Nicholson and Yusif [35] proved that the class of direct-projective modules is equivalent to the class of D_2 modules. An R-module M is said to be a D_2 module if $M/A \cong B \leq^{\oplus} M$ then $A \leq^{\oplus} M$, where A and B are submodules of M.

In 1986, Hiremath [17] introduced the concept of Hopficity in the context of rings and modules. Hopfian modules are those modules for which every non-zero surjective endomorphism is an automorphism. In his work, Hiremath proved that if R is a semisimple Artinian ring, then an R-module M is Hopfian if and only if M has a finite length. The study of Hopfian modules has led to important developments in ring and module theory.

In 2011, Lee, et al. [26] introduced the notion of dual-Rickart modules which is the dual of the class of Rickart module. A module M is said to be dual-Rickart if the image of every endomorphism of M is a direct summand of M. These two classes of modules are closely connected with von Neumann regular rings, and the connection between them was first established by Rangaswamy [39] in 1967. The endomorphism ring $S = End_R(M)$ of a module M is von Neumann regular if and only if the kernel and image of every endomorphism of M are direct summands of M, i.e., M is both a Rickart and a dual-Rickart module. This result has important implications for the theory of rings and modules.

In this thesis, the author studied several rings one of them is a semisimple ring. The semisimple ring was introduced by Emmy Noether [36] in 1921. A semisimple ring is a type of ring that is particularly well-behaved from a module-theoretic perspective. It is defined as a ring that is semisimple as a module over itself. This means that each R-module can be written as a direct sum of simple (irreducible) submodules. Another way to characterize a semisimple ring is in terms of short exact sequences of modules. A ring R is semisimple if and only if any short exact sequence of R-modules splits.

In this thesis, the author also studied von Neumann regular rings. von Neumann regular rings were introduced by von Neumann in his paper entitled "On Regular Rings" [41]. A ring R is said to be von Neumann regular if every element of R is regular. An element a in a ring R is regular if there exists an element x in R such that $axa = a$. A module is said to be Endoregular if its Endomorphism ring is von Neumann regular ring.

In 1958, Irving Kaplansky [21] defined hereditary rings as a ring in which every ideal is projective. Several characterizations of hereditary rings exist in the literature, including the fact that every submodule of a projective R-module is also projective, and every factor module of an injective R-module is injective.

In 1961, Shizuo ENDO [14] generalized the idea of hereditary rings to semi-hereditary rings in his paper entitled "Semi-hereditary rings". Over a semi-hereditary ring, every finitely generated ideal is projective. Moreover, over a semi-hereditary ring, every finitely generated submodule of the projective module is also projective. In addition, every finitely generated factor module of an injective module is also injective.

Recently, there has been a growing interest in studying direct-projective modules and their generalizations. By several mathematicians, for example, Ibrahim et al. [18], Li. et al. [28] etc. systematically developed the theory of such modules. The above sequence of important ideas and works motivates the author to study direct-projective modules and their generalizations like semisimple direct-projective modules, finite direct-injective modules, and pure direct-projective modules. Also, we study finite direct projective covers and envelopes. The thesis has been divided into five chapters. I am giving a brief account of the materials presented in the thesis.

Chapter 1 is the collection of notations, definitions, and basic results which are used in the subsequent chapters.

In **Chapter 2**, we study some properties of direct projective modules in respect of SSP and SIP properties and characterize such modules in terms of endoregular and Hopfian modules. Further, we introduce the concept of Semi-simple direct projective modules, which is a generalization of Direct projective modules, and study their properties.

The following are the main results in Chapter 2;

1. The following statements are equivalent for an R-module M:
    - (a) M is a direct projective and dual Rickart module;
    - (b) M is an endoregular module.
2. Let M be an R-module. Then the following statements are equivalent:
    - (a) M is a direct projective module with $(**)$ property;
    - (b) M is a Hopfian module with $(**)$ property.
3. The following statements are equivalent for an R-module M:

    (a) R is a semi-simple Artinian ring;
    (b) Every 2-generated R-module is a semi-simple direct projective.

In **Chapter 3**, we introduce the concept of finite direct projective modules, which is an extension of the concept of direct projective modules. We investigate the relationships among various types of modules, like Rickart modules, D_3 modules, direct projective modules, finite direct projective modules, and endoregular modules. Finally, we provide the characterization of semi-hereditary rings, S-rings, and semi-simple Artinian rings in terms of finite direct projective modules.

The following are the main results in Chapter 3;

1. The direct summand of a finite direct projective module is a finite direct projective.
2. The following statements are equivalent for a finitely generated module M:
    (a) M is a finite direct projective module;
    (b) M is a direct-projective module;
    (c) M is a D_3 module;
    (d) M is a Rickart module;
    (e) M has SIP property;
    (f) Each submodule of M is a direct summand.
3. The following statements is equivalent for a ring R:
    (a) R is a semi-hereditary ring;
    (b) Each finitely generated submodule of a projective module is a finite direct projective module.

In **Chapter 4**, we study the finite direct projective covers and envelopes. We find a condition for a finite direct projective cover to be a projective cover of a module. Also, we characterize the finite direct projective cover of a module over semi-perfect and semi-regular rings. In the last of this chapter, we discuss the concept of finite direct projective envelopes and establish its equivalence with semisimple Artinian rings and S-rings.

The following are the main results in Chapter 4;

1. The following statements are equivalent for a ring R:
    (a) R is a semi-perfect ring;
    (b) Every finitely generated submodule of an R-module M has a finite direct projective cover.
2. The following statements are equivalent for a ring R:
    (a) R is a semi-regular ring;
    (b) Every finitely presented R-module M has a finite direct projective cover.
3. The following statements are equivalent for a ring R:
    (a) R is a semi-simple Artinian ring;
    (b) Every 2-generated R-module has a finite direct projective envelope.

In **Chapter 5**, we introduce the notion of Pure direct projective modules, which is an extension of the class of direct projective modules and the dual notion of pure direct injective modules. In this chapter, we study the direct sums and direct summands of pure direct projective modules. Further, we characterize semi-simple rings, pure semisimple rings, pure hereditary rings, and von Neumann regular rings in terms of pure direct projective modules.

The following are the main results in Chapter 5;

1. Let M be a pure direct projective module. If $M = M_1 \bigoplus M_2$ with $f : M_1 \longrightarrow M_2$ be a homomorphism such that $Img(f)$ is a direct summand of M_2 and $Ker(f)$ is a pure submodule of M_1. Then $Ker(f)$ is a direct summand of M_1.
2. For a von-Neumann regular ring R, the following statements are equivalent:
   (a) R is a semi-simple ring;
   (b) All R-modules are relatively pure direct projective to any R-module.
3. The following statements are equivalent:
   (a) R is a purely semisimple ring;
   (b) Every 2-generated R module is a pure direct projective module.
4. For hereditary ring R, the following statements are equivalent:
   (a) R is a von-Neumann regular ring;
   (b) Every pure direct projective R-module is a direct projective module;
   (c) Every pure projective R-module is a projective.

Chapter 1

Preliminaries

This chapter is mainly devoted to the collection of definitions and basic results, which are used in the subsequent chapters of the thesis. Throughout the thesis, unless otherwise indicated all modules over a ring R will be understood to be right R-modules and a ring R we shall always mean an associative ring with unity.

Rings, Modules and Module Homomorphism

Definition 1.0.1. *An algebraic structure (R,+,.), where R is a non-empty set together with two binary operations + and . is said to be a ring if the following conditions are satisfied.*

(1) $(R, +)$ *is an abelian group.*

(2) $(R, .)$ *is a semi-group.*

(3) The binary operation ' . ' distributes over ' + ' from the left as well as from the right, i.e., $\forall\ r, s, t \in R$

(i) $r.(s+t) = r.s + r.t$

(ii) $(r+s).t = r.t + s.t$

A ring $(R, +, .)$ is called commutative if $r.s = s.r \; \forall \; r, s \in R$. Further, R is said to be a ring with unity if there exists $1 \in R$ such that $1.r = r.1 = r \; \forall \; r \in R$.

Definition 1.0.2. *(1) Let R be a ring not necessarily containing identity element. An additive abelian group $(M, +)$ is called a right R-module if there exists a mapping from $M \times R$ to M defined by $(m, r) \to mr$, $\forall \; m \in M$, $r \in R$ satisfying the following conditions:*

(i) $(m+n)r = mr + nr$ *for all* $m, n \in M$ *and for all*$r \in R$.

(ii) $m(r+s) = mr + ms$ *for all* $m \in M$ *and* $\forall \; r, s \in R$.

(iii) $m(rs) = (mr)s$ *for all* $m \in M$ *and* $\forall \; r, s \in R$.

A left R-module can be defined by taking action of the ring R from left.

(2) Further, if $m.1 = m$ for all $m \in M$, where 1 is the unity of R, then M is called an ***unital*** *R-module.*

(3) A non-empty subset N of an R-module M is called a submodule of M if N is also an R-module and we denote it by $N \leq M$.

Definition 1.0.3. *Let M and N be R-modules:*

(1) A mapping $\varphi : M \to N$ is called a module homomorphism

$\varphi(m_1 + m_2) = \varphi(m_1) + \varphi(m_2)$ *and* $\varphi(mr) = \varphi(m)r$ *for all* $m_1, m_2 \in M$. *and* $r \in R$.

The set of all R-homomorphisms from M to N is denoted by $\mathrm{Hom}_R(M, N)$.

(2) *For any* $\varphi \in \operatorname{Hom}_R(M, N)$, *the* kernel *and the* image *of* φ *defined as follows* $\operatorname{Ker}(\varphi) = \{m \in M : \varphi(m) = 0\}$ *and* $\operatorname{Im}(\varphi) = \{\varphi(m) \in N : m \in M\}$.

(3) *An R-homomorphism from M to M is called an endomorphism and the set of all endomorphisms of M is denoted by* $\operatorname{End}_R(M)$.

Theorem 1.0.4. (Fundamental theorem of module homomorphisms) *Let M and N be R-modules. If $\varphi : M \to N$ be any R-homomorphism, then* $\varphi(M) \cong M/\operatorname{Ker}(\varphi)$.

Proposition 1.0.5. *Let M and N be R-modules.*

(i) *An onto R-homomorphism $f : M \to N$ splits if there exists an R-homomorphism $g : N \longrightarrow M$ with $f \circ g = I_N$. In this case, g is called a splitting map for f.*

(iii) *An one-one R-homomorphism $f : L \to M$ splits if there exists an R-homomorphism $g : M \longrightarrow L$ with $f \circ g = I_L$. In this case, g is a splitting map for f.*

Direct Sums, Direct Products, and Direct Summands

Definition 1.0.6. *Let $\{M_i\}_{i \in I}$ be a family of R-modules, where I is an arbitrary index set. Then*

(i) $\Pi_{i \in I} M_i = \{(m_1, m_2, \ldots, m_i, \ldots) : \, m_i \in M_i \, \forall \, i \in I\}$ *denotes the direct product of the family of R-modules $\{M_i\}_{i \in I}$.*

(ii) $\bigoplus_{i \in I} M_i = \{(m_1, m_2, \ldots, m_i, \ldots) : \, m_i \in M_i \, and \, finitely \, many \, m_i's \, are \, non - zero\}$ *denotes the direct sum of the family of R-modules $\{M_i\}_{i \in I}$.*

Further, we have $M^{(I)} = \bigoplus_{i \in I} X_i$ and $M^I = \Pi_{i \in I} X_i$, where $X_i = M$ for every $i \in I$.

Definition 1.0.7. *A submodule N of a module M is called a* **direct summand** *of M if there exists a submodule N' of M such that $M = N \oplus N'$, and we denote it by $N \leq^{\oplus} M$. In this case $M = N \oplus N'$ implies $M = N + N'$ and $N \cap N' = 0$.*

Idempotents, Small submodules, Minimal Epimorphism and Exact Sequences

Definition 1.0.8. *An element e of a ring R is called an* **idempotent** *element of R if $e^2 = e$.*

Proposition 1.0.9. *If N_1 and N_2 are submodules of M such that $M = N_1 \oplus N_2$ then there exists an unique idempotent endomorphism $e \in \mathrm{End}_R(M)$ such that $N_1 = eM$ and $N_2 = (1 - e)M$.*

Definition 1.0.10. *A submodule N of M is called small (or superfluous) in M, abbreviated as $N << M$ in case $N + L = M \implies L = M$ for any submodule L of M.*

Definition 1.0.11. *An epimorphism $f : A \longrightarrow B$ is called minimal if $Ker(f)$ is small in A.*

Definition 1.0.12. *Let M_n be an R-module $\forall$ n and φ_n be a homomorphism from M_n to M_{n-1} for all n. Then a sequence $\cdots \to M_{n+1} \xrightarrow{\varphi_{n+1}} M_n \xrightarrow{\varphi_n} M_{n-1} \to \dots$ is called an exact sequence at M_n if $\mathrm{Ker}(\varphi_n) = \mathrm{Im}(\varphi_{n+1})$, while this sequence is called an exact sequence if it is exact at M_n for each n.*

Definition 1.0.13. *Let L, M and N be R-modules. Then,*

(i) The exact sequence $0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$ is called a short exact sequence.

(ii) A sequence $0 \to L \xrightarrow{\varphi} M$ is exact if and only if φ is a monomorphism.

(iii) A sequence $M \xrightarrow{\psi} N \rightarrow 0$ *is exact if and only if* ψ *is an epimorphism.*

Definition 1.0.14. *[6] For an* R*-modules* L, M *and* N*, the following conditions are equivalent:*

(i) An exact sequence $0 \rightarrow L \xrightarrow{\varphi} M \xrightarrow{\psi} N \rightarrow 0$ *splits;*

(ii) There exists a homomorphism $\varphi' : M \rightarrow L$ *such that* $\varphi' o \varphi = I_L$*, where* I_L *is an identity map on* L*;*

(iii) There exists a homomorphism $\psi' : N \rightarrow M$ *such that* $\psi o \psi' = I_N$*, where* I_N *is an identity map on* N*.*

Finitely Generated Modules, Cyclic Modules, Free Modules, Finitely Related Modules, Finitely Presented Modules, Coherent Modules

Definition 1.0.15. *A module* M *is said to be* **finitely generated** *if there exist* $m_1, m_2, \ldots, m_n \in M$ *such that* $M = \sum_{i=1}^{n} m_i R$*. The set* $\{m_1, m_2, \ldots, m_n\}$ *is called the set of generators of* M*. A module generated by a single element is called a* **cyclic** *module. Further, a submodule is called cyclic if a single element generates it.*

Definition 1.0.16. *A* R*-module* M *is called a* **free module** *if it has a basis, i.e., there exists a subset* $B \subseteq M$ *such that each element* $m \in M$ *can be uniquely expressed as a finite sum,* $m = \sum_{i=1}^{n} m_i r_i$ *for some* $r_1, r_2, \ldots, r_n \in R$ *and* $m_1, m_2, \ldots, m_n \in B$*.*

Definition 1.0.17. *A module* N *is said to be* **finitely related** *[24] if there exists an exact sequence* $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ *of* R*-modules, where* M *is a free module (of arbitrary rank) and* L *is finitely generated.*

Definition 1.0.18. *A module N is said to be* **finitely presented** *[24] if there exists an exact sequence $0 \to L \to M \to N \to 0$ of R-modules, where M is a free module (of finite rank) and L is finitely generated (or equivalently, there exists an exact sequence $R^m \to R^n \to N \to 0$ with $m, n \in \mathbb{N}$).*

Definition 1.0.19. *A finitely generated R-module M is said to be* **coherent** *if every finitely generated submodule of M is finitely presented. A ring R is called coherent if R_R is a coherent R-module.*

Artinian and Noetherian Modules and Rings

Definition 1.0.20. *A module M is called* **Noetherian** *if it satisfies the ascending chain condition on its submodules, i.e., if every ascending chain $M_1 \leq M_2 \leq \cdots \leq M_n \leq \ldots$ of submodules of M becomes stationary after finitely many steps. A ring R is called Noetherian if the R-module R_R (${}_R R$) is Noetherian.*

Theorem 1.0.21. *For a module M, the following conditions are equivalent:*

- *(i) M is Noetherian;*
- *(ii) Every submodule of M is finitely generated;*
- *(iii) Every non-empty set A of submodules of M has a maximal element.*

Definition 1.0.22. *A module M is called* **Artinian** *if it satisfies the descending chain condition on its submodules, i.e., if every descending chain $M_1 \geq M_2 \geq \cdots \geq M_n \geq \ldots$ of submodules of M becomes stationary after finitely many steps. A ring R is called Artinian if the R-module R_R (${}_R R$) is Artinian.*

(*)-Property, (**)-Property, Hopfian and co-Hopfian Modules and Dedekind Finite

Definition 1.0.23. *A module M is said to satisfy the (*)-property if every non-zero endomorphism of M is a monomorphism.*

Definition 1.0.24. *A module M is said to satisfy the (**)-property if every non-zero endomorphism of M is epimorphism.*

Definition 1.0.25. *(i) A module M is said to be* **Hopfian** *if any surjective endomorphism of M is an automorphism.*

(ii) A module M is said to be **co-Hopfian** *if any injective endomorphism of M is an automorphism.*

Definition 1.0.26. *A ring R is called Dedekind Finite if for elements a,b of R, $ab = 1$ implies $ba = 1$.*

Definition 1.0.27. *An R-module M is Dedekind Finite if and only if End(M) is Dedekind Finite Ring.*

Pure Submodules, Flat Modules, Pure Split Modules, and Purely Semisimple Rings

Definition 1.0.28. *A short exact sequence $0 \to N_1 \xrightarrow{\phi} N_2 \to N_3 \to 0$ of right R-modules is said to be pure exact if $0 \to N_1 \otimes F \to N_2 \otimes F \to N_3 \otimes F \to 0$ is an exact sequence for any left R-module F [24].*

According to P.M. Cohn [9], a submodule N of an R-module M is said to be a **pure submodule** *of M, abbreviated by* $\mathbf{N \leq^p M}$*, if and only if $0 \to N \otimes L \to M \otimes L$ is* *exact for every left R-module L. Further, an ideal I of a ring R is said to be pure if I is a pure submodule of R_R.*

Definition 1.0.29. *An R-module M is said to be* **flat** *if $0 \to M \otimes N_1 \to M \otimes N_2$ is exact whenever $0 \to N_1 \to N_2$ is exact for left R-modules N_1 and N_2.*

Proposition 1.0.30. *(i) [24, Proposition 4.29] A ring R is Noetherian if and only if all finitely generated R-modules are finitely presented.*

(ii) [24, Theorem 4.30] Let P be a finitely related R-module. Then P is flat if and only if it is projective.

Lemma 1.0.31. *[15, Proposition 8.1]. The following conditions hold:*

(i) Let N be a submodule of an R module M. If M/N is flat, then N is a pure submodule of M. Moreover, for a flat R module M, N is a pure submodule of M if and only if M/N is flat.

(ii) If N is a submodule of M such that every finitely generated submodule of N is a pure submodule of M, then N is a pure submodule of M.

Lemma 1.0.32. *[15, Proposition 7.2]. Suppose $L \subseteq N \subseteq M$ be an R modules. Then*

(i) If $L \leq^p N$ and $N \leq^p M$, then $L \leq^p M$.

(ii) If $L \leq^p M$, then $L \leq^p N$.

(iii) If $L \leq^p N$, then $N/L \leq^p M/L$.

(iv) If $L \leq^p M$ and $N/L \leq^p M/L$, then $N \leq^p M$.

Definition 1.0.33. *A module M is called* **pure split** *if every pure submodule of M is a direct summand of M.*

Definition 1.0.34. *A ring R is called a purely semisimple ring [15] if every pure submodule of an R-module M is a direct summand of M.*

Lemma 1.0.35. *If R is a Noetherian ring and M is a finitely generated R-module, then each pure submodule of M is a direct summand of M.*

Projective Modules, Quasi-Projective module, Projective Cover of a module and Quasi Projective cover of a module

Definition 1.0.36. *(i) An R-module P is projective if given any R-epimorphim $f : M \longrightarrow N$ and a module homomorphism $g : P \longrightarrow N$, there exists a module homomorphism $h : P \longrightarrow M$ such that $f \circ h = g$.*

$$\begin{tikzcd}
& P \arrow[d, "g"] \arrow[dl, dashed, "\exists h"'] \\
M \arrow[r, "f"] & N
\end{tikzcd}$$

(ii) An R-module P is called quasi-projective if given any R-epimorphism $g : P \longrightarrow A$, and a homomorphism $f : P \longrightarrow N$, there is an $h \in End_R(P)$ such that the diagram below

$$\begin{tikzcd}
& P \arrow[d, "g"] \arrow[dl, dashed, "\exists h"'] \\
P \arrow[r, "f"] & N
\end{tikzcd}$$

commutes and $f \circ h = g$.

(iii) A projective module P is called a projective cover of a module N if there exists a minimal epimorphism $f : P \longrightarrow N$..

(iv) A quasi projective module P is called a quasi projective cover of a module N if there exists a minimal epimorphism $f : P \longrightarrow N$.

Direct Projective Modules and Direct Projective Covers of a Modules

Definition 1.0.37. *An R-module M is called direct-projective if for any direct summand N of M with projection $p_N : M \longrightarrow N$ and any epimorphism $g : M \longrightarrow N$, there exists an $f \in End_R(M)$ such that $g \circ f = p_N$.*

$$\begin{tikzcd}
 & M \arrow[dl, dashed, "\exists f"'] \arrow[d, "p_N"] \\
M \arrow[r, "g"] & N
\end{tikzcd}$$

Definition 1.0.38. *A direct projective module P is called a direct projective cover of a module N if there exists a minimal epimorphism $f : P \longrightarrow N$.*

Hereditary, Semi-Hereditary Rings and von Neumann Regular Rings

Definition 1.0.39. *(i) A ring R is called* **hereditary** *if each ideal of R is projective as an R-module.*

(ii) A ring R is called **semi-hereditary** *if each finitely generated ideal of R is projective as an R-module.*

Theorem 1.0.40. *[8, Theorem 5.4] The following statements are equivalent for a ring R:*

(i) R is hereditary ring;

(ii) Every submodule of a projective R-module is projective;

(iii) Every quotient module of an injective R-module is injective.

Definition 1.0.41. *A ring R is called* **von Neumann regular** *if, for each $a \in R$, there exists $b \in R$ such that $a = aba$.*

Definition 1.0.42. *A module M is called* **endoregular** *[27] if the endomorphism ring of M is a von Neumann regular.*

Proposition 1.0.43. *[44, 3.10] The following conditions are equivalent for a ring R:*

- *(i) R is von Neumann regular;*
- *(ii) Every principal ideal is a direct summand;*
- *(iii) Every finitely generated ideal is a direct summand.*

D_1 Modules, D_2-Modules, and D_3-Modules

Consider the following conditions defined in [11] and [32] for an R-module M.

D_1**:** For every submodule N of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 << M$.

D_2**:** If N is a submodule of M and M/N is isomorphic to a direct summand of M, then N is also a direct summand of M.

D_3**:** If L and N are direct summands of M with $M = L + N$, then $L \cap N$ is a direct summand of M.

Definition 1.0.44.

- *(i) A module with D_i-condition is called a D_i-module for every $i = 1, 2, 3$.*
- *(ii) A module with D_1-condition is known as lifting module.*

Note :- In 2003, Nicholson and Yusif [35] proved that the class of direct-projective modules is equivalent to the class of D_2 modules.

Simple Modules, Semisimple Modules, Semisimple Rings and Polysimple Modules

Definition 1.0.45. *Let M be an R-module. Then*

(i) M is called a **simple module** *if it contains no non-trivial proper submodule.*

(ii) M is called an **indecomposable module** *if it can not be written as a direct sum of two proper direct summands of M.*

Definition 1.0.46. *The sum of all simple submodules of an R-module M is called the* **socle** *of M, and it is denoted by* $\mathrm{Soc}(M)$.

Definition 1.0.47. *A non-zero module M is called* **semisimple** *if it is expressible as a sum of simple submodules, while a ring R is called semisimple if the R-module R_R $({}_RR)$ is a semisimple module.*

Proposition 1.0.48. *An R-module M is* **semisimple** *if and only if $Soc(M) = M$.*

Proposition 1.0.49. *The following conditions are equivalent for a ring R:*

(i) R is semisimple;

(ii) Every R-modules are semisimple;

(iii) Every R-modules are injective;

(iv) Every R-modules are projective;

(v) Every ideal of R is a direct summand.

Definition 1.0.50. *A module M is called* ***polysimple*** *if every non-zero submodule of M contains a simple submodule of M. Thus every M is polysimple if $Soc(L) \neq 0$ for every nonzero submodule L of M.*

SSSP (SSP) Modules, SSIP (SIP) Modules

Definition 1.0.51. *An R-module M is said to have summand sum property* (**SSP**) *if the sum of any two direct summands of M is a direct summand of M, while M is said to have strong summand sum property* (**SSSP**) *if the sum of arbitrary direct summands of M is a direct summand of M. A module M is called an SSP ($SSSP$) module if M has SSP ($SSSP$).*

Definition 1.0.52. *An R-module M is said to have summand intersection property* (**SIP**) *if the intersection of any two direct summands of M is a direct summand of M, while M is said to have strong summand intersection property* (**SSIP**) *if the intersection of arbitrary direct summands of M is a direct summand of M. A module M is called an SIP ($SSIP$) module if M has SIP ($SSIP$).*

Rickart Modules, Dual Rickart Modules, Pure Rickart Modules, Pure Dual Rickart Modules

Definition 1.0.53. *Let M be an R-module and $S = End_R(M)$. M is called a Rickart module if and only if for every $\phi \in S$ and $r \in R$, the annihilator of ϕ in M is generated by an idempotent of S, which is equivalent to $rM = Ker(\phi) \leq^{\oplus} M$.*

Definition 1.0.54. *Let M is an R-module and $S = End_R(M)$, then M is a d-Rickart (or dual Rickart) module if and only if for every $\phi \in S$, the image of ϕ in M is a direct summand of M, which is equivalent to $\phi M = Img(\phi) = eM$ for some $e^2 = e \in S$.*

Definition 1.0.55. *An R-module M is called Pure Rickart if, for every $f \in End_R(M)$, $Ker(f)$ is a pure submodule of M (in the sense of Anderson and Fuller [1]). A ring R is called a Pure Rickart Ring if R_R is a Pure Rickart module.*

Definition 1.0.56. *An R-module M is called Pure Dual Rickart if, for every* $f \in End_R(M)$*,* $Img(f)$ *is a pure submodule of* M *(in the sense of Anderson and Fuller [1]). A ring* R *is called a Pure Dual Rickart Ring if* R_R *is a Pure Dual Rickart module.*