Chapter 6

Dynamics of a single particle moving on a Random Lorentz Lattice-gas

6.1 Introduction

Up to now we analyse the motion of the multi active particle system, In this Chapter 6 we analyse the motion of single passive particle on Lorentz lattice gas. Most native species, like microscopic particle move in random or inhomogeneous medium [Cohen & Wang (1995); Kumar & Mishra (2019); Shaebani et al. (2020)] and periodic substrate [Semwal et al. (2021)]. The inhomogeneity can be present on regular lattices [Gates & Westcott (1982); Roerdink & Shuler (1985)] or in many natural systems, it can be random arrangement of inhomogeneity [Picu (2011)]. Lorentz-lattice Gas (LLG) has turned out to be one of the useful way to understand the dynamics of particle on regular or random lattices [Ernst & Binder (1988); Shaebani et al. (2020)]. Most of the previous study is limited to dynamics of a single particle moving on different regular lattice [Gale (1993); Langton (1986); Meng & Cohen (1994); Mishra et al. (2016); Webb & Cohen (2015)]. Random lattices can be encountered in many example systems and can be modeled in various different ways [Grosfils et al. (1999)]. Most of the previous study of LLG is focused on regular lattice [Mishra et al. (2016)], also some of our previous studies are limited to regular lattice [Kumar & Mishra (2019)]. Also most of the natural systems are random in nature viz: network of cytoskeleton inside our cell [Golding & Cox (2006)], human neural network [Edelen et al. (2016)] etc. The dynamics of a pseudo particle, motor in cytoskeleton network and information in neural network can be modeled as dynamics of a particle moving on random network. Such systems motivate us to perform our present study.

We divide our study in two parts, in first part we give basic algorithm to generate the random lattice and in second part we study the dynamics of a single particle moving on the lattice. Lattice is occupied with randomly place left and rights rotators.

Our main results are as follows: We first construct the random lattice, starting from randomly placed points on the two-dimensional lattice and then joining them within some distance and checking that all the points are connected, hence path is available to reach every point on the lattice. In that ways finally we construct the fully connected *two-dimensional random lattice* and check its properties. We find that the mean number of nodes or the coordination number increases on increasing density of points on the lattice and also distribution of degree or the coordination number is Poisson distribution. We further introduced three types of dynamics of a single particle on the lattice: (i) a single particle moving randomly on the lattice; (ii) particle moving on the lattice decorated with randomly placed left and right rotators, which rotate the direction of particle to its extreme left and right available path; (iii) rotators which rotate the direction of particle to one of the possible right and left available paths. We find that the dynamics of particle moving randomly on the lattice is pure random walk. Whereas dynamics of the particle on the lattice with two kinds of rotators is deterministic random walk wth two distinct types of steady state trajectories. The particle dynamics in the presence of rotators can be divided in two parts: early time transient dynamics and late time steady state. System is studied for various ratios of left and right rotators. For all ratios, when particle move in the presence of leftmost and rightmost rotators: early time dynamics is subdiffuive with mean square displacement (MSD) $\Delta(t) \sim t^{2/3}$ and at late time it becomes ballistic with $\Delta(t) \sim t^2$. Whereas dynamics in the presence of left and right rotator is early time diffusive and then becomes ballistic at late times.

In rest of the article: in section 6.2 we first design the random lattice and discuss its characteristics. Section 6.5 we study the dynamics of a single particle moving on the random lattice. Then we come to the detail study of motion in the presence of leftmost and rightmost rotators and section 6.7 discusses the details of the two kinds of models with left and right rotators. We then discuss about the results of the two models in 6.6 and 6.7 and finally we conclude our study in section 6.8.

6.2 Construction of the Random Lattice

We first describe the construction of random lattice on a two-dimensional substrate. The method can be described in the following steps: In first step we generate the random points on a two-dimensional lattice and then check that the whole lattice is connected. The mechanism is given as below:-

Step 1- *Distribution of points randomly in space with given density:* We start with a square system of length L wherein, N lattice points are distributed (density $\rho = \frac{N}{L^2}$). This is done by generating two random numbers, between 0 to L and 0 to L respectively to serve as the x and y coordinates of a point.

Step 2- *Connect all points within a distance R_c of each other, keeping in view the periodic boundary condition (PBC):* We define a cutoff distance R_c such that if distance between any two points is less than R', they would be connected with a bond. If we want to judge which all pairs of particles are to be connected through a bond, we need to check one point's distance with all the other points and decide if the distance between each pair is less than R_c or not. This means we are making $N-1$ comparisons for each of N points. This gives total of N^2-N comparisons. The complexity is going to be $\mathscr{O}(N^2)$ which we strive to improve.

We sort our points with respect to either x or y coordinate, just for once and at the start. Now for each point, we use binary search algorithm to determine the points which are first and last within the interval $[x-R_c, x+R_c]$. This can be done in $\log_2 N$ number of steps each. Now we restrict our search to all points within this interval. Fairness of this optimization is discussed below.

The area of the interval we have truncated our search to is $2\times L\times R'$. Total area is L^2. Assuming points are randomly distributed, the number of points in this interval is $\frac{2R'N}{L}$. Hence for each point, we make exactly $2\log_2 N+\frac{2R'N}{L}$ queries.The mean distance between points R' in general much smaller than L. For a given L, the choice of R' depends on N and it decreases as N increases.

The periodicity is imposed in both the directions: x and y. This can be implemented by changing the above algorithm that connects lattice sites within R' distance between each other.

To begin with, we set the value of R' to the mean distance between points. This value however will be corrected in subsequent steps with the aim to end up with the minimum value of R' which keeps the given distribution of points connected.

Step 3- *Check if we end up with a connected graph:* When $R_c = L$, every lattice point will be connected to every other and we will have a “completely connected graph". When $R_c = 0$, no point will be connected to any other point. For any intermediate values, the graph may or may not be “connected". The graph being connected ensures that there is always path forward while traversing.

For a given distribution of points in space, one can find the threshold value of R_c above which, if we join points within R_c, we get a connected graph. This will be the minimum value of R_c (under defined tolerance level) for which the graph would be connected. To know if our current R_c gives a connected graph, we need to check for each point whether it is connected to all others through at least one path or not. For checking whether one point is connected to another point], we need to find if there exists a path between them. This is to be checked for all point pairs, of a single point and then for all points as well. To avoid such escalating complexity, we use the union find algorithm with weighted union and path compression heuristics Iliev & Kyurkchiev (2018), which makes the problem $\mathcal{O}(N+M)$ practically, where N is the number of points in the lattice and M is equal to the number of edges/bonds.

Step 4- *Binary Search to determine minimum R_c that keeps the graph connected:* To determine the threshold R_c in less trials, a modified binary search is used as described: we know that for $R_c = L$, the graph will surely be connected and it will not be connected for $R_c = 0$. Between $R_c = 0$ and $R_c = L$, there is a threshold value of R_c above which the lattice is connected and below which, it's not. Set $right\_limit = L$ and $left\_limit = 0$. We had begun with $mid = R_c$ which is the mean distance between points. If found connected, we change *right_limit* of binary search to *mid* and if it is found not connected, we update the $left\_limit$ of binary search to *mid*. Below is the algorithm for the same:-

```
while( left_limit <= right_limit && right_limit - left_limit > Tolerance){
           if ( lattice is connected at R_{c}) right_limit=mid
                 else left_limit=mid
           mid=( right_limit + left_limit ) /2
           R_{c}=mid &  Repeat step 2 with revised value of R_{c}
}
```

Hence, we obtain the minimal value of R_c correct to defined tolerance in limited number of steps. The tolerance defined here is one-tenth of the mean distance between points. This ends giving us our desired connected random graph, with periodic boundary condition.

The above algorithm is different from the other previously developed algorithms to generate the random lattice: like algorithm using Voronoi tessellations Green & Sibson (1978), which generates the non crossing bonds. But by making a random lattice like this, we are making a system with crossing bonds. We can also make a system of non-crossing bonds by making a Voronoi network or tessellation [Okabe et al. (2008)]. One way to make such a network is to make the above network, and then removing crossing bonds (remove the longer ones). The effect of such non-crossing bonds on the dynamics of particle can be discussed in future work. Hence in the current study we focus on the dynamics of particle on crossing bond lattice, which can be realised as neural network Edelen et al. (2016) or cytoskeleton network [Fletcher & Mullins (2010)] in our body or many other networks like social network [Kurka et al. (2015)], information propagating network in a real society [Newman et al. (2002)].

Hence in our present study we design a simple network and rules for a single particle moving on that network. The idea of network is taken from classic problems of percolation and random walk [Dziob & Sokołowska (2020); Masoliver et al. (1989)] but final outcome of particle dynamics can be mapped to the above mentioned topics of our everyday use. We discuss different models for the particle dynamics.

6.3 Properties of random lattice

We first characterise the properties of random lattice. Snapshot of a random lattice with all connected bonds is shown in 6.1(a). The degree distribution $P(n)$ of the random lattice is a distribution of degree n (number of connected bonds) for all the points on the lattice (see Fig. 6.1(b)). The peak of the distribution n_0 gives the information about the most probable degree of the random lattice. We fit the degree distribution $P(n)$ and find that it is a *Poisson* distribution. The explanation of Poisson distribution of $P(n)$ can be understood as: if we have a target of a disk of radius “R" around a given point, and points are put down randomly, then the distribution should be a Poisson one, at least in the limit that the density of points on the lattice is small. In general it would technically be a Binomial distribution, but it becomes a Poisson distribution when the density is low. The mean of the Poisson distribution should correspond to the density of points that are put down. Tuning density ρ shift the peak of the degree distribution n_0.

Similarly, R_c depends on the ρ because for a given distribution of points in space, it is up to the algorithm to decide the minimum R_c which results in a connected graph. We note that practically this number is always more than the mean distance $r_0 = \sqrt{\frac{L^2}{N}}$ between points in the lattice.

The bond length (σ) distribution $P(\sigma)$ in Fig. 6.2(a), for $N = 1000$ in 100×100 space, shows that as the bond length (on $x-$axis) increases and becomes closer to R_c, the $P(\sigma)$ also increases. A similar trend is seen for random lattices of any system size or density. This can be explained as follows: Fix any point as origin without loss of generality. The points connected to origin are at various distances (less than R_c) and hence, the bond length is a distribution. Note that any point lying at a circle with center at origin and radius r will yield a count for bond length r. Since the circumference of this circle increases linearly with r, the number of points lying on the circle increases in similar proportion. This explains why the frequency of bonds (within an interval) increases approximately linearly with the bond length σ itself.

The peak of the Poisson degree distribution n_0 shift with increase in the number of particles. When number of particles are increased, R_c decreases but the peak shifts towards the higher end. The plot of n_0 and R_c with N is shown in Fig. 6.2(b)(in all log-log plot always have base 10 log_{10}). The simulations show that the averaged value of the connectivity threshold R_c varies with number of points N: $R_c \propto N^{-0.42}$. This is less than the rate of decrease of the mean distance between points which decreases with $N^{-0.5}$. The deviation of the value 0.5, may be a logarithmic correction. Since the maximum closest distance between points likely grows logarithmically with the size of a system. The n_0 too shifts right with increase in N, with : $n_0 \propto N^{0.18}$. We also characterise the dependence of R_c on the system size L for fixed density $\rho = 0.005, 0.01, 0.02$, and we find that the minimum distance increases slowly with increasing L, because for larger systems there is a larger chance of having points farther from other points. In Fig. 6.2(c) we plot the R_c vs. L for the fixed density $\rho = 0.005, 0.01, 0.02$ on the $\log - \log$ scale and lines are the power with slopes $0.12, 0.11, 0.10$ respectively.

6.4 Model: Definitions

Each vertex of the lattice is occupied by one of the two sets of rotators as shown in Fig. 6.3. The right rotator marked as left/right rotators (filled/empty) circles. Left/right rotators rotate the direction of moving particle to left/right from its incoming direction. We define three types of models, based on characteristics of rotators and its interaction with particle. The moving particle always keep moves forward and does not retract back after hitting a lattice point in any of these models. Hence all the lattice points is occupied by either left or right rotators and none of the site is vacant.

1. Model 1: Pure Random Motion (RM): This model is studied in Section 6.5. In this case, there are no rotators placed on any sites on the random lattice. Particle always chooses a random path out of all available paths and move in that direction. Hence it is like a random walk on a two-dimensional random lattice.

2. Model 2: Leftmost/Rightmost Rotators (ER): This model is studied in Section 6.6. In this case, leftmost Lt and rightmost Rt rotators, as shown in Fig. 6.3(a), are distributed across the N lattice points. r_e is the fraction of Lt rotators $r_e = \frac{N_{Rt}}{N_{Rt}+N_{Lt}}$,

where N_{Rt} and N_{Lt} and number of Rt and Lt rotators respectively and $N = N_{Lt} + N_{Rt}$. $r_e = 0$ implies only left rotators are present and $r_e = 1$ implies only right rotators are present. We vary r_e between 0 to 1 in steps of 0.1 and the dynamics is symmetric with respect to $r_e = 0.5$. The above model is valid for the dynamics of a single particle moving in completely deterministic environment. Where the path of the particle is only determine from the direction it is coming. The example of systems, which can be mapped to this case are like information transfer on social network, neural network etc.

3. Model 3: Left/Right Rotators (R): This model is studied in Section 6.7. In this case, L and R rotators, as shown in Fig. 6.3(b), are distributed across the N lattice points. r_e is the ratio of L rotator as $re = \frac{N_R}{N_R+N_L}$, where N_R and N_L and number of R and L rotators respectively and $N = N_L + N_R$. Again results are symmetric with respect to $r_e = 0.5$. This model is different from the Model 2, where dynamics is purely deterministic, now it has some randomness depends on the number of connected bonds on the lattice. The two Models 2 and 3 we defined above can be compared with the previously studied model of Eulerian walk on two-dimension Euler (1741) which we aim to focus in future work.

6.4.1 Observables and characterisation of the system

The following observables are calculated to characterise the motion of the particle on the lattice,

(i) Mean squared displacement (MSD), which is a measure of the magnitude of displacement of the particle from the starting point. MSD ($\Delta(t)$) is defined as $\Delta(t) = \langle[\mathbf{r}(t) - \mathbf{r}(0)]^2\rangle$ where $\langle.....\rangle$ represents the averaging over many random, independently generated configurations of lattice and rotators on the lattice.

(ii) $P_T(\rho,t)$: probability of the particle being in a traversing trajectory at time t, which is the fraction of realizations that are in traversing trajectories at time t calculated over a large number of realisations. The probability of being in a closed trajectory is denoted as $P_C(\rho,t) = 1 - P_T(\rho,t)$.

To characterise the motion, we extract the MSD exponent β from $\Delta(t)$ by assuming $\Delta(t) \sim t^{\beta}$.
If $\beta > 1$, the motion is called superdiffusion. If $\beta \sim 1$ the motion is diffusive, and if $\beta < 1$, the motion of the particle is subdiffusive. System is studied for various ratios of right/left rotators r_e. The length of the system L is tuned from 100 to 500 and density of points on the lattice is tuned from 0.01 to 0.05. System is studied for ratio 0 to 0.5 (because model is symmetric with respect to $r_e = 0.5$). Simulation is performed for total time 12000 and averaging is done for 1000 independent realisations.

6.5 Pure Random Motion (RM)

The particle is set to move in the random lattice with $N = 100$ in 100×100 space. The particle has equal probability to move forward in any of the other branches except from where it arrived (the probability of tracing back a certain path is 0). In this case we expect the motion is purely a random walk with late time MSD should vary linearly with time. Fig. 6.4 shows the diffusion process. The graph shows the slope is equal to 1. Hence particle dynamics on pure random lattice is diffusive at late time. It further implies the random nature of our underlying lattice. The trajectory of the particle motion on the lattice can be compared with the standard model for percolation in two-dimensions. Details of such comparison is one of our future project.

6.6 Dynamics in the presence of leftmost (Lt) and rightmost (Rt) rotators (ER)

A particle is set to move in a random lattice with *Lt* and *Rt* rotators equally distributed. The aim is to study the dynamics of a real life network where the moving information does not follow a random path everywhere. Some junctions are more right or left deviating than the others. Logarithm of the mean square displacement averaged over multiple system is plotted against the logarithm of the number of steps (or time) in Fig. 6.5(a). Data shows an initial phase with slope $\Delta(t) \sim t^{0.67 \sim 2/3}$, and then at late time $\Delta(t) \sim t^2$ (superdiffusion). It is noted that the time spends in the initial phase of the MSD with slope $\sim 2/3$ depends on the ratio r_e. As r_e approaches 0.5, then the range of MSD with slope 0.67 increases.
The early time subdiffusive behaviour with a unique exponent of 2/3, the similar subdiffusive exponents is found for the motion of mRNA in E. Coli cell also in other studies of particle motion on pure deterministic environment on regular lattices . We later explore the details of the particle's trajectory for the present case. Here the model is purely deterministic. Hence if it repeats a path once, it is always going to repeat, will be trapped in the closed orbit or keep exploring the lattice hence ballistic motion. Hence in this case only two types of particle trajectories can be found, either trapped or traversing. The initial subdiffusive feature is due to the finite fraction of closed trajectories. With time more and more trajectories get closed, but due to periodic boundary condition, there is a finite number of paths available and hence finally particle keep traversing on the lattice. We can say that a network with larger r_e, requires more travel before it get trapped (because it experience more homogeneous distribution of rotators) and hence, the portion with slope 0.67 increases in length. This kind of motion with initial dynamics subdiffusive with slope 0.67 and later with slope 2 (traversing), occurs for all densities ρ.
Now we analyse the trajectory of particle motion: The MSD shows the mean dynamics of

particle for many sets of configuration. When we starts to analyse the individual trajectory, a particle moving in a random lattice, at any instant of time, may either continue to move in the infinite span of the lattice or may get trapped. The phenomena of getting trapped needs more understanding. All the rotators used in this study are deterministic in the sense that they will always deviate the incoming trajectory to the rightmost/ leftmost paths only out of all available paths. This means that if the particle comes from point a to point b and gets deflected to point c (in accordance with the type of rotator at point b), then every other time it traverses from a to b, it will surely go to c only and no other point and in turn, again follow the same path ahead of c as before and arrive at a and b again. This essentially is the definition of a trap. In such cases, the mean square displacement will approach a constant value.

The $\Delta(t)$ plots of independent system are basically of two types: The first case as seen in Fig. 6.6(a) is the trapping when $\Delta(t)$ oscillates and forms a band. The thickness of the band varies according to the size of the trap in space. Sometimes, such a log plot of a trapping case may be extending to the negative $y-$axis of the graph as well. This occurs due to revisiting of the particle to the origin with respect to which the $\Delta(t)$ is calculated or a point that is fairly close to the origin. The second case shown in Fig. 6.6(b) is a usual traversing particle graph in which the $\Delta(t)$ keeps increasing with time.

It will be interesting to know with time how many independent trajectories of particles are trapped. This information is calculated and averaged over multiple system yielding the probability of traversing with time. $P_T(t)$The state of a particular trajectory will end up as one of either traversing or trapped at a particular time. Here, we only discuss the probabilities associated with a general system to be found in one of those states at a given time. Fig. 6.7(a) show the probability of traversing $P_T(t)$ vs. time-steps t, for various configurations. The probability of traversing $P_T(t)$ decay logarithmically at early time and approaches a constant value at late times. In the Fig. 6.7(d) we draw the saturated value of

P_{st} vs. r_e and find it increases with r_e and approach the value close to 0.5 as r_e approaches 0.5. The fact that the probability of traversing becomes almost constant is discussed now: Is it due to the finite size of the system we took? In small systems, the periodic boundaries can dictate things. For example; in a finite system, there can emerge many trajectories which form a deterministic path across and through the system boundaries such that they only keep moving forward in deterministic way. Such trajectories may keep moving forward indefinitely with no probability of getting trapped and contribute to the probability of traversing becoming constant in the later phase. So, to understand the effect of finite size, we simulated larger system but with same density (particle number N is increased but the system size is increased also such that area available per unit particle remains the same) in Fig. 6.7(a–c). We compared the order and the bifurcations of different values of re at increasing system sizes across Fig. 6.7(a–c). Only $r_e = 0$ is seperated in Fig. 6.7(a). The probability of traversing is lowest for $r_e = 0$, followed by $r_e = 0.1$ and $r_e = 0.2$ respectively, which also separate in Fig. 6.7(b). The $r_e = 0.4$ and 0.3 bands are still very close to $r_e = 0.5$. However, while increasing the system size to 300×300 in Fig. 6.7(c) , $r_e = 0.3$ can also be identified separately below $r_e = 0.5$ and 0.4, which are still close together.

Fig. 6.7(d) finally shows the effect of variation of system size on the probability of traversing at late time P_{st}. The spreading of the graph between the lowest ($r_e = 0$) and the top most (r_e close to 0.5) band increases with increase in system size. Since the graphs with $r_e = 0.5$, 0.4 and 0.3 rise quickly, it can be said that the effect of system size is much higher for values of r_e close to 0.5, than those close to $r_e = 0$. Note that the bigger the system sizes, the closer they are to model an infinite random lattice. The curves flatten up with bigger system sizes. This implies that for sufficiently big system, the curves would saturate and the final state will be traversing with certain finite probability. However, the proportion of traversing trajectories depend upon the ratio of rotators r_e and time. Larger the r_e, more homogeneous the system is, hence particle explore more, before it get trapped in a closed orbit. On increasing system size the probability of exploration increases, hence we find increasing P_{st} with L. But since number of bonds per unit area is almost the same, hence after a critical L, the P_st saturates to a constant value. For small r_e, particle experiences more extreme dynamics and hence explore the system less, before it gets trapped. Hence P_st is lower. Next, to note the effect of varying density of the random lattice system, we plot the $P_T(t)$ for various configurations, keeping the system size fixed and increasing N, thereby increasing the density. See Fig. 6.8(a),(b) and (c). Similar to previous plot, the $P_T(t)$ decreases logrithmically and saturate to a constant value. As the density increased, the probability of traversing increased for all bands independently as the bands can be seen rising in height across plots of increasing density (compare heights of curves of same r_e across Fig. 6.8(a),(b) and (c)). This is expected because the increase in density of the lattice would open up many more paths for traversal of a moving entity and in turn, hitting a trap would also become less probable.The bands for various r_e values are closer together for lower density system and the spreading of these bands increases as the density of the random lattice system increases. In Fig. 6.8(a), only $r_e = 0$ and 0.1 are distinct while rest are indistinguishable. While in Fig. 6.8(b) which has greater density, $r_e = 0.2$ also clearly separates. In Fig. 6.8(c), even $r_e = 0.3$ is clearly isolated. $r_e = 0.4$ and 0.5 are, however, still indistinguishable but the trend is clear by now.

When the number of particles increases, the density increases and so does the probability of traversing. Fig. 6.8(d) reveals the exponents (indicated by the slopes of respective regression lines) at which the probability changes with number of particles as the power law. For $r_e = 0$, it is a low value but it changes progressively as r_e changes from 0 to 0.5. A higher curve has higher slope i.e. it shows even greater ascent with increasing density. The increase in spreading is evident again among different values of r_e. As the value of r_e changes from 0 to 0.5, the corresponding curves also appear in same order of height in the graph (low to high). Higher curve implies high probability of traversing for the corresponding r value. This implies when both types of rotators are more evenly present (r_e close to 0.5), the probability of traversing of such system is higher and also increases sharply with increase in density, as compared to the system where only one type of rotator is dominant (r_e close to 0).

In this discussion about the variation in density of the random lattice system, the density is tuned by adjusting N. The same results can be obtained by keeping the density same by changing the system size and keeping the N fixed for larger enough L.

6.7 Dynamics in the presence of left and Right rotators (R)

A particle is set to move in a random lattice with L and R rotators distributed with ratio r_e. As compared to the previous model, the rotators are not deterministic and give a much more relaxed and probabilistic deviation of the incoming trajectory towards any of the right/left available trajectories, depending upon its nature. Fig. 6.9(a) gives an idea of the undergoing process. The slope of the log-log plot of MSD vs. t time is linear initially (simple diffusive), followed by t^2 (super diffusive). No clear trend is observed for various r_e values. Early time diffusive nature of particle dynamics is due to the randomness present in the particle motion.

To decide whether a particle is trapped or not in this model, we look into its $\Delta(t)$ plot and check if it saturates at late time. To check saturation, we consider $\Delta(t)$ values during windows of time steps at late time of the trajectory and see if the trajectory saturates or not by analysing the magnitude of change in $\Delta(t)$ values in this window. For a trapped trajectory, $\Delta(t)$ must remain more or less same while it should continuously increase for a traversing trajectory.

Fig. 6.10(a) show probability of traversing $P_T(t)$ of this model plotted verses time steps for various configurations. For all cases $P_T(t)$ shows an early time logarithmic decay to a constant value. The effect of increasing system size on these probabilities (at late time), keeping density constant, is shown in Fig. 6.9(b). The values corresponding to different r_e flatten up quickly (between $L = 150$ and $L = 200$) which indicates that the system sizes have already become sufficiently big to closely model infinite random lattice and any more increase in system sizes does not cause any appreciable change in nature of the plots of $P_T(t)$ for various r_e. This can be attributed to the rotators being less strictly deterministic in this model (R and L rotators deviate incoming trajectory to all available right/left paths as compared to Rt and Lt rotators which deviate to only rightmost/leftmost). It increases the available paths forward for a trajectory by a large extent and thereby increasing immensely the number of possibilities to traverse a lattice system, making it sufficiently big.
Fig. 6.10(b) captures the effect of increase in density ρ on P_{st}, states that the probability first decreases and then saturates for all r_e. The final state is traversing with its associated probability. The P_{st} increases on increasing r_e, because for the larger r_e, again the lattice is more homogeneous in nature and hence lead to the more probability of traversing trajectory.

6.8 Conclusion

We design a random lattice with crossing bonds with random bond length and bond angle. Our approach and designed lattice is different from the previously introduced algorithm to design random lattices: like Voronoi Tessellation Green & Sibson (1978). We generate a random lattice, which is more general and is more closer to many random lattices found in natural and social networks. We characterise the properties of the random lattice and find that the bond length distribution is Poissonian and mean bond length depends on the density and size of the lattice. We further studied the dynamics of a single particle moving on such lattice and find its dynamics is a random walk with mean square displacement of particle motion varies linearly with time. Hence we conclude the random nature of the underlying lattice. We further decorated the lattice with two types of rotators present and introduced two models: Model 2, where the rotators rotate the incoming particle to its extreme right and left available Rt/Lt paths and Model 3, where it gets deflected in any possible right or left R/L available path with equal probability. For the case of Model 2, for a given configuration model is deterministic but randomness is introduced through different given configuration of Rt/Lt rotators. Model is studied for various ratios r_e of Rt/Lt rotators. For each r_e, initial dynamics of particle is subdiffusive with a unique exponent $2/3$ and late time dynamics is traversing t^2. Hence starting from a given distribution of rotators, particle have a finite probability of traversing.We characterise the probability of traversing $P_T(t)$, which shows an initial logarithmic decay to a finite value. Steady state P_{st} increases as r_e approaches 0.5. For Model 3, when R/L rotators rotate the direction of incoming particle randomly to one of the right or left available paths. Hence now model has less deterministic nature as the case (i). Due to this initial dynamics of particle is diffusive and late time t shows the ballistic motion. Also in the steady state the probability of traversing $P_{st}(t)$ approaches a constant value and $P_{st}(t)$ increases as r_e approached 0.5. Our current study focus on the dynamics of a single particle. Model will be interesting if it is studied for many particles interacting system. The interaction can be introduced of two types. One where particle-particle direct interaction which can be exclusion type (no two particles can sit on top of each other) and second particle motion can affect the underlying characteristics of lattice points and hence when another particle appears on the lattice point it experiences an indirect interaction due to previously passed particle.

Our study revealed a great deal of interesting and extensive information about the dynamics of a particle on random lattices. This is a very general study and can be fine tuned to any particular field where binary traversal of information happens in a network.

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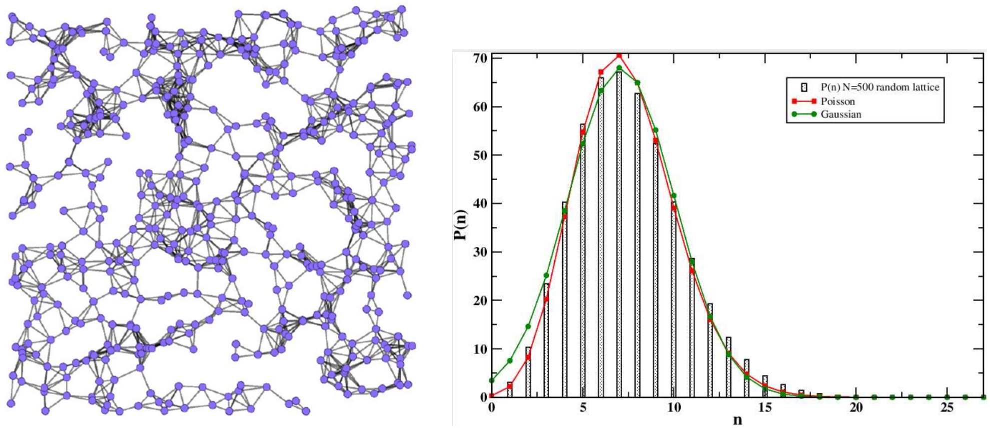
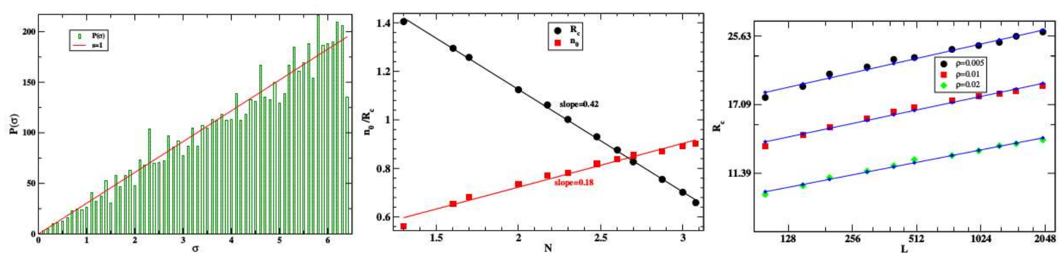


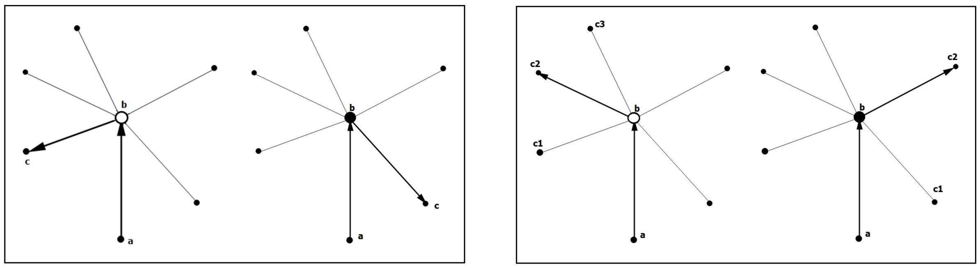
(a) (b)

Fig. 6.1 (a) A connected periodic random lattice resulted by distributing $N = 500$ points in 100×100 lattice. Lines shows the connected bonds due to periodic boundary conditions are not shown here. (b) The degree distribution $P(n)$ for N=500 random lattice averaged over multiple system.



(a) (b) (c)

Fig. 6.2 (a) Bond Length distribution $P(\sigma)$ for $N = 1000$ in 100×100 space. (b) Log-Log plot of the variation of average R' and peak of the degree distribution n_0 with increase in number of lattice points N distributed in 100×100 lattice. (c) Variation of R_c with system size L on log log scale for different values of density, where blue lines are the fitting lines for three different densities and have slopes 0.12, 0.11 and 0.10 for density ρ = 0.005, 0.01 and 0.02 respectively.



(a) (b)

Fig. 6.3 (a) A *Lt* rotator (shown with an open circle) rotates the incoming particle to the leftmost available path forward while a *Rt* rotator (shown with a closed circle) rotates it to the rightmost available path, relative to the initial direction. For example, an incoming particle from a strikes an *Lt* and an *Rt* rotator placed at point b. They deflect it to the leftmost and rightmost paths respectively, towards c. (b)A L rotator (shown with an open circle) rotates the incoming particle, with equal probability, to any of the left paths available while an R rotator (shown with a closed circle) rotates it to any of the right paths available forward, relative to its initial direction. For example, an incoming particle from a strikes an L and an R rotator placed at point b.The L rotator has equal probability of deflecting it towards $c1$ or $c2$, but chooses $c2$ in this particular trial. Similarly the R rotator has equal probability of deflecting it towards $c1$ or $c2$ or $c3$, but chooses $c2$ in this particular trial. If there is no left/right path available, then they turn the trajectory towards the leftmost/rightmost path.

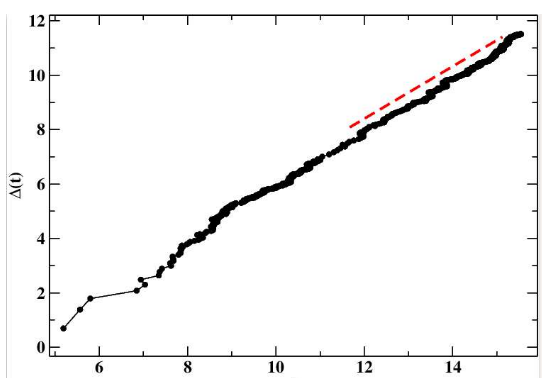
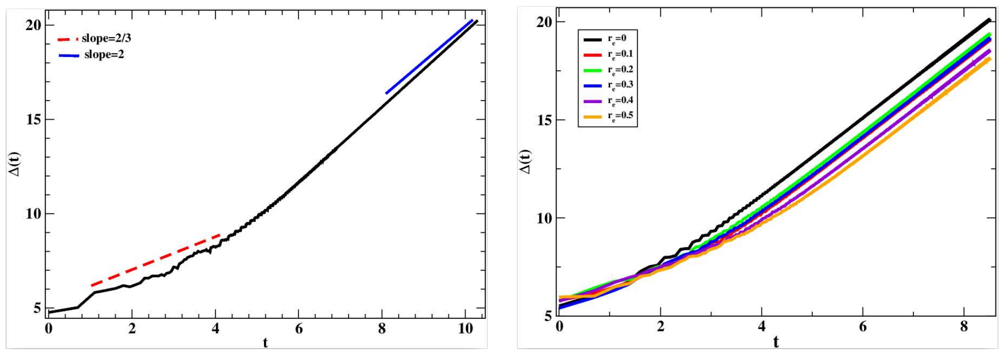
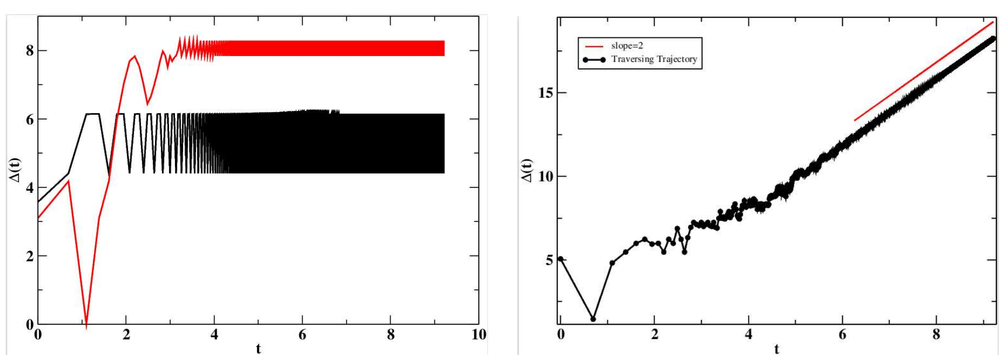


Fig. 6.4 Log-Log plot of the $\Delta(t)$ vs. time steps for $N = 100$ in 100×100 space. Dotted line shows the line with slope 1.



(a) (b)

Fig. 6.5 (a) Mean square displacement $\Delta(t)$ is plotted against number of steps (or time) in log-log scale for $N = 100$ in 100×100 space, with Lt and Rt rotators evenly and equally distributed. (b) Variations with the ratio of Lt and Rt rotators in the log-log plot of Mean Square Displacement verses time steps for $N = 400$ in 200×200 system. $\Delta(t)$ is plotted for various r_e values from 0 to 0.5.



(a) (b)

Fig. 6.6 (a)MSD vs time in log-log of a trapped trajectory system. (b)MSD vs time in log-log of a travelling trajectory.

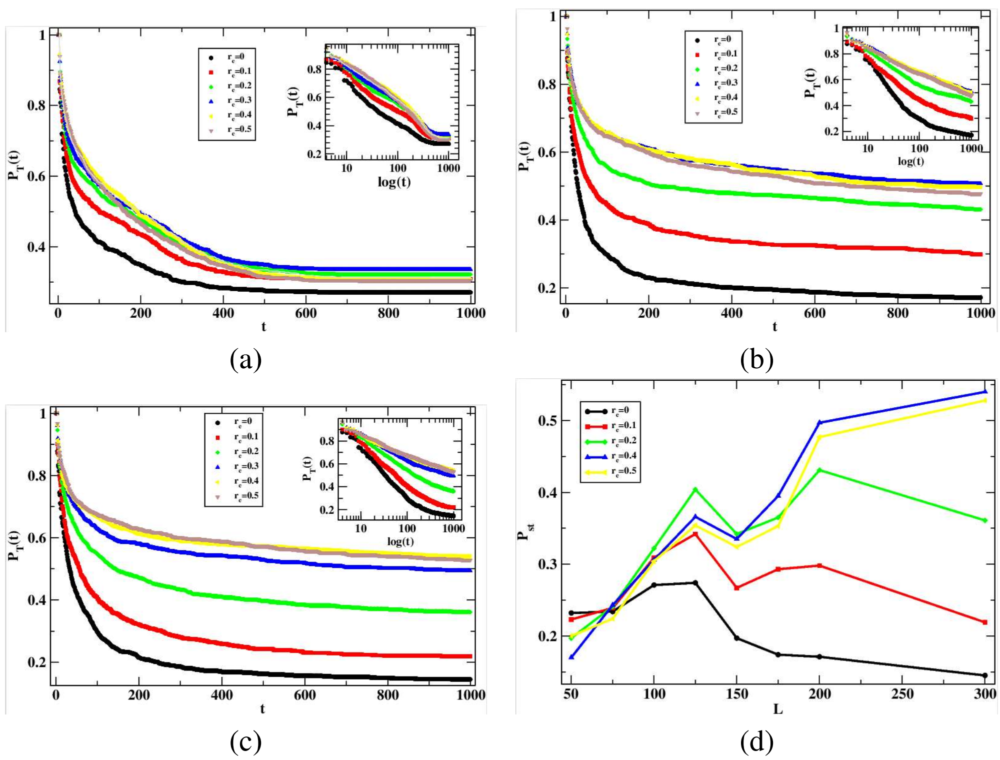


Fig. 6.7 Probability of traversing $P_T(t)$ vs. time-steps t for (a) $N = 100$ in 100×100 space (b) $N = 400$ in 200×200 space (c) $N = 900$ in 300×300 space (d) Variation of steady state probability of traversing P_{st} with respect to change in system size L, with density ρ kept constant.

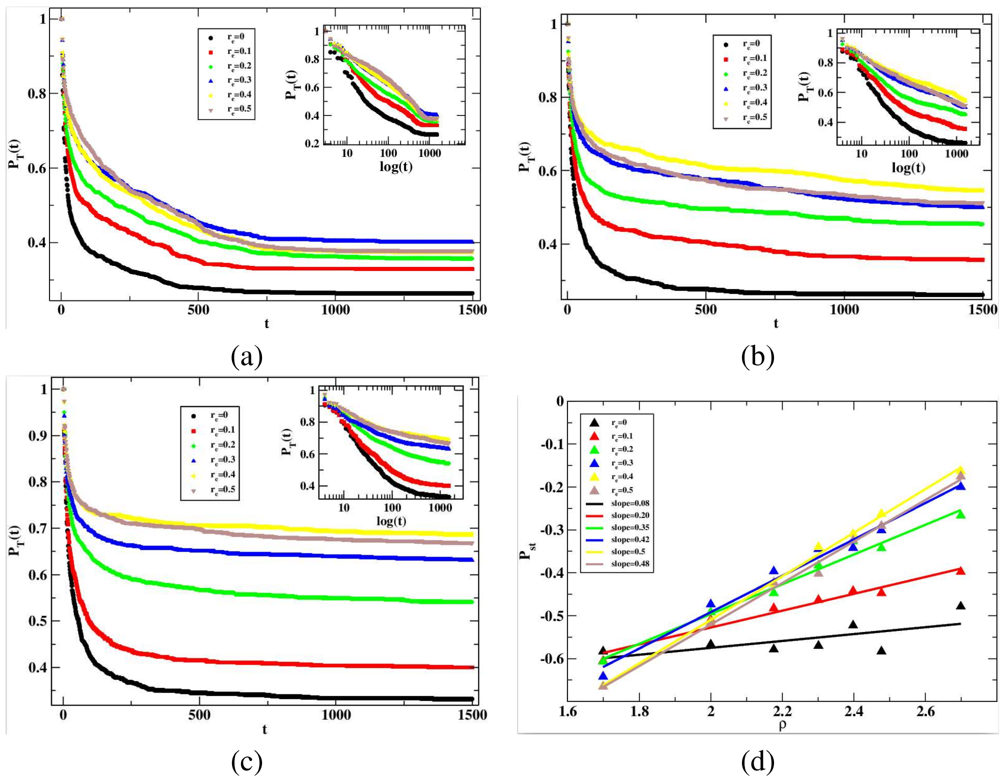


Fig. 6.8 Probability of traversing $P_T(t)$ verses time-steps t for (a) $N = 150$ in 100×100 space (b) $N = 300$ in 100×100 space (c) $N = 500$ in 100×100 space (d) Variation in log-log scale of the P_{st} with respect to $\bar{n}$.

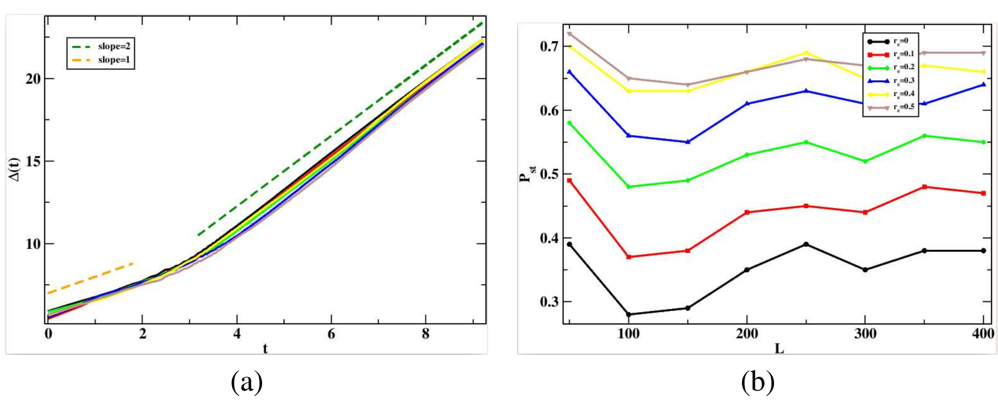


Fig. 6.9 (a)MSD averaged over multiple system is plotted against number of steps (or time) in log-log scale for $N = 400$ in 200×200 space, with L and R rotators distributed by ratio $r = 0$ to 0.5 (b) Variation in Probability of traversing $P_{st}(t)$ (at late time) with respect to change in system size L, with density kept constant.

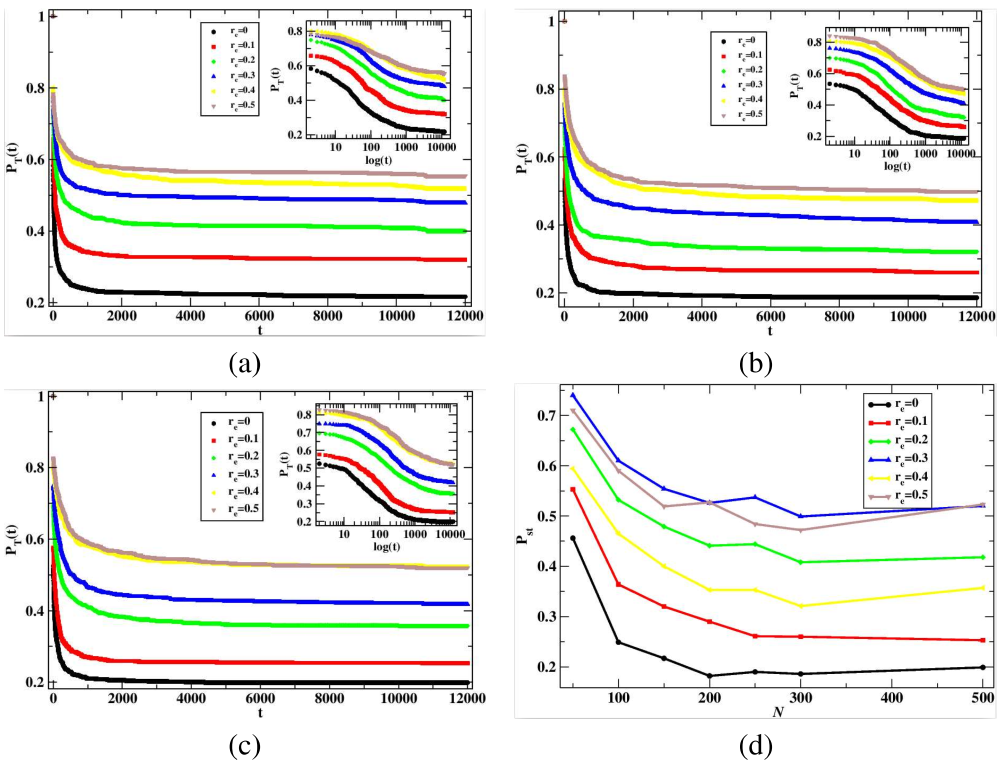


Fig. 6.10 Probability of traversing $P_T(t)$ vs. time-steps t for (a) $N = 150$ in 100×100 space (b) $N = 300$ in 100×100 space (c) $N = 500$ in 100×100 space (d) Variation of the $P_{st}(t)$ (probabilities at steady state/ late time) with respect to change in number of particles $\mathscr{N}$ distributed in constant space.