Chapter 2

Nonsmooth PI Controller for Uncertain Systems

The key contribution of this chapter is based on the non-trivial Lyapunov analysis of the closed loop system controlled by nonsmooth PI controller. For feedback stabilization of the higher order uncertain chain of integrators, an integral of the discontinuous function is being used in place of integral term of Proportional-Integral (PI) controller. By doing so, the overall control is absolutely continuous rather than discontinuous and the property of invariance concerning the matched Lipschitz uncertainty is still preserved. The efficacy of the proposed control algorithm has been demonstrated using numerical simulation on the magnetic suspension system.

2.1 Introduction

Stability of a perturbed system, is one of the classical problems in the control literature [51]. There are several ways to address this problem. For example, consider the system $\dot{\chi} = F(\chi, \rho(t)) + G(\chi, \rho(t))u; \quad \sigma = h(\chi)$ where u is the control signal, χ is the states of the system, σ is the output, and $\rho(t)$ represents unknown external perturbations or model uncertainties. In several practical scenarios, one of the main objectives is to construct a feedback control law u such that the output σ robustly tracks a reference signal σ_0, despite unknown external perturbations or model uncertainties. There are several different methodologies are already reported in the literature to simplify the above-mentioned problem for the design of a feedback control u. One such strategy is known as a normal form [57], [58]

$$
\begin{aligned}
\dot{\zeta} &= f_0(\zeta, x, d(t)) \\
\dot{x}_i &= x_{i+1}, \quad i = 1, \cdots, n-1 \\
\dot{x}_n &= f(\zeta, x, t) + g(\zeta, x, t)u + d(t) \\
\sigma &= x_1
\end{aligned}
\tag{2.1}
$$

where $\zeta \in \mathbb{R}^p$ and $x \in \mathbb{R}^n$ are the states, $f : \mathbb{R}^p \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ and $g : \mathbb{R}^p \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ are known nominal nonlinearities, $u \in \mathbb{R}$ is the control input, subsystem $\dot{\zeta} = f_0(\zeta, x, d(t))$ with $f_0 : \mathbb{R}^p \times \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^p$ represents the zero dynamics of the system [57], and $d(t)$ corresponds to uncertainties/perturbations. It has been already reported in literature that if subsystem $\dot{\zeta} = f_0(\zeta, x, d(t))$ is Input-to-State stable [51] with respect to x and $d(t)$, then the above mentioned tracing problem (2.1) can be reduced to stabilization of uncertain chain of integrator

$$
\begin{aligned}
\dot{x}_i &= x_{i+1}, \quad i = 1, \cdots, n-1 \\
\dot{x}_n &= f(\zeta, x, t) + g(\zeta, x, t)u + d(t) \\
\sigma &= x_1
\end{aligned}
\tag{2.2}
$$

about the equilibrium point $x = 0$.

The main intention of control researchers is to investigate the problem that, if the above system with know $f(\zeta, x, t)$, $g(\zeta, x, t) \neq 0$ is uniformly asymptotically stable at the origin, then what can be said about the stability and behavior of the perturbed system for $d(t) \neq 0$ at the equilibrium point $x = 0$? There are several approaches available in the literature to solve this problem. In [51], it is reported that if $d(t)$ vanishes at the equilibrium point, then classical state feedback can guarantee the asymptotic stability. However, for nonvanishing disturbances at the origin the memoryless state feedback control (proportional control) doesn't ensure the uniform asymptotical stability of the system [51]. It is found that some classes of nonvanishing perturbations are taken care by the dynamic state feedback like simple PI (proportional-integral feedback control), but it fails to handle the time varying perturbations [52]- [55].

Several delicate controllers have been investigated to enhance control performance and robustness. Sliding mode control (SMC) [56] is one of them. It is one of the most promising control technique for controlling plants under uncertain conditions. However, this controller is still not quite popular in industries because of their discontinuous nature.

In the last two decades, some methods have been proposed to construct the continuous control action based on sliding mode. One of such idea is coined by Prof. Levant [62], [64] known as "Higher order sliding mode". The main idea is to introduce one or more integrators in the system such that the control signal becomes a continuous function [64]. For instance, to obtain the absolute continuous control signal for the system $\dot{x} = f(t,x) + g(t,x) + u,\ x \in \mathbb{R}$, an integrator is introduced to increase the order of the system by one and then discontinuous higher order sliding mode algorithm can be used. However, the implementation of these controllers required the knowledge of $\dot{x}$. In this case, we can reconstruct perturbation, by computing $g(t,x) = \dot{x} - f(t,x) - u$, and it would be possible to compensate it without a discontinuous control [65].

To avoid the above-mentioned drawbacks, a nonlinear PI controller (the Super-twisting algorithm [62], [66]) has been proposed. This controller gives finite time stability in the presence of continuous Lipschitz perturbation for relative degree one system concerning control. However, generalization as well as practical implementation of Super-twisting is not so straightforward because one has to maintain homogeneity in order ensured finite time stability also it is difficult to implement the fractional power in an industrial environment [59]. It is important to mention here that various Lyapunov function that has been suggested for the popular Super-twisting and its variant [60] is not applicable if the nonlinear proportional term is replaced by linear one. Therefore, it is important to look into that if a proportional term of Super-twisting is replaced by linear one then how to give mathematical guarantee for the convergence of the modified algorithm? The further question of interest: is it possible to extend the same structure for a higher order uncertain case with the mathematical guarantee for the convergence?

Motivating from the above fact and wide applicability and acceptability of PI and its variant in the industries, it seems that some more work is required in the area of the classical PI controller for following goals,

- modify classical PI control and give sound mathematical proof to tune gains such that it can handle all kind of Lipschitz disturbances either vanishing or nonvanishing at the origin. (One can further note that if a disturbance is discontinuous, no continuous control can handle it).

- design control such that overall control is absolutely continuous.

- propose a controller such that, it does not require any extra information other than the state variables.

For achieving the specified goal integral part of PI controller is replaced by a discontinuous integrator. Adding this extra integrator overall control is still absolutely continuous rather than first order sliding mode control, but the property of invariance concerning Lipschitz matched uncertainties is still preserved. Finally, we prove the stability of the closed-loop system via a homogeneous, continuously differentiable and strict Lyapunov function.

The rest of the chapter is organized as follows. The notions and preliminaries and problem formulation are established in Section 2.2 and 2.4 respectively. The main results of the chapter and discussion about the proposed controller are presented in Section 2.4. The construction of Lyapunov Function along with the proof of main Theorems and numerical simulation are documented in 2.5 and 2.6 respectively. Finally, some concluding remarks are included in Section 2.7.

2.2 Notions and Preliminaries

Our notations are standard. We let $\mathbb{R}$ denote the real numbers and $\mathbb{R}^+$ denote the nonegative reals. The dilation operator for $x = [x_1, \cdots, x_n] \in \mathbb{R}^n$ is defined as $\Delta_\lambda^{\mathbf{r}} := (\lambda^{r_1} x_1, \cdots, \lambda^{r_n} x_n), \forall \lambda > 0$, where $r_i > 0$ with $i = 1, \cdots, n$ are the weights of of the individual coordinate of x. A functional $V : \mathbb{R}^n \to \mathbb{R}$ is said to be (weighted) $\mathbf{r}$-homogeneous of degree $h \in \mathbb{R}$ if the following identity $V(\Delta_\lambda^{\mathbf{r}} x) = \lambda^h V(x)$ holds. Homogeneous functions have several elegant properties, we are going to recall a result about continuous real-valued homogeneous functions ([68], Lemma 4.2), which will be used in the proof of the main Theorems of this note.

Lemma 1 *Suppose V_1 and V_2 are continuous real-valued functions $V_i : \mathbb{R}^n \to \mathbb{R}$, homogeneous with the same weights and degrees $l_1 > 0$ and $l_2 > 0$, respectively, and V_1 is positive-definite. Then for every $x \in \mathbb{R}^n$,*

$$\left[\min_{\{z:V_1(z)=1\}}\right] [V_1(x)]^{\frac{l_2}{l_1}} \le V_2(x) \le \left[\max_{\{z:V_1(z)=1\}}\right] [V_1(x)]^{\frac{l_2}{l_1}} .$$

Also we have used the Young's inequality in order to show the positive definiteness of Lyapunov function, which can be stated as:

Lemma 2 *The following inequality is always satisfied* $ab \leq c^p \frac{a^p}{p} + c^{-q} \frac{b^q}{q}$, *for any positive real numbers* $a, b, c > 0$ *and* $p, q > 1$, *with* $\frac{1}{p} + \frac{1}{q} = 1$.

Apart from the above two Lemma we use the following notion. For a positive integer n. The signum vector function $SIGN(x)$ is a function $\mathbb{R}^n \to \mathbb{R}^n$ whose behavior in each coordinate is as per the signum function. Explicitly, it is defined as the function $(x_1, \cdots, x_n) \mapsto (\text{sign}(x_1), \cdots, \text{sign}(x_n))$.

2.3 Background and Problem Formulation

In this chapter, we consider the n^{th} order uncertain chain of integrators, given as

$$\begin{aligned} \dot{x}_i &= x_{i+1}, \; i = 1, \cdots, n-1, \\ \dot{x}_n &= u + d(t), \end{aligned} \tag{2.3}$$

where $\mathbf{X}^\top = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix} \in \mathbb{R}^{1 \times n}$ are the states, $u \in \mathbb{R}$ is the control input and $d(t)$ represents the Lipschitz uncertainties/perturbations. Our main is to design a *continuous or at least absolutely controller* u such that system (2.3) stabilizes at the origin despite of the Lipschitz uncertainties/perturbations $d(t)$.

For illustration of the proposed control strategy consider a simplified model of the motion of an underwater vehicle $\dot{v} + v|v| = \bar{u} + d$ where $v \in \mathbb{R}$ is the vehicle velocity and $\bar{u} \in \mathbb{R}$ is the control input (the thrust provided by a propeller) and d is unknown disturbance due to water wave [69]. One can assume that the disturbance d in arbitrary combination of $\sin(t)$ and $\text{cost}(t)$. Suppose that control objective is to maintain constant velocity v_d in-spite of d. Now suppose $x_1 := v - v_d$ and $\bar{u} := u + v|v|$, then one can write

$$\dot{x}_1 = u + d, \quad x_1 \in \mathbb{R}. \tag{2.4}$$

In the absence of uncertainties d, a simple proportional continuous feedback control $u := \alpha(x_1), \alpha : \mathbb{R} \to \mathbb{R}$ is able to stabilize the system (2.4) at the origin. However, above mentioned feedback controller is fail to stabilize system (2.4) in the presence of non-vanishing perturbation $d \neq 0$, because at the origin feedback is zero but perturbations d are nonzero. Now, suppose the case where perturbation d is non-vanishing but some unknown constant, then simple continuous PI controller $u := \alpha(x_1) + \int_0^t k_I x_1(\tau) d\tau$ or $u := k_P x_1 + \int_0^t k_I x_1(\tau) d\tau$ with proper k_P and k_I can stabilize the origin because of the following facts:

The closed-loop system can be written as $\dot{x}_1 = \alpha(x_1) + \int_0^t k_I x_1(\tau)d\tau + d$, which can be further simplified as

$$\dot{x}_1 = \alpha(x_1) + z, \quad \dot{z} = k_I x_1 \tag{2.5}$$

where $z := \int_0^t k_I x_1(\tau)d\tau + d$. The closed-loop system (2.5) is a second order disturbance-free system in the transformed domain. Therefore, using the proper selection of α and k_I it is possible to show that origin is asymptotically stable. However, if d is not constant then closed loop system is no longer remains disturbance-free. In such case, we need to rethink about some more appropriate memory based continuous integral controllers. It is important to note here that memoryless discontinuous control like first order sliding mode can be a choice for the solving the above-mentioned problem provided d is bounded. However, we aim to find some class of continuous controllers keeping in mind that it is very similar to the existing PI controller and also able to tackle large classes of non vanishing disturbances.

2.4 Main Results

2.4.1 Nonsmooth PI for the First Order System for $n = 1$ in (1.3):

In this note we are going to show that the following Theorem gives the asymptotic stability of the system (2.4) about the origin if nonlinear proportional term of Super-twisting controller $u := -k_1|x_1|^{1/2}\text{sign}(x_1) - k_2\int_0^t \text{sign}(x_1(\tau))d\tau$ is replaced by linear one $u = -k_1 x_1 - k_2 \int_0^t \text{sign}(x_1(\tau))d\tau$ where k_1 and k_2 are the designed parameters.

Theorem 1 *Consider the system* (2.4) *and a matched Lipschitz continuous perturbation* $d(t)$ *with Lipschitz constant* $d_0 > 0$. *Then the nonsmooth control law*

$$u = -k_1 x_1 - k_2 \int_0^t sign(x_1(\tau))d\tau \tag{2.6}$$

stabilizes the origin asymptotically in spite of disturbance $d(t)$ *for any* $k_1 > 0$ *and* $d_0 \leq k_2 \leq L(t)\left(\pi_1 + \frac{2^{\frac{3}{2}}}{3}\pi_2\right)$ *with* $\pi_1 \geq \frac{2^2 2^{\frac{5}{6}}}{3^2}\pi_2$ *where* $\pi_i; i = 1,\ 2$ *are some positive constants and* $L : \mathbb{R}^+ \to \mathbb{R}^+$ *is the some positive continuously differential function with* $|L(t)| \leq c_1$, $|L^{-1}(t)| \leq c_2$ *and* $|\dot{L}(t)| \leq c_3$ *where* c_1, c_2 *and* c_3 *are fixed constants.*

Remark 1 *System* (2.4) *can be re-written as*

$$\dot{x}_1 = -k_1 x_1 + z, \dot{z} = -\left(k_2 - \dot{d}\,(t)\, sign(x_1)\right) sign(x_1).$$

Therefore, stability of $\dot{x}_1 = -k_1 x_1 + z,\ \dot{z} = -k^{'}(t)\, sign(x_1),\ k^{'} : \mathbb{R}^+ \to \mathbb{R}^+$ *implies the stability of system* (2.6)*. The same is reflected in Theorem 1.*

2.4.2 Nonsmooth PI for the Higher Order Uncertain Chain of Integrators

The following Theorem gives the asymptotic stability of the system (2.3).

Theorem 2 *Consider the system* (2.3) *and a matched Lipschitz continuous perturbation* $d(t)$ *with Lipschitz constant* $d_0 > 0$*. Then the nonsmooth control law*

$$u = -\mathbf{K_p X} - \int_0^t K_I sign\left(\mathbf{K_P X}\right) d\tau \tag{2.7}$$

stabilizes the origin asymptotically in spite of disturbance $d(t)$ *if* $\mathbf{K_P}$ *and* K_I *are selected such that*

- *all the eigenvalues of matrix* $Q := (\mathbf{A} - \mathbf{BK_P})$ *must be negative and real for any proper selection of* $\mathbf{K_p} = \begin{bmatrix} k_1 & k_2 & \dots & k_n \end{bmatrix}$ *and*

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

- *and gain* K_I *satisfy* $d_0 \leq K_I \leq L(t)\left(-\pi_1 \|\mathbf{B}\| + \frac{2^{\frac{3}{2}}}{3}\pi_\mathbf{2}\mathbf{B}\right)$ *with* $\frac{2^{\frac{3}{2}}}{3}\|\pi_\mathbf{2}\| \geq \pi_1 \geq \frac{2^2 2^{\frac{5}{6}}}{3^2}\|\pi_\mathbf{2}\|$ *where* π_1 *is any positive constant and* $L : \mathbb{R}^+ \to \mathbb{R}^+$ *is the some positive continuously differential function with* $|L(t)| \leq c_1,\ |L^{-1}(t)| \leq c_2$ *and* $|\dot{L}(t)| \leq c_3$ *where* c_1, c_2 *and* c_3 *are fixed constants and* $\pi_\mathbf{2} = \begin{bmatrix} \pi_{21} & \pi_{22} & \dots & \pi_{2n} \end{bmatrix}$ *with some positive constants* $\pi_{2i} > 0$ *for* $i = 1, \dots, n$*.*

Remark 2 *The main benefits of the proposed controller over the first order sliding mode is that overall control is continuous and property of insensitivity concerning Lipschtiz disturbance is retained. Another advantage of the proposed controller over first order sliding mode is that it is also able to reject ramp time-varying disturbances.*

2.5 Construction of Lyapunov Function and Proof of Main Theorems

Next result states the detailed proof of Theorem 1.

Proof : On substitution of the proposed controller (2.6) into (2.8), the closed loop system is given by

$$\dot{x}_1 = -k_1 x_1 + z,\ \dot{z} = -k_2 \mathrm{sign}(x_1) + \dot{d} \tag{2.8}$$

where $z(t) := -k_2 \int_0^t \mathrm{sign}(x_1(\tau))d\tau + d$. The solution of (2.8) is understood in the sense of Fillipov [67]. By introducing time-varying change of variables

$$z_1(t) = \frac{x(t)}{L(t)},\ z_2(t) = \frac{z(t)}{L(t)},\ L(t) > 0, \quad \forall t \geq 0 \tag{2.9}$$

In the new co-ordinates, system (2.8) is given by

$$\dot{z_1} = -\left(k_1 + \frac{\dot{L}}{L}\right) z_1 + z_2, \quad \dot{z_2} = -\frac{k_2}{L}\mathrm{sign}(z_1) + \frac{\dot{d}}{L} - z_2\frac{\dot{L}}{L} \tag{2.10}$$

In general, an algebraic equivalence of systems (2.8) and (2.10) does not preserve the stability properties of a dynamical system. For this it is necessary and sufficient to have topological equivalence: algebraic equivalence plus the condition $|L(t)| \leq c_1$ and $|L^{-1}(t)| \leq c_2$ where c_1 and c_2 are fixed constants [61]. Also, system (2.8) and (2.10) are not homogenous or weighted homogeneous. Still one can use the weighted homogeneous Lyapunov function to prove stability. Also, the various Lyapunov function that has been suggested for the Super-twisting is not straightforwardly applicable to the proposed non-smooth PI. Therefore, we have looked at some different Lyapunov analysis which is also the main technical contribution of this chapter. Consider the following Lyapunov function in the new coordinates

$$V(z) = \left(\pi_1 |z_1| + \frac{1}{2}z_2^2\right)^{\frac{3}{2}} + \pi_2 z_1 z_2 \geq \left(\pi_1 |z_1| + \frac{1}{2}z_2^2\right)^{\frac{3}{2}} - \pi_2|z_1||z_2| \tag{2.11}$$

Applying norm inequality to $\left(\pi_1 |z_1| + \frac{1}{2}z_2^2\right)^{\frac{3}{2}}$ and Young's inequality (see Lemma 2) to term $\pi_2|z_1||z_2|$, we are going to show that proposed Lyapunov function (2.11) is bounded from below by zero.

$$\begin{aligned} V(z) &\geq (\pi_1 |z_1|)^{\frac{3}{2}} + \left(\frac{1}{2}z_2^2\right)^{\frac{3}{2}} - \pi_2\left(\frac{2}{3}g^{\frac{3}{2}}|z_1|^{\frac{3}{2}} + \frac{1}{3}g^{-3}|z_2|^3\right) \\ &= \left(\pi_1^{\frac{3}{2}} - \frac{2}{3}\pi_2 g^{\frac{3}{2}}\right)|z_1|^{\frac{3}{2}} + \left(\left(\frac{1}{2}\right)^{\frac{3}{2}} - \frac{1}{3}\pi_2 g^{-3}\right)|z_2|^3. \end{aligned}$$

where $\quad g \geq 0$.

It is important to note here that $V \geq 0$ for all z if each of $\left(\pi_1^{\frac{3}{2}} - \frac{2}{3}\pi_2 g^{\frac{3}{2}}\right)$ and $\left(\frac{1}{2}\right)^{\frac{3}{2}} - \frac{1}{3}\pi_2 g^{-3}$ should be greater than 0. Suppose

$$2^{\frac{1}{2}} \left(\frac{\pi_2}{3}\right)^{\frac{1}{3}} < g < \left(\frac{3}{2\pi_2}\right)^{\frac{2}{3}} \pi_1,$$

which implies

$$\left(\frac{3}{2\pi_2}\right)^{\frac{2}{3}} \pi_1 \geq 2^{\frac{1}{2}} \left(\frac{\pi_2}{3}\right)^{\frac{1}{3}}.$$

Thus, $\pi_1 \geq \frac{2^{\frac{1}{2}} 2^{\frac{2}{3}}}{3}\pi_2$. Selecting g to be the linear combination of $2^{\frac{1}{2}} \left(\frac{\pi_2}{3}\right)^{\frac{1}{3}}$ and $\left(\frac{3}{2\pi_2}\right)^{\frac{2}{3}} \pi_1$ will lead to $V \geq 0$. Thus $g = \alpha 2^{\frac{1}{2}} \left(\frac{\pi_2}{3}\right)^{\frac{1}{3}} + (1-\alpha)\left(\frac{3}{2\pi_2}\right)^{\frac{2}{3}} \pi_1, \quad 0 \leq \alpha \leq 1$. Now our next aim is to show $\dot{V} < 0$,

$$\begin{aligned}
\dot{V} &= \left\{\frac{3}{2}\left(\pi_1 |z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}} \pi_1 \mathrm{sign}(z_1) + \pi_2 z_2\right\} \dot{z}_1 + \left\{\frac{3}{2}\left(\pi_1 |z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}} z_2 + \pi_2 z_1\right\} \dot{z}_2 \\
&= -\frac{3}{2}\left(\pi_1 |z_1| + \frac{1}{2}z_2^2\right)\chi + \pi_2 z_2^2 - \pi_2\left(k_1 + \frac{\dot{L}}{L}\right) z_1 z_2 - \pi_2 \frac{k_2}{L}\mathrm{sign}(z_1) z_1 + \pi_2 z_1 \frac{\dot{d}}{L} \\
&\quad - \pi_2 z_1 z_2 \frac{\dot{L}}{L},
\end{aligned}$$

where $\chi := \pi_1 \mathrm{sign}(z_1)\left(\left(k_1 + \frac{\dot{L}}{L}\right) z_1 - z_2\right) + z_2\left(\frac{k_2}{L}\mathrm{sign}(z_1) + z_2\frac{\dot{L}}{L} - \frac{\dot{d}}{L}\right)$. One can also rewrite $\dot{V}$ as,

$$\dot{V} = -W_1(z)\left(\frac{\dot{L}}{L}\right) + W_2(z)\left(\frac{\dot{d}}{L}\right) - W_3^*(z), \tag{2.12}$$

where

$$\begin{aligned}
W_1(z) &= \frac{3}{2}\left(\pi_1 |z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}} \left(\pi_1 |z_1| + z_2^2\right) + 2\pi_2 z_1 z_2 \\
W_2(z) &= \frac{3}{2}\left(\pi_1 |z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}} z_2 + \pi_2 z_1 \\
W_3^*(z) &= \left(\frac{3}{2}\left(\pi_1 |z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}} k_1 \pi_1 + \pi_2 \frac{k_2}{L}\right) |z_1| \\
&\quad + \frac{3}{2}\left(\pi_1 |z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}} \left(\frac{k_2}{L} - \pi_1\right) \mathrm{sign}(z_1 z_2) |z_2| - \pi_2 z_2^2 + \pi_2 k_1 z_1 z_2.
\end{aligned} \tag{2.13}$$

We are going to show that $W_3^*(z)$ would dominate over $W_2(z)$, given that $\left|\dot{d}\right| < k_2$. Since,

$$\frac{3}{2}\left(\pi_1 |z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}} |z_2| \geq \frac{3}{2}\left(\frac{1}{2}\right)^{\frac{1}{2}} z_2^2,$$

therefore,

$$\pi_2 (2)^{\frac{1}{2}} \left(\pi_1 |z_1| + \frac{1}{2} z_2^2\right)^{\frac{1}{2}} |z_2| \geq -\pi_2 z_2^2 \text{ and } W_3^* \leq W_3^{'},$$

where,

$$W_3^{'} = \left\{\frac{3}{2}\left(\pi_1 |z_1| + \frac{1}{2} z_2^2\right)^{\frac{1}{2}} k_1 \pi_1 + \pi_2 \frac{k_2}{L}\right\} |z_1| + \left\{\frac{3}{2}\left(\pi_1 + \frac{1}{2} z_2^2\right)^{\frac{1}{2}} \chi_1\right\} |z_2| + \pi_2 k_1 z_1 z_2, \tag{2.14}$$

where $\chi_1 := \left(\frac{k_2}{L}\text{sign}\,(z_1 z_2) - \pi_1 \text{sign}\,(z_1 z_2) + \frac{2^{\frac{3}{2}}}{3}\pi_2\right)$. Again, as $z_1 z_2 \leq \frac{2}{3} c^{\frac{3}{2}} |z_1|^{\frac{3}{2}} + \frac{1}{3} c^{-3} |z_2|^3$, so $W_3^{'} \leq W_3$, where W_3 can be written as

$$\begin{aligned} W_3 = &\left\{\frac{3}{2}\left(\pi_1 |z_1| + \frac{1}{2} z_2^2\right)^{\frac{1}{2}} k_1 \pi_1 + \pi_2 \frac{k_2}{L}\right\} |z_1| + \left\{\frac{3}{2}\left(\pi_1 |z_1| + \frac{1}{2} z_2^2\right)^{\frac{1}{2}} \chi_1\right\} |z_2| \\ &+ \pi_2 k_1 \left(\frac{2}{3} c^{\frac{3}{2}} |z_1|^{\frac{3}{2}} + \frac{1}{3} c^{-3} |z_2|^3\right). \end{aligned} \tag{2.15}$$

Since $\left(\pi_1 |z_1| + \frac{1}{2} z_2^2\right)^{\frac{1}{2}} |z_1| \geq \pi_1^{\frac{1}{2}} |z_1|^{\frac{3}{2}}$,
$2\left(\pi_1 |z_1| + \frac{1}{2} z_2^2\right) |z_2| \geq |z_2|^3$, $W_3^f \geq W_3$, where W_3^f can be written as,

$$\begin{aligned} W_3^f = &\left\{\chi_2 \left(k_1 \pi_1 + \frac{4}{9} \pi_2 k_1 \pi_1^{-\frac{1}{2}} c^{\frac{3}{2}}\right) + \pi_2 \frac{k_2}{L}\right\} |z_1| \\ &+ \chi_2 \left(\chi_1 + \frac{4}{9} \pi_2 k_1 c^{-3} \left(\pi_1 |z_1| + \frac{1}{2} z_2^2\right)^{\frac{1}{2}}\right) |z_2|, \end{aligned} \tag{2.16}$$

where $\chi_2 := \frac{3}{2}\left(\pi_1 |z_1| + \frac{1}{2} z_2^2\right)^{\frac{1}{2}}$. For W_3^f to be greater than zero $\forall z$, both the coefficients of equation (3.11) should be independently greater than zero, that is, if

$$\chi_2 \left(k_1 \pi_1 + \frac{4}{9} \pi_2 k_1 \pi_1^{-\frac{1}{2}} c^{\frac{3}{2}}\right) + \pi_2 \frac{k_2}{L} \geq 0, \tag{2.17}$$

$$\left(\frac{k_2}{L} - \pi_1\right) \text{sign}\,(z_1 z_2) + \frac{2^{\frac{3}{2}}}{3} \pi_2 + \frac{8}{27} \pi_2 k_1 c^{-3} \chi_2 \geq 0. \tag{2.18}$$

These two inequalities are satisfied if,

$$\begin{aligned} \frac{k_2}{L} - \pi_1 + \frac{2^{\frac{3}{2}}}{3} \pi_2 + \frac{8}{27} \pi_2 k_1 c^{-3} \chi_2 \geq 0 \\ -\frac{k_2}{L} + \pi_1 + \frac{2^{\frac{3}{2}}}{3} \pi_2 + \frac{8}{27} \pi_2 k_1 c^{-3} \chi_2 \geq 0, \end{aligned} \tag{2.19}$$

which can be rewritten as

$$0 \leq k_2 \leq L \left(\pi_1 + \frac{2^{\frac{3}{2}}}{3} \pi_2\right). \tag{2.20}$$

Since,

$$W_3^f(z) \geq \overset{*}{W}{}_3^f(z) \triangleq \alpha\,|z_1| + \beta\left(\pi_1\,|z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}|z_2|\,, \tag{2.21}$$

with

$$\begin{aligned} \alpha &= \min_z\left[\chi_2\left(k_1\pi_1 + \frac{4}{9}\pi_2 k_1 \pi_1^{-\frac{1}{2}} c^{\frac{3}{2}}\right) + \pi_2\frac{k_2}{L}\right] \geq 0 \\ \beta &= \min_z\left[\chi_1 + \frac{8}{27}\pi_2 k_1 c^{-3}\chi_2\right] \geq 0, \end{aligned} \tag{2.22}$$

$\overset{*}{W}{}_3^f(z)$ is a continuous and homogeneous positive definite function. According to Lemma 3, it follows that $\forall z \in \mathbb{R}^2$, $W_2(z) \leq \gamma \overset{*}{W}{}_3^f(z)$ is satisfied, with $\gamma = \max_{\left\{z:\overset{*}{W}{}_3^f(z)=1\right\}} > 0$, because both $W_2(z)$ and $\overset{*}{W}{}_3^f(z)$ are continuous and homogeneous with same weights and degree. Finally,

$$\begin{aligned} W_1(z) &= \frac{3}{2}\left(\pi_1\,|z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}\left(\pi_1\,|z_1| + z_2^2\right) + 2\pi_2 z_1 z_2 \\ &\geq \frac{3}{2}\left(\pi_1\,|z_1|\right)^{\frac{1}{2}}\pi_1\,|z_1| + \frac{3}{2}\left(\frac{1}{2}z_2^2\right)^{\frac{1}{2}} z_2^2 - 2\pi_2\left(\frac{2}{3}g^{\frac{3}{2}}\,|z_1|^{\frac{3}{2}} + \frac{1}{3}g^{-3}\,|z_2|^3\right) \\ &= \left(\frac{3}{2}\pi_1^{\frac{3}{2}} - \frac{4}{3}\pi_2 g^{\frac{3}{2}}\right)|z_1|^{\frac{3}{2}} + \left(\frac{3}{2^{\frac{3}{2}}} - \frac{2\pi_2}{3}g^{-3}\right)|z_2|^3 \end{aligned} \tag{2.23}$$

$W_1(z)$ is positive-definite if $\frac{3}{2}\pi_1^{\frac{3}{2}} - \frac{4}{3}\pi_2 g^{\frac{3}{2}} > 0$ and $\frac{3}{2^{\frac{3}{2}}} - \frac{2\pi_2}{3}g^{-3} > 0$ or equivalently

$$\frac{2^{\frac{5}{6}}}{3^{\frac{2}{3}}}\pi_2^{\frac{1}{3}} < g < \frac{3^{\frac{4}{3}}}{2^2}\frac{\pi_1}{\pi_2^{\frac{2}{3}}}$$

and such a g exists if $\pi_1 > \frac{2^{\frac{5}{6}}2^2}{3^2}\pi_2$. Thus π_1 should be selected such that $\pi_1 > \frac{2^{\frac{5}{6}}2^2}{3^2}\pi_2$. It can be noted that it also fulfills $\pi_1 \geq \frac{2^{\frac{1}{2}}2^{\frac{2}{3}}}{3}\pi_2$ required for $V \geq 0$. This completes the proof. □

2.5.1 Proof of Theorem 2

proof : After applying proposed controller (2.7) into (2.3), the closed loop system is given by

$$\begin{aligned} \dot{\mathbf{X}} &= \mathbf{AX} - \mathbf{BK_PX} + \mathbf{B}Z \\ \dot{Z} &= -K_I \mathrm{sign}\left(\mathbf{K_PX}\right) + \dot{d} \end{aligned} \tag{2.24}$$

where, $Z = -\int_0^t K_I \text{sign}\left(\mathbf{K_P X}\right) d\tau + d$. On applying the following time-varying change of variables, $\mathbf{Z_1}(t) := \frac{\mathbf{X(t)}}{L(t)}$, $Z_2(t) := \frac{Z(t)}{L(t)}$, one can rewrite the (2.24) as

$$\begin{aligned}\dot{\mathbf{Z}}_\mathbf{1} &= -\left(\frac{\dot{L}}{L}\mathbf{I} + \mathbf{BK_P} - \mathbf{A}\right)\mathbf{Z_1} + \mathbf{B}Z_2 \\ \dot{Z}_2 &= -\frac{\dot{L}}{L}Z_2 - \frac{K_I}{L}\frac{\mathbf{K_P Z_1}}{\|\mathbf{K_P Z_1}\|} + \frac{\dot{d}}{L},\end{aligned} \tag{2.25}$$

where $\mathbf{I}$ is an identity matrix and $L(t)$ is some continuously differentiable time varying positive function $\mathbb{C}^1$ i.e., $L(t) > 0 \quad \forall t \geq 0$ and $\dot{L} > 0$ exists. Also, to maintain topological equivalence so that stability of transferred system (2.25) implies stability of original closed loop system (2.24) we are further assuming that $|L(t)| \leq c_1$ and $|L^{-1}(t)| \leq c_2$ (where c_1 and c_2 are fixed constants [61]).

Now consider the $V(Z)$ be a Lyapunov function in the new co-ordinates

$$V(Z) = \left(\pi_1 ||\mathbf{Z_1}|| + \frac{1}{2}Z_2^2\right)^{\frac{3}{2}} + \pi_\mathbf{2}\mathbf{Z_1}Z_2, \tag{2.26}$$

where $Z := [\mathbf{Z_1}^\top \; Z_2]^\top$, $\pi_1 > 0$ and $\pi_\mathbf{2} = \begin{bmatrix} \pi_{21} & \pi_{22} & \dots & \pi_{2n} \end{bmatrix}$ with $\pi_{2i} > 0$ for $i = 1, \dots, n$. Next using Young's (see Lemma 2) and norm inequalities, we are going to show that proposed Lyapunov function (2.26) is bounded from below by zero.

$$\begin{aligned}V(Z) &\geq (\pi_1\|\mathbf{Z_1}\|)^{\frac{3}{2}} + \left(\frac{1}{2}Z_2^2\right)^{\frac{3}{2}} - \|\pi_\mathbf{2}\|\left(\frac{2}{3}g^{\frac{3}{2}}\|\mathbf{Z_1}\|^{\frac{3}{2}} + \frac{1}{3}g^{-3}|Z_2|^3\right), \quad g \geq 0 \\ &= \left(\pi_1^{\frac{3}{2}} - \frac{2}{3}\|\pi_\mathbf{2}\|g^{\frac{3}{2}}\right)\|\mathbf{Z_1}\|^{\frac{3}{2}} + \left(\left(\frac{1}{2}\right)^{\frac{3}{2}} - \frac{1}{3}\|\pi_\mathbf{2}\|g^{-3}\right)|Z_2|^3.\end{aligned} \tag{2.27}$$

For $V \geq 0; \quad \forall Z$, we have selected $\pi_1 \geq \frac{2^{\frac{1}{2}}2^{\frac{2}{3}}}{3}\|\pi_\mathbf{2}\|$ and $g = \alpha 2^{\frac{1}{2}}\left(\frac{\pi_\mathbf{2}}{3}\right)^{\frac{1}{3}} + (1-\alpha)\left(\frac{3}{2\|\pi_\mathbf{2}\|}\right)^{\frac{2}{3}}\pi_1$, $0 \leq \alpha \leq 1$ same as first order case.

Time derivative of Lyapunov function (2.26) along the system trajectory (2.25)

$$\begin{aligned}\dot{V}(Z) = &\left(\Theta\pi_1\text{SIGN}\left(\mathbf{Z_1}^\top\right) + \pi_\mathbf{2}\mathbf{Z_2}\right)\dot{\mathbf{Z}}_\mathbf{1} + \left(\Theta Z_2 + \pi_\mathbf{2}\mathbf{Z_1}\right)\dot{Z}_2 = \left(\Theta\pi_1\text{SIGN}\left(\mathbf{Z_1}^\top\right) + \pi_\mathbf{2}\mathbf{Z_2}\right) \\ &\left(-\left(\frac{\dot{L}}{L}\mathbf{I} + \mathbf{BK_P} - \mathbf{A}\right)\mathbf{Z_1} + \mathbf{B}Z_2\right) + \left(\Theta Z_2 + \pi_\mathbf{2}\mathbf{Z_1}\right)\left(-\frac{\dot{L}}{L}Z_2 - \frac{K_I}{L}\frac{\mathbf{K_P Z_1}}{\|\mathbf{K_P Z_1}\|} + \frac{\dot{d}}{L}\right),\end{aligned} \tag{2.28}$$

where

$$\Theta := \frac{3}{2}\left(\pi_1\|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}$$

or

$$\dot{V}(Z) = -W_1\left(\frac{\dot{L}}{L}\right) + W_2\left(\frac{\dot{d}}{L}\right) - W_3^*, \tag{2.29}$$

where,

$$W_1 = \frac{3}{2}\left(\pi_1 ||\mathbf{Z_1}|| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}\left(\pi_1 ||\mathbf{Z_1}|| + Z_2^2\right) + 2Z_2\pi_\mathbf{2}\mathbf{Z_1} \tag{2.30a}$$

$$W_2 = \frac{3}{2}\left(\pi_1 ||\mathbf{Z_1}|| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}} Z_2 + \pi_\mathbf{2}\mathbf{Z_1} \tag{2.30b}$$

$$W_3^* = \frac{3}{2}\left(\pi_1 \|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}\Xi + \pi_2 Z_2\left(\mathbf{B}\mathbf{K_P} - \mathbf{A}\right)\mathbf{Z_1} - Z_2^2\pi_\mathbf{2}\mathbf{B} + \frac{K_I}{L}\pi_\mathbf{2}\mathbf{Z_1}\text{sign}\left(\mathbf{K_P}\mathbf{Z_1}\right)$$

where

$$\Xi := \pi_1\frac{\mathbf{Z_1}^\top\left(\mathbf{B}\mathbf{K_P} - \mathbf{A}\right)\mathbf{Z_1}}{\|\mathbf{Z_1}\|} - \pi_1 Z_2\frac{\mathbf{Z_1}^\top\mathbf{B}}{\|\mathbf{Z_1}\|} + Z_2\frac{K_I}{L}\text{sign}\left(\mathbf{K_P}\mathbf{Z_1}\right).$$

We are going to show that W_3^* would dominate over W_2, given that $\left|\dot{d}\right| < K_I$.

Since, we have selected $\mathbf{K_P}$ such that $\mathbf{Q} := \mathbf{B}\mathbf{K_P} - \mathbf{A}$ has positive eigenvalues. Then using Rayleigh inequality one can write $\lambda_{\min}\{\mathbf{Q}\}\|\mathbf{Z_1}\|^2 \leq \mathbf{Z_1}^\top\mathbf{Q}\mathbf{Z_1} \leq \lambda_{\max}\{Q\}\|\mathbf{Z_1}\|^2$, where $\lambda_{\min}\{\mathbf{Q}\}$ and $\lambda_{\max}\{\mathbf{Q}\}$ are the minimum and maximum eigenvalues of the matrix $\mathbf{Q}$. One can further write $\pi_1\frac{\mathbf{Z_1}^\top(\mathbf{B}\mathbf{K_P}-\mathbf{A})\mathbf{Z_1}}{\|\mathbf{Z_1}\|} \leq \pi_1\lambda_{\max}\{Q\}\|\mathbf{Z_1}\|$. Furthermore, $\frac{K_I}{L}\pi_\mathbf{2}\mathbf{Z_1}\text{sign}\left(\mathbf{K_P}\mathbf{Z_1}\right) \leq \frac{K_I}{L}\|\pi_\mathbf{2}\|\|\mathbf{Z_1}\|$, provided $K_I \geq 0$ and $\pi_1 Z_2\frac{\mathbf{Z_1^T}\mathbf{B}}{\|\mathbf{Z_1}\|} \leq \pi_1\|\mathbf{B}\||Z_2|$. Therefore, $W_3^* < W_3^{''}$, where

$$W_3^{''} := \frac{3}{2}\left(\pi_1 \|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}\Theta_1 + \pi_\mathbf{2}Z_2\left(\mathbf{B}\mathbf{K_P} - \mathbf{A}\right)\mathbf{Z_1} - Z_2^2\pi_\mathbf{2}\mathbf{B} + \frac{K_I}{L}\alpha_1\|\mathbf{K_P}\|\|\mathbf{Z_1}\|. \tag{2.31}$$

where $\alpha_1 := \pi_\mathbf{2}\mathbf{K_P}^\top\left(\mathbf{K_P}\mathbf{K_P}^\top\right)^{-1} > 0$ and $\Theta_1 := \pi_1\lambda_{\max}\{Q\}\|\mathbf{Z_1}\| + \pi_1\|\mathbf{B}\|Z_2 + \frac{K_I}{L}\text{sign}\left(Z_2\mathbf{K_P}\mathbf{Z_1}\right)|Z_2|$.
Since, $\frac{3}{2}\left(\pi_1\|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}|Z_2| \geq \frac{3}{2}\left(\frac{1}{2}\right)^{\frac{1}{2}}z_2^2$, therefore,

$$W_3^{iv} = \frac{3}{2}\left(\pi_1 \|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}\Theta_2 + \pi_\mathbf{2}Z_2\left(\mathbf{B}\mathbf{K_P} - \mathbf{A}\right)\mathbf{Z_1} + \frac{K_I}{L}\alpha_1\|\mathbf{K_P}\|\|\mathbf{Z_1}\|, \tag{2.32}$$

where

$$\Theta_2 := \pi_1\lambda_{\max}\{Q\}\|\mathbf{Z_1}\| + \pi_1\|\mathbf{B}\|Z_2 + \frac{K_I}{L}\text{sign}\left(Z_2\mathbf{K_P}\mathbf{Z_1}\right)|Z_2| + \frac{2^{\frac{3}{2}}}{3}\pi_\mathbf{2}\mathbf{B}|Z_2|. \tag{2.33}$$

Again, as

$$\|\mathbf{Z_1}\||Z_2| \leq \frac{2}{3}c^{\frac{3}{2}}\|\mathbf{Z_1}\|^{\frac{3}{2}} + \frac{1}{3}c^{-3}|Z_2|^3; c > 0, \tag{2.34a}$$

$$\left(\pi_1\|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}\|\mathbf{Z_1}\| \geq \pi_1^{\frac{1}{2}}\|\mathbf{Z_1}\|^{\frac{3}{2}}, \tag{2.34b}$$

$$2\left(\pi_1 ||\mathbf{Z_1}|| + \frac{1}{2}Z_2^2\right)|Z_2| \geq |Z_2|^3. \tag{2.34c}$$

So $W_3^{iv} \leq W_3^{vi}$, where W_3^{vi} can be written as

$$W_3^{vi} = \left(\Theta\Theta_3 + \frac{K_I}{L}\alpha_1 \|\mathbf{K_P}\|\right)\|\mathbf{Z_1}\| + \Theta\left(\Theta_4 + \frac{8}{27}c^{-3}\Theta\left(\|\pi_{\mathbf{2}}\| \|\mathbf{BK_P} - \mathbf{A}\|\right)\right)|Z_2|, \quad (2.35)$$

where,

$$\begin{aligned} \Theta_3 &:= \pi_1 \lambda_{\max}\{Q\} + \frac{4}{9}c^{\frac{3}{2}}\pi_1^{\frac{-1}{2}}\left(\|\pi_{\mathbf{2}}\| \|\mathbf{BK_P} - \mathbf{A}\|\right), \\ \Theta_4 &:= \pi_1 \|\mathbf{B}\| \operatorname{sign}(Z_2) + \frac{K_I}{L}\operatorname{sign}(Z_2\mathbf{K_P}\mathbf{Z_1}) + \frac{2^{\frac{3}{2}}}{3}\pi_{\mathbf{2}}\mathbf{B} \end{aligned} \quad (2.36)$$

For W_3^{vi} to be greater than zero $\forall Z$, both the coefficients Θ_3 and Θ_4 of equation (2.35) should be independently greater than zero , that is, if

$$\pm\pi_1 \|\mathbf{B}\| \pm \frac{K_I}{L} + \frac{2^{\frac{3}{2}}}{3}\pi_{\mathbf{2}}\mathbf{B} \geq 0. \quad (2.37)$$

For inequality $\Theta_4 \geq 0$ to be satisfied, all of inequalities (2.37) have to be satisfied, which can be re-written as

$$0 \leq K_I \leq L\left(-\pi_1 \|\mathbf{B}\| + \frac{2^{\frac{3}{2}}}{3}\pi_{\mathbf{2}}\mathbf{B}\right). \quad (2.38)$$

Furthermore, (2.38) is satisfied only if $\frac{2^{\frac{3}{2}}}{3}\|\pi_{\mathbf{2}}\| \geq \pi_1$. Since,

$$W_3^{vi}(Z) \geq \overset{*}{W_3^{vi}}(Z) \triangleq \alpha\|\mathbf{Z_1}\| + \beta\left(\pi_1\|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}|Z_2|, \quad (2.39)$$

with

$$\begin{aligned} \alpha &= \min_Z\left[\Theta\Theta_3 + \frac{K_I}{L}\alpha\|\mathbf{K_P}\|\right] \geq 0 \\ \beta &= \min_Z\left[\Theta_4 + \frac{8}{27}c^{-3}\Theta\left(\|\pi_{\mathbf{2}}\| \|\mathbf{BK_P} - \mathbf{A}\|\right)\right] \geq 0, \end{aligned} \quad (2.40)$$

$\overset{*}{W_3^{vi}}(Z)$ is a continuous and homogeneous positive definite function. According to Lemma 3, it follows that $\forall Z \in \mathbb{R}^{n+1}$, $W_2(Z) \leq \gamma \overset{*}{W_3^{vi}}(Z)$ is satisfied, with $\gamma = \max_{\left\{Z:\overset{*}{W_3^{vi}}(Z)=1\right\}} > 0$, because both $W_2(Z)$ and $\overset{*}{W_3^{vi}}(Z)$ are continuous and homogeneous with same weights and degree.

$$\begin{aligned} W_1(Z) &= \frac{3}{2}\left(\pi_1\|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}\left(\pi_1\|\mathbf{Z_1}\| + Z_2^2\right) + 2Z_2\pi_{\mathbf{2}}\mathbf{Z_1} \\ &\geq \frac{3}{2}\left(\pi_1||\mathbf{Z_1}||\right)^{\frac{1}{2}}\pi_1||\mathbf{Z_1}|| + \frac{3}{2}\left(\frac{1}{2}Z_2^2\right)^{\frac{1}{2}}Z_2^2 \\ &- 2||\pi_{\mathbf{2}}||\left(\frac{2}{3}g^{\frac{3}{2}}||\mathbf{Z_1}||^{\frac{3}{2}} + \frac{1}{3}g^{-3}|Z_2|^3\right) \\ &= \left(\frac{3}{2}\pi_1^{\frac{3}{2}} - \frac{4}{3}\|\pi_{\mathbf{2}}\| g^{\frac{3}{2}}\right)\|\mathbf{Z_1}\|^{\frac{3}{2}} + \left(\frac{3}{2^{\frac{3}{2}}} - \frac{2\|\pi_{\mathbf{2}}\|}{3}g^{-3}\right)|Z_2|^3. \end{aligned} \quad (2.41)$$

Table 2.1: Model Parameters for Magnetic-levitation System

Parameters	**Symbols**	**Values**
Winding Resistance	R	1 [Ω]
Winding Inductance	L_1	$0.02[H]$
Gravitational Accleration	gc	$9.81[m/s^2]$
Magnetic Force Constant	k_M	0.0005 [mH]
Frictional Constant	k_F	0.01 [$N/m/sec$]
Ferromagnetic ball's mass	m	0.1 [kg]
Desired ball's position	z_1	$0.1[m]$

$W_1(z)$ is positive-definite if $\frac{3}{2}\pi_1^{\frac{3}{2}} - \frac{4}{3}\|\boldsymbol{\pi_2}\| g^{\frac{3}{2}} > 0$ and $\frac{3}{2^{\frac{3}{2}}} - \frac{2\|\boldsymbol{\pi_2}\|}{3} g^{-3} > 0$ or equivalently $\frac{2^{\frac{5}{6}}}{3^{\frac{2}{3}}}\|\boldsymbol{\pi_2}\|^{\frac{1}{3}} < g < \frac{3^{\frac{4}{3}}}{2^2}\frac{\pi_1}{\|\boldsymbol{\pi_2}\|^{\frac{2}{3}}}$ and such a g exists if $\pi_1 > \frac{2^{\frac{5}{6}}2^2}{3^2}\|\boldsymbol{\pi_2}\|$. Thus π_1 should be selected such that $\pi_1 > \frac{2^{\frac{5}{6}}2^2}{3^2}\|\boldsymbol{\pi_2}\|$. It can be noted that it also fulfills $\pi_1 \geq \frac{2^{\frac{1}{2}}2^{\frac{2}{3}}}{3}\|\boldsymbol{\pi_2}\|$ required for $V \geq 0$. This completes the proof. □

2.6 Simulation

We demonstrate the robustness of nonsmooth PI control for the third order uncertain chain of integrators containing constant or time-varying matched disturbances. Consider the following magnetic suspension system [51]

$$
\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -\frac{k_F}{m}z_2 + g_c - \frac{k_M}{2m}\left(\frac{z_3}{z_1+q}\right)^2 \\
\dot{z}_3 &= -\frac{R}{L(z_1)}z_3 + \frac{k_M}{L(z_1)}\left(\frac{z_2 z_3}{(z_1+q)^2}\right) + \frac{u}{L(z_1)}
\end{aligned}
\tag{2.42}
$$

where $L(z_1) = L_1 + \frac{K_L}{q+z_1}$ with $K_L = 0.01[H.m], q = 0.05[m]$, $z_1 = \sigma \in \mathbb{R}_+$ is the vertical distance of the ball measured from the coil, $z_2 = \dot{\sigma}$ is the velocity, $z_3 = i$ is the electrical current and the control u is the voltage applied and the control objective is to bring the ball position to $z_1 = 0.1[m]$. Other model parameters are given in the Table 2.1. In order to convert system (2.42) into normal form following transformation is defined

$$x := \Theta(z) = \begin{bmatrix} \sigma(z) \\ \dot{\sigma}(z) \\ \ddot{\sigma}(z) \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ -\frac{k_F}{m} z_2 + g_c - \frac{k_M}{2m}\left(\frac{z_3}{z_1+q}\right)^2 \end{bmatrix} \tag{2.43}$$

where $\sigma(z) := z_1$ is the output of system. It can be verified that $\Theta(z)$ is a diffeomorphism in $D := \{z_1 + q > 0 \text{ and } z_3 > 0\}$. In the transformed co-ordinate system can be represented as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -\frac{k_F}{m} x_3 - \frac{k_M z_3}{mL(z_1)(z_1+q)^2}\left[-Rz_3 + \frac{L_1 z_2 z_3}{(z_1+q)} + u\right] \end{aligned} \tag{2.44}$$

Now selecting control u as

$$u = Rz_3 - \frac{L_1 z_2 z_3}{(q+z_1)} + \frac{mL(z_1)(z_1+q)^2}{k_M z_3}\left(\nu + \frac{k_F}{m} x_3\right) \tag{2.45}$$

to obtain

$$\dot{x}_1 = x_2,\ \dot{x}_2 = x_3,\ \dot{x}_3 = \nu + d(t),$$

where $d(t)$ is comes into picture due to uncertainties in gravity, systems parameters and external noise. For the current simulation we have considered $d(t) = 0.5 + 0.1t + \sin(t)$ and controller ν as

$$\begin{aligned} \nu :=& -k_1(x_1 - 0.1) - k_2 x_2 - k_3 x_3 \\ &+ \int_0^t -k_4 \text{sign}(k_1(x_1 - 0.1) + k_2 x_2 + k_3 x_3) d\tau \end{aligned} \tag{2.46}$$

in order to track $x_1 = z_1 := z_{1d} = 0.1$, where $k_1 = 2000, k_2 = 400$, $k_3 = 30$, and $k_4 = 10$. In order to simulate the system (2.42), the initial conditions are chosen as $Z_1 = 0.01, z_2 = 0$, and $z_3 = 2.2$. Tracking of position $z_1 = 0.1$ is shown in the Fig. 2.1 using nonsmooth PI in the presence of disturbance $d = 0.5 + 0.1t + \sin(t)$. It is also confirmed from the simulation shown in Fig. 2.2, that the control is continuous.

2.7 Summary

The stabilization of systems with nonsmooth PI controller under uncertainty is studied in this paper. The proposed method completely rejects the Lipschitz matched disturbances.

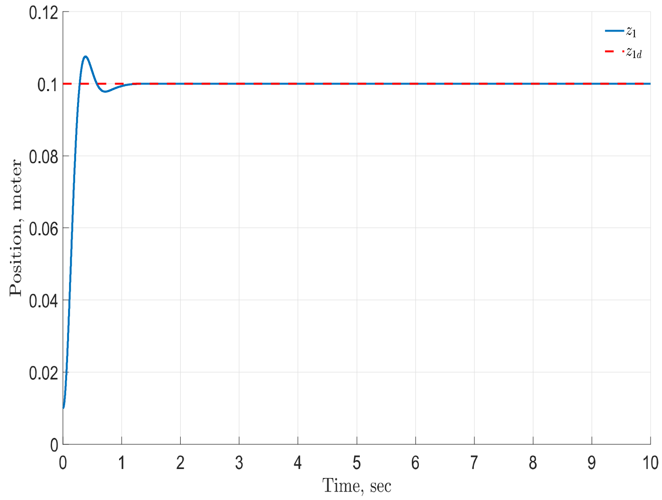

Figure 2.1: Evolution of position with respect to time

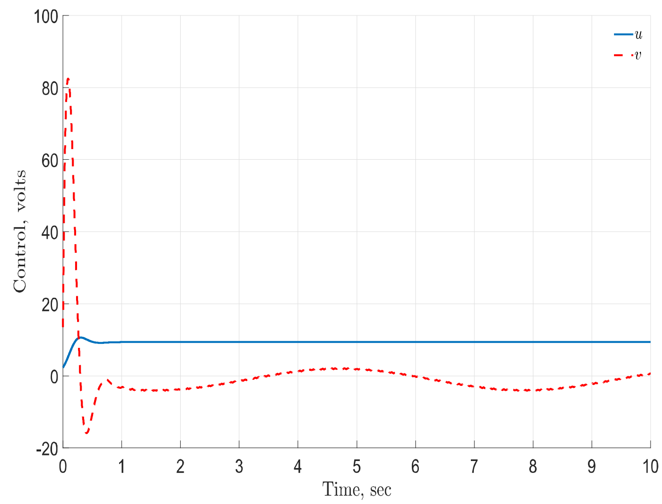

Figure 2.2: Evolution of control with respect to time

The insensitivity to the disturbances is obtained by incorporating the nonsmooth part in the controller. The stability of the proposed nonsmooth PI controller for an uncertain chain of an integrator is established for first order system via non-trivial strict Lyapunov function; then it is extended to an uncertain chain of an integrator. Finally, the performances of the controller are demonstrated using Matlab simulations of a magnetic suspension system.

In next Chapter, we study the design of a artificial delayed output twisting algorithm for the uncertain systems with relative degree two. The stability analysis of closed loop system and the controller gains have been obtained by Lyapunov-Razumikhin method.