Preface

The aim of this thesis is to study the asymptotic behaviour of Lambert series associated to certain automorphic forms and the asymptotic behaviour of moments of generalized divisor function associated to Fourier coefficients of Rankin-Selberg L-functions. A Lambert series is a q-series involving certain sequence of complex numbers. Lambert series frequently occurred in the works of Ramanujan, who established an identity reflecting the relation between Lambert series associated to Möbius function and non-trivial zeros of Riemann zeta function. However, it was later discovered that Ramanujan's identity was incorrect, and this error was rectified by Hardy and Littlewood. Further, Zagier has studied the asymptotic expansion of Lambert series associated to the constant term of an automorphic function and conjectured that the non-trivial zeros of Riemann zeta function can be numerically evaluated using Fourier coefficients of automorphic function in consideration. Moreover, Zagier predicted the oscillatory behaviour of the Lambert series from the presence of the cosine function in its asymptotic expansion. In 2000, Hafner and Stopple proved the above mentioned conjecture of Zagier utilizing certain heat kernel of Ramanujan tau function and associated symmetric square L-function by employing Rankin-Selberg method. Recently, Chakraborty, Kanemitsu and Maji proved the similar result for normalized Hecke eigenform where the authors obtained the asymptotic behaviour of certain Lambert series associated to a normalized Hecke eigenform. Analogues results have been proved for Lambert series associated to various automorphic forms by several authors.

Motivated from the conjecture of Zagier and following the methodology given by Hafner and Stopple to prove Zagier's conjecture, we study certain Lambert series associated to Siegel cusp forms and normalized Hecke eigenforms.

In another direction, we also study the asymptotic behaviour of moments of generalized divisor function to prove the oscillatory behaviour of Fourier coefficients of product of Rankin-Selberg type L-functions over a sparse set supported at primitive integral positive definite binary quadratic forms. Study of the asymptotic behaviour of summatory functions of arithmetical functions is one of the classical problems in analytic number theory. Such problems are referred as divisor problems. Generalized divisor function $d_\ell(n)$ is closely related with $\zeta^\ell(s)$, where $\zeta(s)$ is the famous Riemann zeta function. Several authors have considered divisor problems and their analogues in the case of various automorphic forms like Hecke eigenforms and Maass cusp forms over various sparse sets like square sum of two integers and primes represented by binary quadratic forms. Summatory functions associated to Fourier coefficients of Hecke eigenform was first studied by Rankin and Selberg. Later, their result was extended for higher moments by Kanemitsu et. al. Further, analogues results have been derived by several authors including Xu, Zhai, Hua and Vaishya. In our work we have extended the work of Hua for higher moments.

There are five chapters in this thesis. The thesis is organized as follows.

Chapter 1 discusses important definitions and results that are essential for understanding this thesis. It covers definitions and examples of arithmetical functions, Lambert series, Gamma function, and Riemann zeta function, along with their analytic properties and related results. The chapter also discusses Meijer G-function, Whittaker function, confluent hypergeometric function, and their connections. Finally, definition and basic properties of modular forms, Jacobi forms and Siegel modular forms with their Fourier-Jacobi expansion, and the L-function associated with them have been discussed.

In Chapter 2, we study the asymptotic behaviour of certain Lambert series involving Fourier-Jacobi coefficients of Siegel cusp forms of weight k and degree 2. The proof of the main result uses the analytic properties of certain Dirichlet series (studied by Kohnen and Skoruppa) associated to Siegel cusp forms in consideration, and the method of contour integration. Further, it has been observed that the corresponding Lambert series has relation with non-trivial zeros of the Riemann zeta function.

In Chapter 3, we extend the result of Chapter 2 by studying the asymptotic behaviour of similar Lambert series associated to Siegel cusp forms of weight k and degree n. We also consider the twist of this Lambert series by a Dirichlet character and study its asymptotic behaviour.

Next in Chapter 4, we study the asymptotic behaviour of certain Lambert series associated to normalized Hecke eigenforms of different weights over $SL_2(\mathbb{Z})$. The proof uses the analytic properties of the Dirichlet series studied by Winnie Li. This generalizes the results of Hafner and Stopple, Chakraborty et. al. and corroborates with Zagier's conjecture.

Finally, in Chapter 5 we analyze the asymptotic behaviour of the higher moments of the generalized divisor function. This analysis is used to demonstrate the oscillatory pattern in the Fourier coefficients of the product of L-functions over a sparse set of integers.