Chapter 7

Conclusion

In this chapter, we are giving the concluding remarks on the contents of this thesis, which are given below:

The Weinstein transform has a rich calculus and its nice mathematical background contains applications in Wavelets, Quantum Calculus and other areas of Mathematical Sciences. Utilising the theory of the Weinstein transform, the continuous Weinstein Wavelet transform was developed considerably by (for example) [8, 9, 11, 13, 32, 34, 38, 58–61], thereby finding potentially useful results. Exploiting the theory of theWeinstein convolution on ultradistributions, the present authors have discussed various properties of the Weinstein transform on spaces of the types $D_\omega, D'_\omega, \mathcal{E}_\omega$ and S_ω and have obtained many results and related observations. This theory can be applied to study the corresponding results in the Sobolev type spaces and will thereby find many interesting results.

We choose to conclude our investigation by presenting references to a number of recent works on the Fourier and other integral transforms, the Bessel and related special functions and polynomials, function spaces, distributional analysis, wavelet analysis, et cetera, including (for example) [63–70, 78], in each of which the reader can find some other presumably novel directions of further researches along the lines which we have developed herein.

The concept of convolution is essential in various fields, like signal processing, image analysis, and differential equations. When extended to generalized distributions (or generalized functions), convolution provides a powerful framework for dealing with objects that are not necessarily functions in the traditional sense, such as the Dirac delta function. Motivated by the above, in the present thesis, by utilizing the theory of the Weinstein transform, we examined various properties of the convolution of generalized distributions. Sobolev spaces are a class of function spaces that generalize the concepts of differentiability and integrability. They play a crucial role in the analysis of partial differential equations (PDEs) and have wide applications in wavelets, quantum calculus, functional analysis, and other related areas of the mathematical sciences. Utilizing this theory, many researchers Dachraoui [4], Hassen and Belgacem [35], Mehrez [30], Salem [56], Saoudi [60, 61], and Trimèche [81], observed useful results regarding the Weinstein transform in the theory of Sobolev-type spaces. In the present thesis, the authors provided a proper mathematical framework for regrading the generalised Sobolev space with the help of the definitions and properties of the Weinstein transform.

Pseudo-differential operators provide a powerful framework that extends the capabilities of partial differential operators, allowing for more complex and nuanced manipulation of functions, particularly in the context of solving differential equations and analyzing signals. Utilizing the theory of the Fourier transform and the Hankel transform, many authors (See [47, 48, 51, 52, 84, 88] discussed pseudo-differential operator and found useful observations). In this thesis, authors developed pseudo-differential operator $A(x, D)$ associated with the symbol class $S_M^{r,l}$ on the space $S_\omega(\mathbb{R}_+^{n+1})$ and obtained many results and interesting properties by utilizing the theory of the Weinstein transform. This theory is extensively used in solving partial differential equations problems like Sobolev spaces and others.

The theory of ultradistributions is the origin of many research works. Exploiting the theory of ultradistributions, characterizations of convolution, and other properties of functional spaces is investigated by many authors [29, 47] by using the theory of Fourier and Hankel transform. In these connections, Beurling [5], Björck [8], Hormander [22], Petzsche [24], Komatsu [29] and Roumieu [53] gave significant contributions to develope the theory of ultradistributions and found fundamental results. Zemanian [89] in his book introduced the theory of Banach-space-valued testing functions to the distributions and studied many interesting results. Later on, Tiwari [79] discussed the properties of Banach space-valued distributions using the theory of Mellin transform. Koh and Lie [27] found various properties of the Hankel transformation of Banach-Space-Valued Generalized functions in space $[H_\mu(A)]'$. Upadhyay [83] investigated the Hankel transformation of Banach-Space-Valued ultradistributions in space $[H^\omega_\mu(A)]'$. The Banach space-valued ultradistibutions associated with the Weinstein transform is a rich and complex area, combining elements of functional analysis, distribution theory, and integral transforms. Motivated by the above result, in the present thesis, the authors introduced the Banach space-valued test functions of Beurling type ultradistributions in space $[H^\beta_\omega(A)]'$ for the weight function ω and obtained various properties by exploiting the theory of the Weinstein transform.

From an application point of view, Beurling-type ultradistributions associated with the Weinstein transform provide an appropriate structure for defining and exploring various characteristics of pseudo-differential operators of Banach space-valued functions associated with certain symbols. The authors can explore several features of Sobolev spaces with help of Banach space-valued functions by applying the theory of pseudo-differential operators involving Weinstein transform. Using the aforesaid theory, we can analyze wavelet transforms with Banach space-valued functions as input or output.

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