Chapter 5

A Legendre Collocation Method for New Generalized Fractional Diffusion Equation

5.1 Introduction

In this chapter, we present the numerical solution of GFADE governed by

$$(\mathcal{D}^{\gamma}_{0+;[z;\omega;2]})u(x,w,\tau) + \mathcal{P}u(x,w,\tau) = \nabla^2 u(x,w,\tau) + \rho(x,w,\tau),\ (x,w) \in \Omega,\quad \tau \in \mathcal{I}_3, \tag{5.1}$$

with initial and boundary conditions given as,

$$\begin{cases} u(x,w,0) = f(x,w,0), & (x,w) \in \Omega, \\ u(x,1,\tau) = g_2(x,\tau),\ u(x,0,\tau) = g_1(x,\tau), & (x,\tau) \in \mathcal{I}_1 \times \mathcal{I}_3, \\ u(1,w,\tau) = h_2(w,\tau)\ u(0,w,\tau) = h_1(w,\tau), & (w,\tau) \in \mathcal{I}_2 \times \mathcal{I}_3, \end{cases} \tag{5.2}$$

where $\Omega = \mathcal{I}_1 \times \mathcal{I}_2,\ \mathcal{I}_1 = [x_r, x_s],\ \mathcal{I}_2 = [w_d, w_u],\ \mathcal{I}_3 = [\tau_k, \tau_l]$,
$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial w^2}$,

$$\mathcal{P}u(x, w, \tau) = \nu_1(x, w, \tau)\frac{\partial u(x, w, \tau)}{\partial x} + \nu_2(x, w, \tau)\frac{\partial u(x, w, \tau)}{\partial w},$$

and $(\mathcal{D}^{\gamma}_{0+;[z;\omega;2]}u)$ denotes the GFD of function u of order γ as

$$\left(\mathcal{D}^{\gamma}_{0+;[z;\omega;2]}u\right)(x, w, \tau) = \frac{[\omega(\tau)]^{-1}}{\Gamma(1-\gamma)} \int_0^{\tau} \frac{\frac{\partial}{\partial s}[\omega(s)u(x, w, s)]}{[z(\tau) - z(s)]^{\gamma}} ds, \quad 0 < \gamma < 1. \tag{5.3}$$

The main theme of this chapter is to develop an accurate and efficient method to solve GFADE. The present method is based on the collocation method where Legendre polynomials are implemented as bases. In Section 5.2, some mathematical preliminaries on fractional derivative, integration, and Legendre polynomials are provided. In Section 5.3, we discuss the approximation by Legendre polynomials expansion for functions with multiple variables. We study the collocation method for a given problem and provide a brief explanation to solve one and two-dimensional GFADE in Section 5.4. The convergence analysis and error estimate of the proposed method is studied in Section 5.5. In Section 5.6, we presented three numerical examples, these are based on the one-dimensional and two-dimensional GFADE. For example, some of them have known solutions and others of them do not have. Finally, conclusion and discussion are drawn in Section 5.7.

5.2 Preliminaries

Definition 5.2.1. Legendre polynomial function is defined as,

$$\mathcal{L}_i(\tau) = \sum_{i_1=0}^{i} (-1)^{i-i_1} \frac{\Gamma(i+i_1+1)}{\Gamma(i_1+1)\Gamma(i-i_1+1)(i_1)!}\tau^{i_1}, \ \tau \in [0,1], \tag{5.4}$$

and satisfy the following recurrence relation in the interval $[0,1]$,

$$\mathcal{L}_0(\tau) = 1,$$
$$\mathcal{L}_1(\tau) = 2\tau - 1,$$

and

$$\mathcal{L}_{i+1}(\tau) = \frac{(1+2i)(2\tau-1)}{1+i}\mathcal{L}_i(\tau) - \frac{i}{1+i}\mathcal{L}_{i-1}(\tau), i = 1,2,3,\cdots. \tag{5.5}$$

The orthogonality relation with respect to weight function $w(\tau)$ is,

$$\int_0^1 \mathcal{L}_{i_1}(\tau)\mathcal{L}_{i_2}(\tau)w_1(\tau)d\tau = \tilde{h}_{i_2}\delta_{i_1 i_2}, \tag{5.6}$$

where $\tilde{h}_{i_2} = \frac{1}{2i_2+1}$, $w_1(\tau) = 1$ and $\delta_{i_1 i_2}$ is Kronecker delta function.

The Legendre polynomial defined above is for one-dimensional function. However we can extend it for higher dimension with weight function $w_1(\tau) = 1$. The orthogonality relation of Legendre polynomial of higher dimension is defined similarly as

$$\int_0^1\int_0^1\int_0^1 \mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(x)\ \mathcal{L}_{j_1}(y)\mathcal{L}_{j_2}(y)\ \mathcal{L}_{k_1}(\tau)\mathcal{L}_{k_2}(\tau)\,dx\,dy\,d\tau = \begin{cases} \frac{1}{(2i_1+1)(2j_1+1)(2k_1+1)} & \text{if } i_1 = i_2, j_1 = j_2, k_1 = k_2; \\ 0 & \text{otherwise.} \end{cases} \tag{5.7}$$

5.3 Function Approximation Using Legendre Polynomials

This section divided into two subsections: In Subsection 5.3.1 we study the one-dimensional approximation of unknown function in the domain $\mathcal{I}_1 \times \mathcal{I}_3$. While in Subsection 5.3.2 we presented the two-dimensional approximation of unknown function in the domain $\Omega \times \mathcal{I}_3$. Without loss of generality, we consider $\Omega = [0,1] \times [0,1]$ and $\mathcal{I}_3 = [0,1]$ for upcoming sections.

5.3.1 Approximation of a Two Variable Function Using Legendre Polynomials

As discussed in [135], for a function $f(x,\tau) \in L^2(\mathcal{I}_1 \times \mathcal{I}_3)$ can be written as,

$$f(x,\tau) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} a_{i_1,i_2} \mathcal{L}_{i_1}(x) \mathcal{L}_{i_2}(\tau), \tag{5.8}$$

where coefficients a_{i_1,i_2} is determined by the formula,

$$a_{i_1,i_2} = \frac{1}{\|\mathcal{L}_{i_1}(x)\|^2 \|\mathcal{L}_{i_2}(\tau)\|^2} \int_0^1 f(x,\tau) \mathcal{L}_{i_1}(x) \mathcal{L}_{i_2}(\tau)\, dx\, d\tau, \quad i_1 = 0,1,2,3,\cdots, \quad i_2 = 0,1,2,3\cdots. \tag{5.9}$$

Again, for the seek of the good approximation of the function $f \in L^2(\mathcal{I}_1 \times \mathcal{I}_3)$, such that

$$f(x,\tau) \approx f_{\mathcal{N},\mathcal{K}}(x,\tau) = \sum_{i_1=0}^{\mathcal{N}} \sum_{i_2=0}^{\mathcal{K}} a_{i_1,i_2} \mathcal{L}_{i_1}(x) \mathcal{L}_{i_2}(\tau), \tag{5.10}$$

where $a_{i_1,i_2} = [a_{00}, a_{01}, a_{02}, \ldots a_{\mathcal{N}\mathcal{K}}]^T$, $\mathcal{L}_{i_1}(x)$ and $\mathcal{L}_{i_2}(\tau)$ are shifted Legendre polynomials with $\mathcal{L}_{i_1}(x) = [\mathcal{L}_0(x), \mathcal{L}_1(x), \mathcal{L}_2(x), \cdots, \mathcal{L}_{\mathcal{N}}(x)]$, $\mathcal{L}_{i_2}(\tau) = [\mathcal{L}_0(\tau), \mathcal{L}_1(\tau), \mathcal{L}_2(\tau), \cdots, \mathcal{L}_{\mathcal{K}}(\tau)]$.

5.3.2 Approximation of a Three Variable Function Using Legendre Polynomials

Let $f(x,w,\tau)$ be function defined on $0 \leq x < 1,\ 0 \leq w < 1$ and $0 \leq \tau \leq 1$ can be expressed in the linear combination of Legendre polynomial as,

$$f(x,w,\tau) = \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty} a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau), \tag{5.11}$$

and can be approximated by the triple shifted Legendre polynomials as,

$$f(x,w,\tau) \approx f_{\mathcal{N},\mathcal{M},\mathcal{K}}(x,w,\tau) = \sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}} a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau), \tag{5.12}$$

where coefficients a_{i_1,i_2,i_3} are determined by the relation,

$$a_{i_1,i_2,i_3} = \frac{1}{\|\mathcal{L}_{i_1}(x)\|^2\|\mathcal{L}_{i_2}(w)\|^2\|\mathcal{L}_{i_3}(\tau)\|^2}\int_0^1\int_0^1\int_0^1 f(x,w,\tau)\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\,dx\,dw\,d\tau. \tag{5.13}$$

$i_1 = 0,1,2,3,...\mathcal{N},\ i_2 = 0,1,2,3,...\mathcal{M},\ i_3 = 0,1,2,3,...\mathcal{K}$,
where $a_{i_1,i_2,i_3} = [a_{000}, a_{001}, ...a_{00\mathcal{K}}, a_{010}, a_{011}, ...a_{0\mathcal{M}\mathcal{K}}, ...a_{\mathcal{N}\mathcal{M}\mathcal{K}}]$,

$\mathcal{L}_{i_1}(x) = [\mathcal{L}_0(x), \mathcal{L}_1(x), ...\mathcal{L}_{\mathcal{N}}(x)]^\top$, $\mathcal{L}_{i_2}(w) = [\mathcal{L}_0(w), \mathcal{L}_1(w), \mathcal{L}_3(w), ...\mathcal{L}_{\mathcal{M}}(w)]^\top$, $\mathcal{L}_{i_3}(\tau) = [\mathcal{L}_0(\tau), \mathcal{L}_1(\tau), \mathcal{L}_2(\tau), ...\mathcal{L}_{\mathcal{K}}(\tau)]^\top$, and a_{i_1,i_2,i_3} are unknown coefficients.

5.4 Numerical Algorithm

In this part, we provide the complete collocation method for one-dimensional as well as two-dimensional GFADE. The details are presented in two parts below.

5.4.1 Collocation Method for One-Dimensional GFADE

Consider one-dimensional GFADE,

$$(\mathcal{D}^{\gamma}_{0+;[z;w;2]})u(x,\tau) + \nu_1(x,\tau)\frac{\partial u(x,\tau)}{\partial x} = \frac{\partial^2 u(x,\tau)}{\partial x^2} + s(x,\tau), \tag{5.14}$$

with initial and boundary condition,

$$\begin{aligned} u(x,0) &= f_1(x), \\ u(0,\tau) = g_1(\tau),\ u(1,\tau) &= g_2(\tau), \end{aligned} \tag{5.15}$$

where $\gamma \in (0,1)$ is real number. $s(x,\tau)$ is source term and $\nu_1(x,\tau)$ is the average fluid velocity. We assume the solution of Eq. (5.14) of the form,

$$u_{\mathcal{N},\mathcal{K}}(x,\tau) = \sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{K}} a_{i_1,i_2}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(\tau). \tag{5.16}$$

From Eq. (5.14), we have,

$$\begin{aligned} (\mathcal{D}^{\gamma}_{0+;[z;w;2]})u_{\mathcal{N},\mathcal{K}}(x,\tau) + \nu_1(x,\tau)\frac{\partial u_{\mathcal{N},\mathcal{K}}(x,\tau)}{\partial x} &= \frac{\partial^2 u_{\mathcal{N},\mathcal{K}}(x,\tau)}{\partial x^2} + s(x,\tau), \\ &(x,\tau) \in \mathcal{I}_1 \times \mathcal{I}_3, \end{aligned} \tag{5.17}$$

and initial and boundary conditions of Eq. (5.15) given as,

$$\begin{cases} u_{\mathcal{N},\mathcal{K}}(x,0) = \sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{K}} a_{i_1,i_2}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(0) = f_1(x),\ x \in \mathcal{I}_1, \\ u_{\mathcal{N},\mathcal{K}}(0,\tau) = \sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{K}} a_{i_1,i_2}\mathcal{L}_{i_1}(0)\mathcal{L}_{i_2}(\tau) = g_1(\tau),\ \tau \in \mathcal{I}_3, \\ u_{\mathcal{N},\mathcal{K}}(1,\tau) = \sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{K}} a_{i_1,i_2}\mathcal{L}_{i_1}(1)\mathcal{L}_{i_2}(\tau) = g_2(\tau),\ \tau \in \mathcal{I}_3. \end{cases} \tag{5.18}$$

Now, Eqs.(5.17) and (5.18) are converted into a linear system of $(\mathcal{N}+1)$ equations in $(\mathcal{K}+1)$ unknowns. For solving this system by collocation method, we consider collocation points as root of shifted Legendre polynomial of $\mathcal{L}_{i_1}(x)$ $(0 \leq i_1 \leq \mathcal{N})$ and $\mathcal{L}_{i_2}(\tau)$ $(0 \leq i_2 \leq \mathcal{K}-1)$ respectively. So we have the points of the form (x_{i_1}, τ_{i_2}) are the collocation points.

$$(\mathcal{D}^{\gamma}_{0+;[z;w;2]})u_{\mathcal{N},\mathcal{K}}(x_{i_1}, \tau_{i_2}) + \nu_1(x_{i_1}, \tau_{i_2})\frac{\partial u_{\mathcal{N},\mathcal{K}}(x_{i_1}, \tau_{i_2})}{\partial x} = \frac{\partial^2 u_{\mathcal{N},\mathcal{K}}(x_{i_1}, \tau_{i_2})}{\partial x^2} + s(x_{i_1}, \tau_{i_2}), \tag{5.19}$$

and initial and boundary conditions of Eq. (5.15) given as,

$$\begin{aligned} u_{\mathcal{N},\mathcal{K}}(x_{i_1}, 0) &= \sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{K}} a_{i_1,i_2}\mathcal{L}_{i_1}(x_{i_1})\mathcal{L}_{i_2}(0) = f_1(x_{i_1}), \\ u_{\mathcal{N},\mathcal{K}}(0, \tau_{i_2}) &= \sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{K}} a_{i_1,i_2}\mathcal{L}_{i_1}(0)\mathcal{L}_{i_2}(\tau_{i_2}) = g_1(\tau_{i_2}), \\ u_{\mathcal{N},\mathcal{K}}(1, \tau_{i_2}) &= \sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{K}} a_{i_1,i_2}\mathcal{L}_{i_1}(1)\mathcal{L}_{i_2}(\tau_{i_2}) = g_2(\tau_{i_2}). \end{aligned} \tag{5.20}$$

In this way, we have a system of $(\mathcal{N}+1)\times(\mathcal{K}+1)$ order linear algebraic equations in unknown coefficients a_{i_1,i_2} from Eqs.(5.19) and (5.20). We can solve these equations by any standard method such as Gaussian elimination method and conjugate gradient method.

5.4.2 Collocation Method for Two-Dimensional GFADE

Similarly for two-dimensional problems, we define the approximate solution of Eq.(5.1) as

$$u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x, w, \tau) = \sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}} a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau). \tag{5.21}$$

From Eqs. (5.1) and (5.2), we have

$$
\begin{aligned}
&(\mathcal{D}^{\gamma}_{0+;[z;\omega;2]})\left(\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}} a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\right)\\
&+\nu_1(x,y,\tau)\frac{\partial}{\partial x}\left(\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}} a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\right)\\
&+\nu_2(x,w,\tau)\frac{\partial}{\partial w}\left(\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}} a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\right)\\
=\;&\frac{\partial^2}{\partial x^2}\left(\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}} a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\right)\\
&+\frac{\partial^2}{\partial w^2}\left(\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}} a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\right)+\rho(x,w,\tau),
\end{aligned}
\tag{5.22}
$$

with initial and boundary conditions are converted to

$$u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x,w,0) = \sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}} a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(0) = f_1(x,w), \tag{5.23}$$

$$u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x,0,\tau) = \sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}} a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(0)\mathcal{L}_{i_3}(\tau) = g_1(x,\tau), \tag{5.24}$$

$$u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x,1,\tau) = \sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}} a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(1)\mathcal{L}_{i_3}(\tau) = g_2(x,\tau), \tag{5.25}$$

$$u_{\mathcal{N},\mathcal{M},\mathcal{K}}(0,w,\tau) = \sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}} a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(0)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau) = h_1(w,\tau), \tag{5.26}$$

$$u_{\mathcal{N},\mathcal{M},\mathcal{K}}(1,w,\tau) = \sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}} a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(1)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau) = h_2(w,\tau). \tag{5.27}$$

After simplify rearranging the Eq. (5.22), we get,

$$
\begin{aligned}
\mathcal{F}(i_1,i_2,i_3,x,w,\tau)=\rho(x,w,\tau),\ i_1=0,1,2,...\mathcal{N},\ i_2=0,1,2,...\mathcal{M},\\
i_3=0,1,2,...\mathcal{K},
\end{aligned}
\tag{5.28}
$$

where,

$$\mathcal{F}(i_1,i_2,i_3,x,w,\tau)=(\mathcal{D}^{\gamma}_{0+;[z;w;2]})\left(\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}}a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\right)+$$
$$\nu_1(x,w,\tau)\frac{\partial}{\partial x}\left(\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}}a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\right)$$
$$+\nu_2(x,w,\tau)\frac{\partial}{\partial w}\left(\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}}a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\right) \quad (5.29)$$
$$-\frac{\partial^2}{\partial x^2}\left(\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}}a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\right)$$
$$-\frac{\partial^2}{\partial w^2}\left(\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}}a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\right),$$

Collocating Eq. (5.28) at collocation points $(x_{i_1}, w_{i_2}, \tau_{i_3})$ as the roots of $\mathcal{N}+1$, $\mathcal{M}+1$ and $\mathcal{K}+1$ shifted Legendre polynomials respectively. We get the following sets of equations,

$$\mathcal{F}(i_1,i_2,i_3,x_{i_1},w_{i_2},\tau_{i_3})=\rho(x_{i_1},w_{i_2},\tau_{i_3}),\ 1\le i_1\le\mathcal{N}-1,\ 1\le i_2\le\mathcal{M}-1,\quad 1\le i_3\le\mathcal{K}, \quad (5.30)$$

initial and boundary conditions are given as,

$$u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x_{i_1},w_{i_2},0) = f_1(x_{i_1},w_{i_2}),\ 0\le i_1\le\mathcal{N},\ 0\le i_2\le\mathcal{M}, \quad (5.31)$$
$$u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x_{i_1},0,\tau_{i_3}) = g_1(x_{i_1},\tau_{i_3}),\ 1\le i_1\le\mathcal{N}-1,\ 1\le i_3\le\mathcal{K}, \quad (5.32)$$
$$u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x_{i_1},1,\tau_{i_3}) = g_2(x_{i_1},\tau_{i_3}),\ 1\le i_1\le\mathcal{N}-1,\ 1\le i_3\le\mathcal{K}, \quad (5.33)$$
$$u_{\mathcal{N},\mathcal{M},\mathcal{K}}(0,w_{i_2},\tau_{i_3}) = h_1(w_{i_2},\tau_{i_3}),\ 0\le i_2\le\mathcal{M},\ 1\le i_3\le\mathcal{K}, \quad (5.34)$$
$$u_{\mathcal{N},\mathcal{M},\mathcal{K}}(1,w_{i_2},\tau_{i_3}) = h_2(w_{i_2},\tau_{i_3}),\ 0\le i_2\le\mathcal{M},\ 1\le i_3\le\mathcal{K}. \quad (5.35)$$

Eqs.(5.30)-(5.35) forms a system of $(\mathcal{N}+1)\times(\mathcal{M}+1)\times(\mathcal{K}+1)$ linear algebraic equations in unknown coefficients a_{i_1,i_2,i_3}. By any standard method (e.g., Gaussian elimination and conjugate gradient methods) we can solve these equations for coefficients a_{i_1,i_2,i_3}, we get the approximate solution by Eq.(5.21).

5.5 Convergence Analysis

In this section, we study the convergence analysis of the proposed collocation method.

Theorem 5.5.1. Let $u(x, w, \tau)$ be the sufficiently smooth exact solution of the Eq. (5.1) in $L^2[\Omega \times \mathcal{I}_3]$ and $u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x, w, \tau)$ denotes the approximate solution of Eq. (5.1) in the form of the shifted Legendre expansion of $u(x, w, \tau)$ as,

$$u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x, w, \tau) = \sum_{i_1=0}^{\mathcal{N}} \sum_{i_2=0}^{\mathcal{M}} \sum_{i_3=0}^{\mathcal{K}} a_{i_1,i_2,i_3} \mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau), \tag{5.36}$$

Then unknown coefficients satisfies the following condition

$$|a_{i_1,i_2,i_3}| \leq \zeta \frac{(2i_1+1)}{\pi^3(i_1+i_1^2)} \frac{(2i_2+1)}{(i_2+i_2^2)} \frac{(2i_3+1)}{(i_3+i_3^2)}, \quad i_1, \ i_2, \ i_3 > 0, \tag{5.37}$$

where ζ is determine by,

$$\sup|u(x, w, \tau)| \leq \zeta.$$

Proof. For any function $u(x, w, \tau)$ defined on $L^2[\Omega \times \mathcal{I}_3]$ can be approximated by triple shifted Legendre polynomials given as,

$$u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x, w, \tau) = \sum_{i_1=0}^{\mathcal{N}} \sum_{i_2=0}^{\mathcal{M}} \sum_{i_3=0}^{\mathcal{K}} a_{i_1,i_2,i_3} \mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau), \tag{5.38}$$

from Eq. (5.13), we have

$$
\begin{aligned}
a_{i_1,i_2,i_3} &= \frac{1}{\|\mathcal{L}_{i_1}(x)\|^2\|\mathcal{L}_{i_2}(w)\|^2\|\mathcal{L}_{i_3}(\tau)\|^2}\int_0^1\int_0^1\int_0^1 u(x,w,\tau)\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)dx\,dw\,d\tau,\\
&\leq \frac{\sup|(u(x,w,\tau))|}{\|\mathcal{L}_{i_1}(x)\|^2\|\mathcal{L}_{i_2}(w)\|^2\|\mathcal{L}_{i_3}(\tau)\|^2}\int_0^1\int_0^1\int_0^1 \mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)dx\,dw\,d\tau,\\
&\leq \frac{\zeta}{\|\mathcal{L}_{i_1}(x)\|^2\|\mathcal{L}_{i_2}(w)\|^2\|\mathcal{L}_{i_3}(\tau)\|^2}\int_0^1\int_0^1\int_0^1 \mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)dx\,dw\,d\tau,\\
&\leq \frac{1}{\|\mathcal{L}_{i_1}(x)\|^2\|\mathcal{L}_{i_2}(w)\|^2\|\mathcal{L}_{i_3}(\tau)\|^2}\zeta\int_0^1 \mathcal{L}_{i_1}(x)dx\int_0^1 \mathcal{L}_{i_2}(w)dw\int_0^1 \mathcal{L}_{i_3}(\tau)d\tau,
\end{aligned}
\tag{5.39}
$$

since,

$$
\int_0^1 \mathcal{L}_{i_1}(x)dx = \frac{\sin(i_1\pi)}{(i_1+i_1^2)\pi} \leq \frac{1}{(i_1+i_1^2)\pi}.
$$

Similarly we can find for $\int_0^1 \mathcal{L}_{i_2}(w)dw$ and $\int_0^1 \mathcal{L}_{i_3}(\tau)d\tau$. Then from Eq. (5.39) we get,

$$
|a_{i_1,i_2,i_3}| \leq (2i_1+1)(2i_2+1)(2i_3+1)\zeta\frac{1}{(i_1+i_1^2)\pi}\frac{1}{(i_2+i_2^2)\pi}\frac{1}{(i_3+i_3^2)\pi}, \tag{5.40}
$$

after simplifying Eq. (5.40),

$$
|a_{i_1,i_2,i_3}| \leq \zeta\frac{(2i_1+1)}{\pi^3(i_1+i_1^2)}\frac{(2i_2+1)}{(i_2+i_2^2)}\frac{(2i_3+1)}{(i_3+i_3^2)}, \tag{5.41}
$$

□

Theorem 5.5.2. Let $u(x,w,\tau)$ be sufficiently smooth exact solution of Eq. (5.1) and $u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x,w,\tau)$ denote the numerical approximation of $u(x,w,\tau)$, and $\sup|u(x,w,\tau)| \leq \zeta$ then $u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x,w,\tau)$ converges to $u(x,w,\tau)$ uniformly.

Moreover,

$$
|u(x,w,\tau) - u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x,w,\tau)| \;\rightarrow\; 0 \; as \; \mathcal{N},\mathcal{M},\mathcal{K} \rightarrow \infty.
$$

Proof. From Eqs. (5.21) and (5.11), we get

$$
\begin{aligned}
&u(x,w,\tau)-u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x,w,\tau)\\
&=\sum_{i_1=\mathcal{N}+1}^{\infty}\sum_{i_2=\mathcal{M}+1}^{\infty}\sum_{i_3=\mathcal{K}+1}^{\infty}a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\\
&+\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=\mathcal{M}+1}^{\infty}\sum_{i_3=\mathcal{K}+1}^{\infty}a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\\
&+\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=\mathcal{K}+1}^{\infty}a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\\
&+\sum_{i_1=\mathcal{N}+1}^{\infty}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=\mathcal{K}+1}^{\infty}a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\\
&+\sum_{i_1=\mathcal{N}+1}^{\infty}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}}a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\\
&+\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=\mathcal{M}+1}^{\infty}\sum_{i_3=\mathcal{K}+1}^{\infty}a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\\
&+\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=\mathcal{M}+1}^{\infty}\sum_{i_3=0}^{\mathcal{K}}a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau),
\end{aligned}
\tag{5.42}
$$

which implies,

$$
\begin{aligned}
&\|u(x,w,\tau)-u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x,w,\tau)\|^2 && (5.43)\\
=&||\int_0^1\int_0^1\int_0^1\Bigg[\sum_{i_1=\mathcal{N}+1}^{\infty}\sum_{i_2=\mathcal{M}+1}^{\infty}\sum_{i_3=\mathcal{K}+1}^{\infty}a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau) && (5.44)\\
&+\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=\mathcal{M}+1}^{\infty}\sum_{i_3=\mathcal{K}+1}^{\infty}a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\\
&+\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=\mathcal{K}+1}^{\infty}a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\\
&+\sum_{i_1=\mathcal{N}+1}^{\infty}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=\mathcal{K}+1}^{\infty}a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\\
&+\sum_{i_1=\mathcal{N}+1}^{\infty}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}}a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\\
&+\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=\mathcal{M}+1}^{\infty}\sum_{i_3=\mathcal{K}+1}^{\infty}a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\\
&+\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=\mathcal{M}+1}^{\infty}\sum_{i_3=0}^{\mathcal{K}}a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\Bigg]||^2. && (5.45)
\end{aligned}
$$

$$
\begin{aligned}
&\|u(x,w,\tau)-u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x,w,\tau)\|^2\\
\le&\left\|\int_0^1\int_0^1\int_0^1\sum_{i_1=\mathcal{N}+1}^{\infty}\sum_{i_2=\mathcal{M}+1}^{\infty}\sum_{i_3=\mathcal{K}+1}^{\infty}a_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\right\|^2,\\
=&\sum_{i_1=\mathcal{N}+1}^{\infty}\sum_{i_2=\mathcal{M}+1}^{\infty}\sum_{i_3=\mathcal{K}+1}^{\infty}a^2_{i_1,i_2,i_3}(\|\mathcal{L}_{i_1}(x)\|)^2(\|\mathcal{L}_{i_2}(w)\|)^2(\|\mathcal{L}_{i_3}(\tau)\|)^2. && (5.46)
\end{aligned}
$$

From Eq. (5.41) we have

$$
|a_{i_1,i_2,i_3}|\le\frac{\zeta(2i_1+1)}{\pi^3(i_1+i_1^2)}\frac{(2i_2+1)}{(i_2+i_2^2)}\frac{(2i_3+1)}{(i_3+i_3^2)}. \tag{5.47}
$$

Substituting the value of a_{i_1,i_2,i_3} from Eq. (5.47) to Eq. (5.46) and taking L^2 norm, we get

$$\begin{aligned} &\|u(x,w,\tau)-u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x,w,\tau)\|^2 \\ \leq \sum_{i_1=\mathcal{N}+1}^{\infty}\sum_{i_2=\mathcal{M}+1}^{\infty}\sum_{i_3=\mathcal{K}+1}^{\infty} &\left(\zeta^2\frac{(2i_1+1)^2}{\pi^6(i_1+i_1^2)^2}\frac{(2i_2+1)^2}{(i_2+i_2^2)^2}\frac{(2i_3+1)^2}{(i_3+i_3^2)^2}\right) \\ &\left(\|\mathcal{L}_{i_1}(x)\|)^2(\|\mathcal{L}_{i_2}(w)\|)^2(\|\mathcal{L}_{i_3}(\tau)\|\right)^2, \end{aligned} \tag{5.48}$$

$$\|u(x,w,\tau)-u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x,w,\tau)\|^2 \leq \frac{\zeta^2}{(1+\mathcal{N})^2(1+\mathcal{M})^2(1+\mathcal{K})^2}. \tag{5.49}$$

From Eq. (5.49) it is clear that as we increase the value of $\mathcal{N},\mathcal{M},\mathcal{K}\to\infty$ $u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x,w,\tau)$ converges uniformly to $u(x,w,\tau)$ under L^2 norm. □

Now, we turn to discuss the error approximation of proposed collocation for GFADE.

Theorem 5.5.3. Let $u(x,w,\tau)$ be sufficiently differentiable function in $L^2[\Omega\times\mathcal{I}_3]$, and $(\mathcal{D}^{\gamma}_{0+;[z;\omega;2]})u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x,w,\tau)$ denote the approximation of $(\mathcal{D}^{\gamma}_{0+;[z;\omega;2]})u(x,w,\tau)$. Assume that, $\sup|(\mathcal{D}^{\gamma}_{0+;[z;\omega;2]})u(x,w,\tau)|\leq\lambda$, $\forall(x,w,\tau)\in\Omega\times\mathcal{I}_3$, then $\mathcal{E}_n\to 0$ as $\mathcal{N},\mathcal{M},\mathcal{K}\to\infty$,

where,

$$\begin{aligned} &\mathcal{E}_n \\ =&\|(\mathcal{D}^{\gamma}_{0+;[z;\omega;2]})u(x,w,\tau)-(\mathcal{D}^{\gamma}_{0+;[z;\omega;2]})u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x,w,\tau)\|_2^2, \\ =&\int_0^1\int_0^1\int_0^1 \|((\mathcal{D}^{\gamma}_{0+;[z;\omega;2]})u(x,w,\tau)-(\mathcal{D}^{\gamma}_{0+;[z;\omega;2]})u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x,w,\tau))\|^2\,dx\,dw\,d\tau. \end{aligned} \tag{5.50}$$

Proof. Since

$$(\mathcal{D}^{\gamma}_{0+;[z;w;2]})u(x,w,\tau)=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{i_3=0}^{\infty}b_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau), \tag{5.51}$$

where b_{i_1,i_2,i_3} is given by,

$$b_{i_1,i_2,i_3} = \frac{1}{\tilde{h}_i\tilde{h}_j\tilde{h}_k}\int_0^1\int_0^1\int_0^1 (\mathcal{D}^{\gamma}_{0+;[z;w;2]})u(x,w,\tau)\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\,dx\,dw\,d\tau, \quad (5.52)$$

we approximate the derivative of function $u(x,w,\tau)$ by,

$$(\mathcal{D}^{\gamma}_{0+;[z;w;2]})u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x,w,\tau) = \sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}} b_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau). \quad (5.53)$$

From Eqs. (5.51) and (5.53) we get,

$$(\mathcal{D}^{\gamma}_{0+;[z;\omega;2]})u(x,w,\tau) - (\mathcal{D}^{\gamma}_{0+;[z;\omega;2]})u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x,w,\tau) \quad (5.54)$$

$$= \int_0^1\int_0^1\int_0^1 \Bigg[\sum_{i_1=\mathcal{N}+1}^{\infty}\sum_{i_2=\mathcal{M}+1}^{\infty}\sum_{i_3=\mathcal{K}+1}^{\infty} b_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau) \quad (5.55)$$

$$+\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=\mathcal{M}+1}^{\infty}\sum_{i_3=\mathcal{K}+1}^{\infty} b_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)$$

$$+\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=\mathcal{K}+1}^{\infty} b_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau) \quad (5.56)$$

$$+\sum_{i_1=\mathcal{N}+1}^{\infty}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=\mathcal{K}+1}^{\infty} b_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)$$

$$+\sum_{i_1=\mathcal{N}+1}^{\infty}\sum_{i_2=0}^{\mathcal{M}}\sum_{i_3=0}^{\mathcal{K}} b_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)+ \quad (5.57)$$

$$\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=\mathcal{M}+1}^{\infty}\sum_{i_3=\mathcal{K}+1}^{\infty} b_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)$$

$$+\sum_{i_1=0}^{\mathcal{N}}\sum_{i_2=\mathcal{M}+1}^{\infty}\sum_{i_3=0}^{\mathcal{K}} b_{i_1,i_2,i_3}\mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)\Bigg] \quad (5.58)$$

by taking the L^2 norm of Eq. (5.58), we have

$$\mathcal{E}_n \tag{5.59}$$

$$= \int_0^1 \int_0^1 \int_0^1 \Bigg[\sum_{i_1=\mathcal{N}+1}^{\infty} \sum_{i_2=\mathcal{M}+1}^{\infty} \sum_{i_3=\mathcal{K}+1}^{\infty} b_{i_1,i_2,i_3} \mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)+ \tag{5.60}$$

$$\sum_{i_1=0}^{\mathcal{N}} \sum_{i_2=\mathcal{M}+1}^{\infty} \sum_{i_3=\mathcal{K}+1}^{\infty} b_{i_1,i_2,i_3} \mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)$$

$$+ \sum_{i_1=0}^{\mathcal{N}} \sum_{i_2=0}^{\mathcal{M}} \sum_{i_3=\mathcal{K}+1}^{\infty} b_{i_1,i_2,i_3} \mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau) \tag{5.61}$$

$$+ \sum_{i_1=\mathcal{N}+1}^{\infty} \sum_{i_2=0}^{\mathcal{M}} \sum_{i_3=\mathcal{K}+1}^{\infty} b_{i_1,i_2,i_3} \mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)$$

$$+ \sum_{i_1=\mathcal{N}+1}^{\infty} \sum_{i_2=0}^{\mathcal{M}} \sum_{i_3=0}^{\mathcal{K}} b_{i_1,i_2,i_3} \mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau) \tag{5.62}$$

$$+ \sum_{i_1=0}^{\mathcal{N}} \sum_{i_2=\mathcal{M}+1}^{\infty} \sum_{i_3=\mathcal{K}+1}^{\infty} b_{i_1,i_2,i_3} \mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau)$$

$$+ \sum_{i_1=0}^{\mathcal{N}} \sum_{i_2=\mathcal{M}+1}^{\infty} \sum_{i_3=0}^{\mathcal{K}} b_{i_1,i_2,i_3} \mathcal{L}_{i_1}(x)\mathcal{L}_{i_2}(w)\mathcal{L}_{i_3}(\tau) \Bigg]^2 \, dx\, dw\, d\tau,$$

$$= \sum_{i_1=\mathcal{N}+1}^{\infty} \sum_{i_2=\mathcal{M}+1}^{\infty} \sum_{i_3=\mathcal{K}+1}^{\infty} b^2_{i_1,i_2,i_3} \int_0^1 |\mathcal{L}^2_{i_1}(x)| dx \int_0^1 |\mathcal{L}^2_{i_2}(w)| dw \int_0^1 |\mathcal{L}^2_{i_3}(\tau)| d\tau. \tag{5.63}$$

Using Eq. (5.52) in Eq. (5.63), we get

$$|\mathcal{E}_n| \leq \sum_{i_1=\mathcal{N}+1}^{\infty} \sum_{i_2=\mathcal{M}+1}^{\infty} \sum_{i_3=\mathcal{K}+1}^{\infty} \frac{\sup|(\mathcal{D}^{\gamma}_{0+;[z;w;2]})u(x,w,\tau)|^2 (2i_1+1)(2i_2+1)(2i_3+1)}{((i_1+i_1^2)\pi)^2((i_2+i_2^2)\pi)^2((i_3+i_3^2)\pi)^2},$$

$$\leq \sum_{i_1=\mathcal{N}+1}^{\infty} \sum_{i_2=\mathcal{M}+1}^{\infty} \sum_{i_3=\mathcal{K}+1}^{\infty} \frac{\lambda^2(2i_1+1)(2i_2+1)(2i_3+1)}{((i_1+i_1^2)\pi)^2((i_2+i_2^2)\pi)^2((i_3+i_3^2)\pi)^2},$$

$$\leq \frac{\lambda^2}{\pi^6(1+\mathcal{N})^2(1+\mathcal{M})^2(1+\mathcal{K})^2}. \tag{5.64}$$

Clearly, we obtained the following result,

$$|\mathcal{E}_n| \leq \left(\frac{\lambda^2}{\pi^6(1+\mathcal{N})^2(1+\mathcal{M})^2(1+\mathcal{K})^2}\right). \tag{5.65}$$

It is clear from the Eq. (5.65) that $\mathcal{E}_n \to 0$ as $\mathcal{N}, \mathcal{M}, \mathcal{K} \to \infty$. □

5.6 Numerical Results

Now, we present the numerical validity of the proposed collocation method by some examples. Examples 5.6.1 is based on the one-dimensional GFADE, while Examples 5.6.2, and 5.6.3 on two-dimensional GFADE. Some of considered examples are of known solution and the rest of them may not admit known solutions. Numerical results of examples are presented by varying the parameters. The maximum absolute error (MAE) and absolute error (AE) are also calculated respectively by the formulae,

$$E_N = \max|u(x, w, \tau_k) - u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x, w, \tau_k)|,\ x, w \in (0, 1),$$

$$AE = |u(x_p, w_q, \tau_k) - u_{N,M,K}(x_p, w_q, \tau_k)|,\ x_p \in (0, 1),\ w_q \in (0, 1).$$

Here, as before, $u(x, w, \tau)$ and $u_{\mathcal{N},\mathcal{M},\mathcal{K}}(x, w, \tau)$ denote the exact the approximate solution, respectively.

Example 5.6.1. Here, we take the one-dimensional GFADE with the exact solution is not known. Consider the following one dimensional GFADE,

$$(\mathcal{D}^{\gamma}_{0+;[z;\omega;2]})u(x, \tau) + \nu_1(x, \tau)\frac{\partial u(x, \tau)}{\partial x} = \frac{\partial^2 u(x, \tau)}{\partial x^2} + s(x, \tau),\ \tau \in \mathcal{I}_3, \tag{5.66}$$

with initial and boundary conditions are given as,

$$\begin{cases} u(x,0) = 0, \ x \in \mathcal{I}_1, \\ u(0,\tau) = 0, \\ u(1,\tau) = 0, \end{cases} \tag{5.67}$$

where $s(x,\tau) = x\tau(1-x)$.

We performed the numerical simulation with: 1) $z(\tau) = \tau, \ \tau^2, \ \tau^3$ and $w(\tau) = 1$ 2) $w(\tau) = 1, \ e^{4\tau}, \ e^{-4\tau}$ and $z(\tau) = \tau$ in Fig. 5.1,by Green, light Green, Violet, light Red and Red color lines respectively with fixed $\mathcal{N} = \mathcal{K} = 7$ and $\gamma = 0.2$. In Fig. 5.1, light Green and Violet color lines show that the solution surface of Eq. (5.6.1) having contract and stretched behaviour for increasing weight function $w(\tau) = e^{4\tau}$ and decreasing weight function $w(\tau) = e^{-4\tau}$ respectively. It is clear from Fig. 5.1 (Red lines) that obtained numerical solution of the problem is shifted towards upward for the increasing scale function $z(\tau) = \tau^3$. Furthermore, we presented the graph of numerical solution of Eq. (5.6.1) for various values of $\mathcal{N} = \mathcal{K}$ in Fig. 5.2. It is clear from the Fig. 5.2 that numerical solution looking similar for different values of $\mathcal{N} = \mathcal{K}$, which proves the validity our proposed method. Finally, we conclude that our proposed method provide better results not only in case of known exact solution but also in case we do not have exact solution.

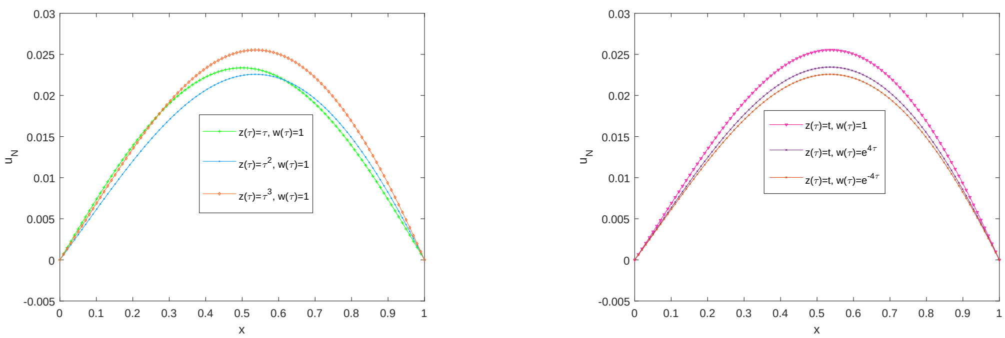


FIGURE 5.1: The numerical solution of Example 5.6.1 for different values of $z(\tau)$ and $w(\tau)$ at $\tau = 1$.

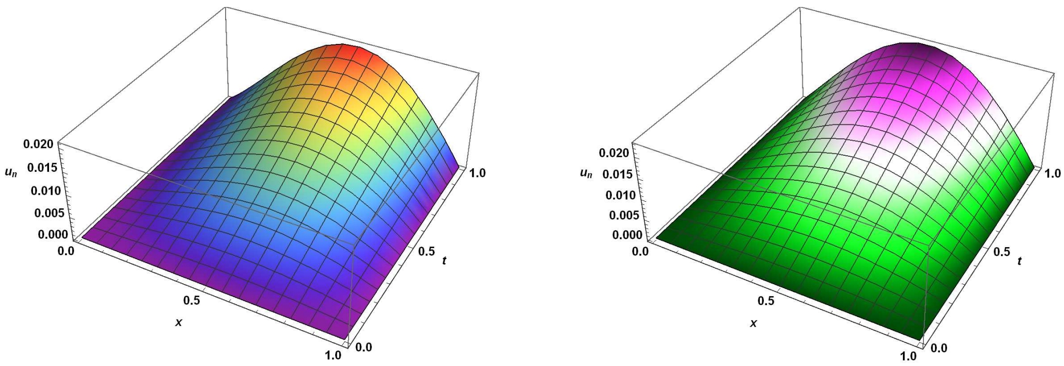


FIGURE 5.2: The numerical solution for different $\mathcal{N} = \mathcal{K} = 4(left), 6(right)$, $\gamma = 0.2$ and $z(\tau) = \tau^3$ for Example 5.6.1.

Example 5.6.2. Consider the following equation with initial and boundary condition,

$$(\mathcal{D}^{\gamma}_{0+;[z;\omega;2]})u(x, w, \tau) + \mathcal{P}u(x, w, \tau) = \nabla^2 u(x, w, \tau) + \rho(x, w, \tau),\ \tau \in \mathcal{I}_3, \tag{5.68}$$

$$\begin{cases} u(x, w, 0) = 0,\ x \in \Omega, \\ u(x, 0, \tau) = 0,\ u(x, 1, \tau) = 0, \\ u(0, w, \tau) = 0,\ u(1, w, \tau) = 0, \end{cases} \tag{5.69}$$

and

$$\begin{aligned} &\rho(x, w, \tau) \\ &= 16\tau \left(-3(-1 + w)w + x^2(-3 + 2w) + x(3 - 4w + 2w^2) - \frac{\tau^{-\gamma}(-1 + x)x(-1 + w)w}{(-1 + \gamma)\Gamma(1 - \gamma)} \right), \\ &\gamma < 1. \end{aligned} \tag{5.70}$$

The exact solution of this problem (5.6.2) is given by $u(x, w, \tau) = 16\tau xw(1 - w)(1 - x)$. In Table 5.1, MAE is obtained by our proposed method with $w(\tau) = 1$ and $z(\tau) = \tau$ for the various value of $\mathcal{N} = \mathcal{M} = \mathcal{K}$. In Table 5.2, we have calculated the AE at the grid point in he defined domain by keeping variable τ fixed. We see in

Tables 5.1 and 5.2 that as we increase the value of $\mathcal{N} = \mathcal{M} = \mathcal{K}$, MAE and AE is decreased continuously and after fixed value of $\mathcal{N} = \mathcal{M} = \mathcal{K}$ it becomes constant. In Fig 5.3, we have plotted the AE graph for different value of τ. In Fig. 5.7 (left), we have shown the graphs of AE with x direction keeping the variables w, and τ fixed. In Fig. 5.7 (right), we have shown the behavior of exact and approximate solution for the various values of $\mathcal{N} = \mathcal{M} = \mathcal{K}$. Fig. 5.5 shows the comparison of exact and approximate solutions with space variable x. In Figs. 5.7 and 5.5, we observe that numerical solution converges to exact solution as the value of $\mathcal{N} = \mathcal{M} = \mathcal{K}$ increases, which proves that our proposed method provide the best approximation.

TABLE 5.1: MAE of the Example 5.6.2 with $\gamma = 0.2$.

N	M	K	$\tau_k = 1/4$	$\tau_k = 1/2$	$\tau_k = 3/4$	$\tau_k = 1$
2	2	2	5.689×10^{-16}	6.039×10^{-16}	2.359×10^{-16}	9.437×10^{-16}
4	4	4	1.899×10^{-13}	1.361×10^{-14}	1.351×10^{-14}	1.272×10^{-14}
6	6	6	2.495×10^{-5}	4.519×10^{-9}	6.712×10^{-7}	1.659×10^{-6}
8	8	8	5.372×10^{-3}	8.444×10^{-3}	9.657×10^{-3}	8.592×10^{-3}
10	10	10	1.485×10^{-9}	4.044×10^{-11}	5.594×10^{-11}	5.455×10^{-10}

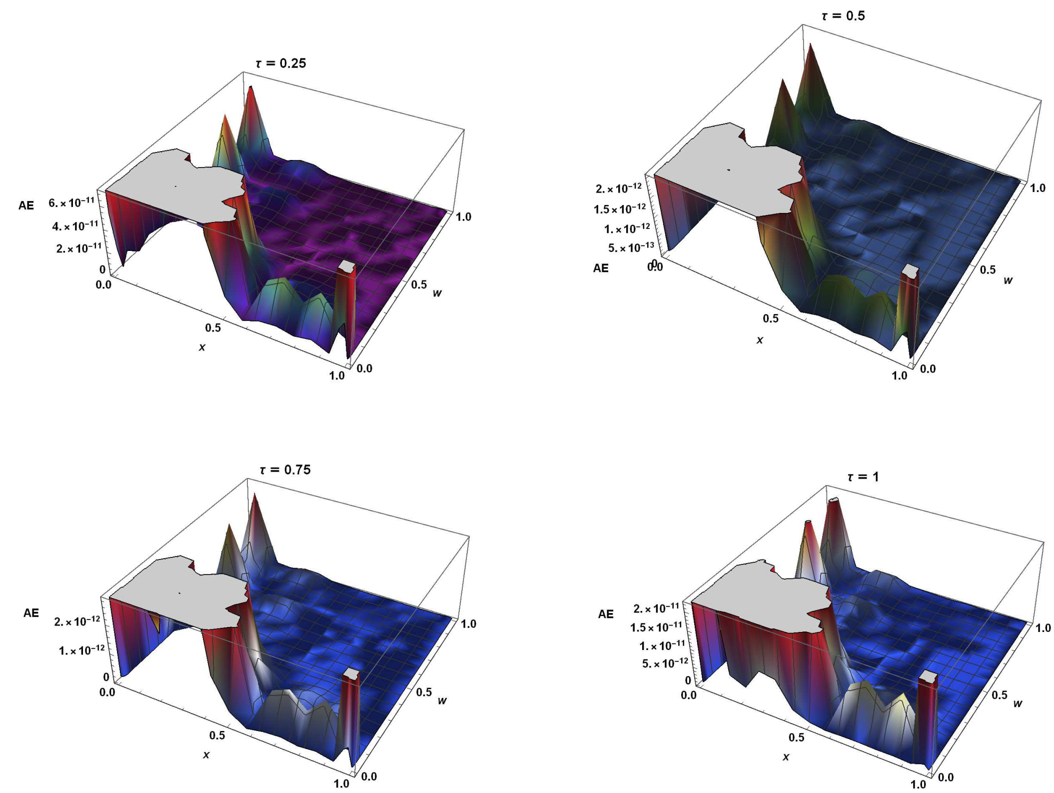


FIGURE 5.3: AE function at different values of t with $\mathcal{N} = \mathcal{M} = \mathcal{K} = 10$ and $\gamma = 0.2$ for Example 5.6.2.

TABLE 5.2: AE of the Example 5.6.2 with $\gamma = 0.2$.

Grid points	$\mathcal{N} = \mathcal{M} = \mathcal{K} = 4$	$\mathcal{N} = \mathcal{M} = \mathcal{K} = 6$	$\mathcal{N} = \mathcal{M} = \mathcal{K} = 8$	$\mathcal{N} = \mathcal{M} = \mathcal{K} = 10$
(0.1,0.1)	1.277×10^{-14}	1.659×10^{-6}	8.592×10^{-3}	5.4599×10^{-10}
(0.2,0.2)	4.386×10^{-15}	1.128×10^{-6}	1.986×10^{-3}	1.225×10^{-10}
(0.3,0.3)	7.661×10^{-15}	2.334×10^{-7}	8.492×10^{-5}	5.224×10^{-11}
(0.4,0.4)	1.144×10^{-14}	1.389×10^{-8}	9.391×10^{-7}	5.496×10^{-13}
(0.5,0.5)	9.104×10^{-15}	9.723×10^{-9}	2.949×10^{-8}	1.248×10^{-13}
(0.6,0.6)	4.996×10^{-15}	2.183×10^{-10}	9.704×10^{-9}	7.249×10^{-14}
(0.7,0.7)	3.109×10^{-15}	3.491×10^{-11}	3.479×10^{-8}	1.589×10^{-13}
(0.8,0.8)	1.777×10^{-15}	9.287×10^{-12}	2.982×10^{-8}	1.216×10^{-13}
(0.9,0.9)	5.829×10^{-16}	7.403×10^{-13}	8.605×10^{-9}	1.445×10^{-13}

Effect of weight function $\omega(\tau)$ on numerical solution In This part, we choose the various weight function and observe the changes in the numerical solutions of Example 5.6.2.

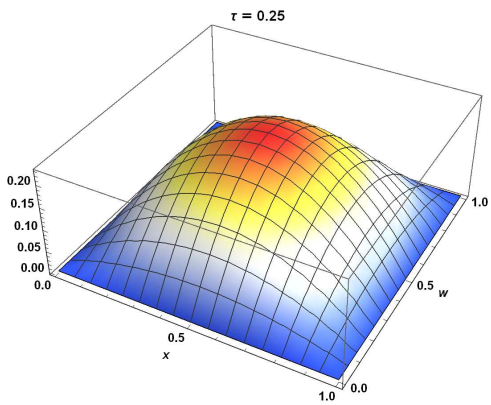
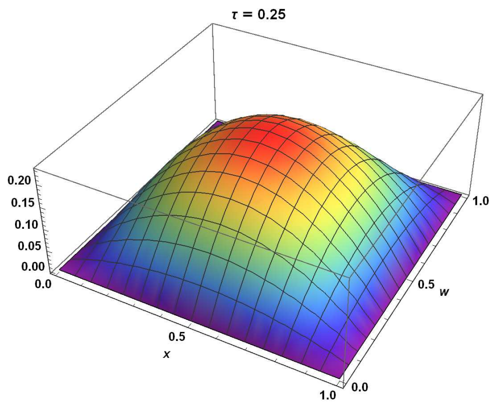
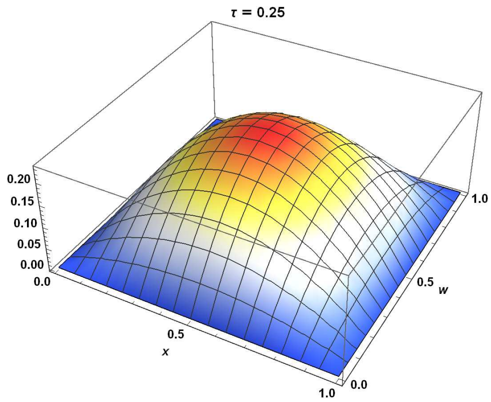


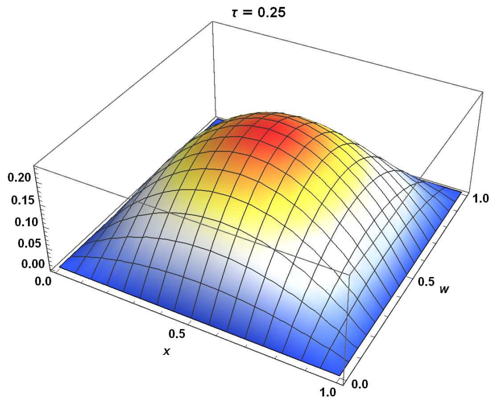
FIGURE 5.4: $\omega(\tau) = e^{4\tau}, \gamma = 0.2$



(b) $\omega(\tau) = e^{\tau},\ \gamma = 0.2$



(c) $\omega(\tau) = e^{-3\tau},\ \gamma = 0.2$



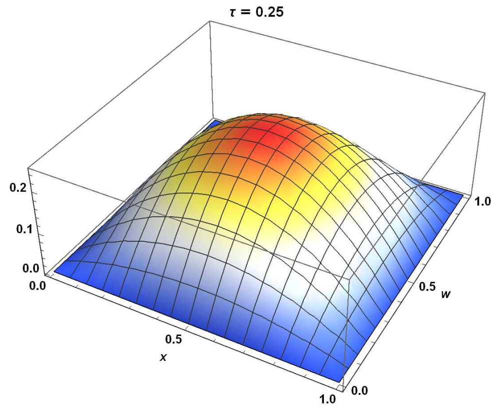
(d) $\omega(\tau) = e^{2\tau},\ \gamma = 0.2$

FIGURE 5.5: Plot of the approximate solution for different choices of weight functions $w(\tau)$ and $z(\tau) = \tau$ for Example 5.6.2.

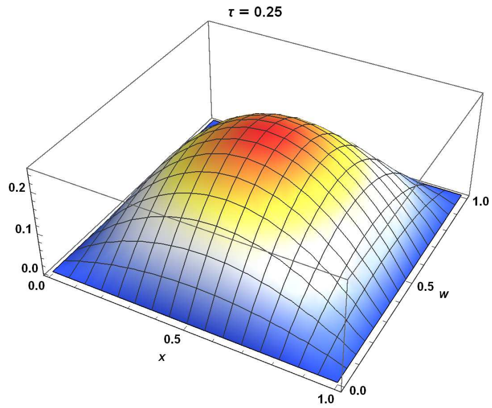
In Fig. 5.5, we found the approximate solutions of problem (5.6.2) by changing the different weight function $\omega(\tau) = e^{4\tau},\ e^{\tau},\ e^{-3\tau}$ and $e^{2\tau}$.

In Fig. 5.5, we observe that that weight function effects the numerical solution inversely, i.e. if we take increasing weight function we get in results contract in solution. Whereas, if we take decreasing weight function we get the response in the solution expand in solution.

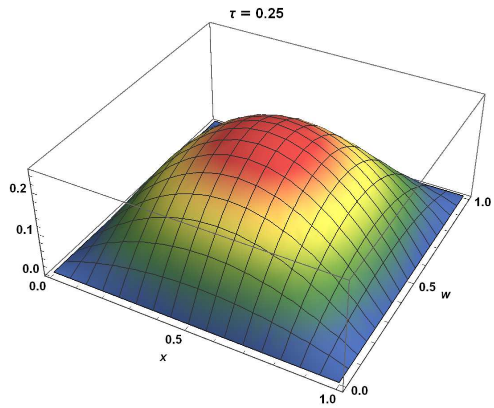
Effect of scale function $z(\tau)$ on the numerical solution In this part, we will changes the various scale function and corresponding changes are discussed below of Example 5.6.2.



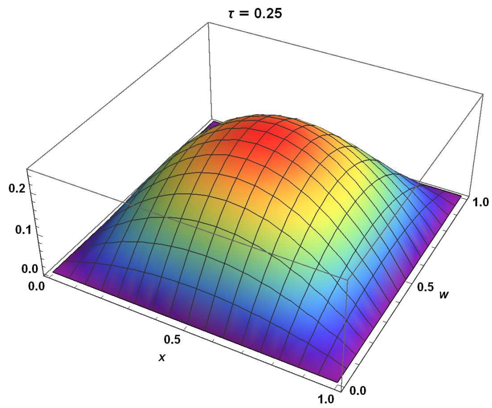
$(a)\ z(\tau) = 0.05\tau,\ \gamma = 0.2$



$(b)\ z(\tau) = \tau^{1/8},\ \gamma = 0.2$



$(c)\ z(\tau) = 20\tau,\ \gamma = 0.2$



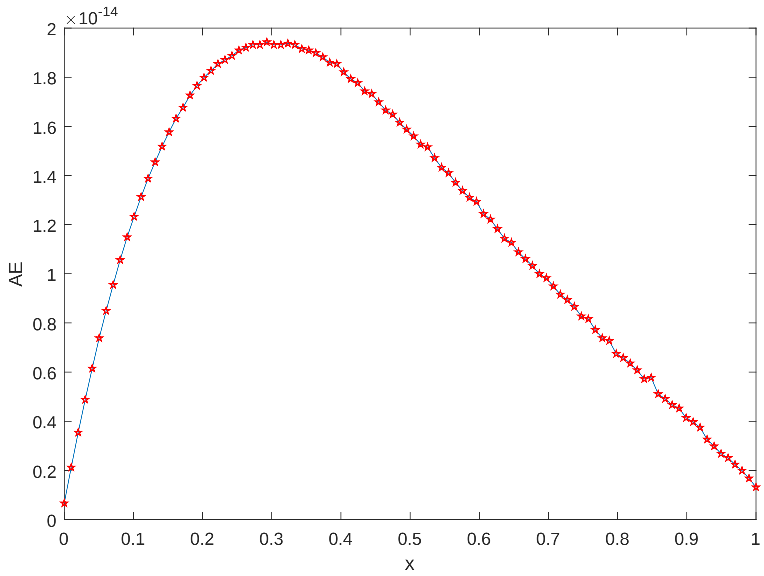
$(d)\ z(\tau) = \tau^{8},\ \gamma = 0.2$

FIGURE 5.6: The approximate solution for different choice of scale functions $z(\tau)$ and $\omega(\tau) = 1$ for Example 5.6.2.

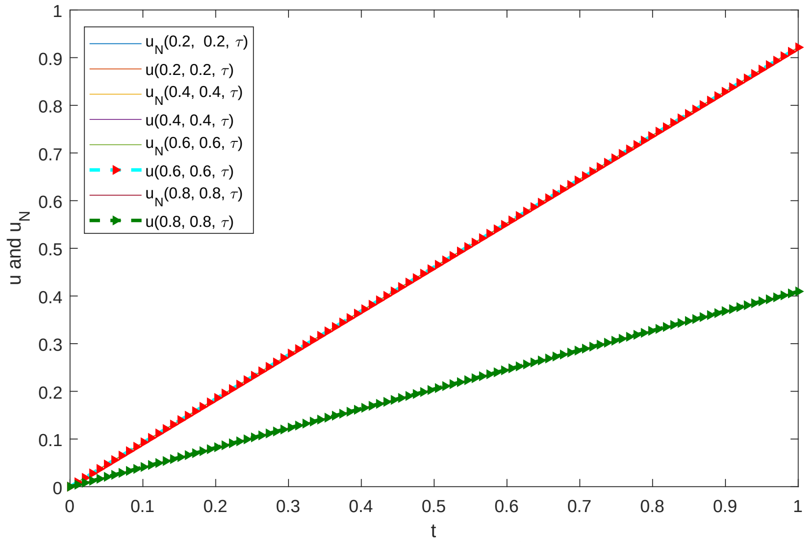
To analyze the property of the numerical solution, we have taken the scale function $z(t)$ increasing and decreasing function respectively. Numerical solutions are obtained with $\gamma = 0.2,\ w(\tau) = 1,\ \nu_1 = 1,\ \nu_2 = 2$ and apply the initial and boundary conditions given in 5.6.2. Further, choose $z(\tau) = 20\tau,\ 0.05\tau,\ \tau^8$ and $\tau^{1/8}$. We can see that the first two and last two shown the linear and the non-linear scale functions, respectively. Results are shown in Fig. 5.6. From the figures, we conclude that.

1) Comparing 5.4 with Figs. 5.6 we observe that enlarging (contracting) choice of $z(\tau)$ effect the solution to contract (enlarge). This enlarging (contracting) is uniform to solution in case when we have taken linear scale enlarging (contracting) functions.

2) Comparing Fig. 5.4 with Figures 5.6, we observe that if we take non-linear enlarging (contracting) of scale functions causes the solution contract (enlarge).



(a)



(b)

FIGURE 5.7: (a) AE for $\mathcal{N} = \mathcal{M} = \mathcal{K} = 10$ with x axis, (b) The comparison of the approximate solution to exact solution for different choice of $x,\ w \in \mathcal{I}_1 \times \mathcal{I}_2$ with τ of Example 5.6.2.

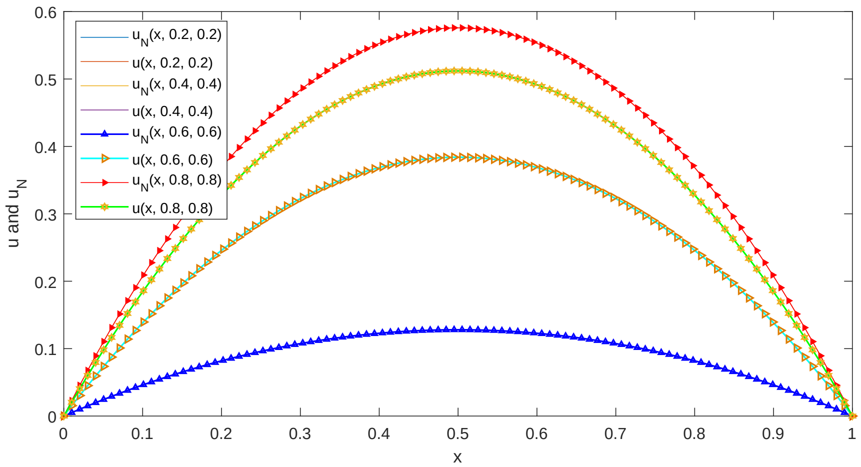


FIGURE 5.8: The approximate solution and the exact solution for different choice of $x,\ w$ with spatial axis x of Example 5.6.2.

Example 5.6.3. Consider the following equation with initial and boundary condition,

$$(\mathcal{D}^{\gamma}_{0+;[z;\omega;2]})u(x,w,\tau) + \mathcal{P}u(x,w,\tau) = \nabla^2 u(x,w,\tau) + \rho(x,w,\tau),\ \tau \in \mathcal{I}_3, \tag{5.71}$$

$$\begin{cases} u(x,w,0) = 0, \quad x \in \Omega, \\ u(x,0,\tau) = 0, \ u(x,1,\tau) = 0, \\ u(0,w,\tau) = 0, \ u(1,w,\tau) = 0, \end{cases} \tag{5.72}$$

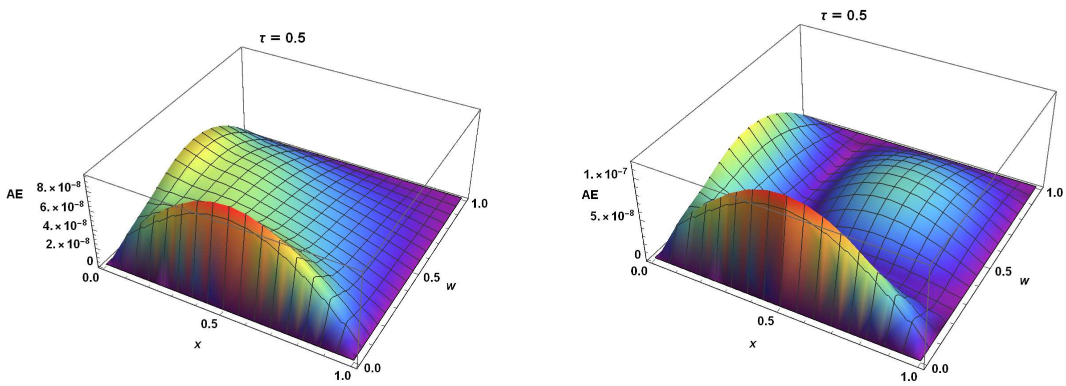
and

$$\begin{aligned} &\rho(x,w,\tau) \\ &= \tau^{1+\beta}\sin(\pi x)\left(\pi\cos(\pi y) + \frac{(1+\beta)t^{-\gamma}\Gamma(1+\beta)\sin(\pi y)}{\Gamma(2+\beta-\gamma)} + \pi\left(\frac{\cos(\pi x)+2\pi\sin(\pi x)}{\sin(\pi x)}\right)\sin(\pi y)\right), \\ &\gamma < 1. \end{aligned} \tag{5.73}$$

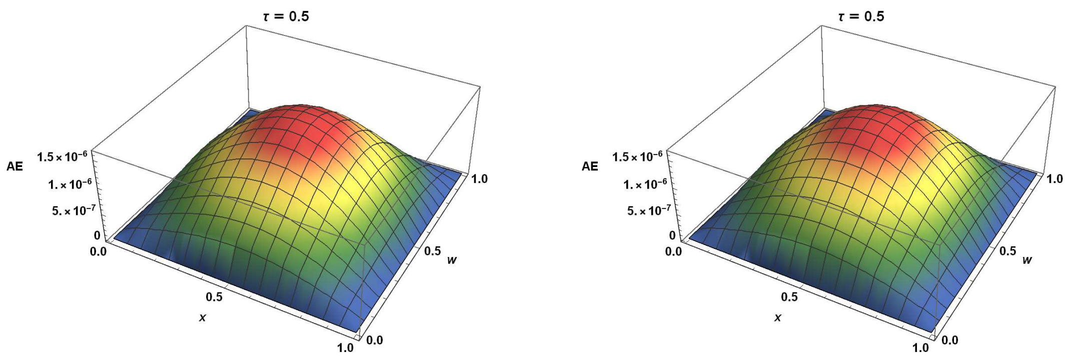
We consider Example 5.6.3, solution is fractional order with exact solution $u(x,w,\tau) = \tau^{1+\beta}\sin(\pi x)\sin(\pi w)$ in domain $\Omega \times \mathcal{I}_3$. To obtained the approximate solution we consider the different values of the parameter β and $w(\tau) = 1$, $z(\tau) = \tau$. First we have calculated the approximate solution for different value of $\mathcal{N} = \mathcal{M} = \mathcal{K} = 2, 4, 6, 8, 10$ and $\beta = 0.5, 1, 3/2, 2$ corresponding MAE shown In Table 5.3. Further, In Table 5.4, we have fixed the value of $\mathcal{N} = \mathcal{M} = \mathcal{K} = 10$ and find numerical solutions for various values of β for a fixed value of $\tau = 0.5$ and calculate AE for $x, \ w \in [0,1]$. In Fig. 5.9, we have shown the comparison graph of AE for different value of β and fixed $\mathcal{N} = \mathcal{M} = \mathcal{K} = 10, \ \gamma = 0.2$. In Fig. 5.10, the comparison between exact and numerical solutions for different value of $\mathcal{N} = \mathcal{M} = \mathcal{K}$ and β are shown. We observe in Fig. 5.10 that all figures are looking similar for various value of β which validate the accuracy of our proposed methods. In Fig. 5.12, we have plotted the exact and approximate solutions for different value of $x, \ w \in [0,1]$ and β. In Fig. 5.12, we have given the AE graph for different values of $\mathcal{N} = \mathcal{M} = \mathcal{K}$ and β. From the figures and tables we observe that our proposed method provide best approximation not only for integer order solution but also of fractional order solution.

TABLE 5.3: MAE of the Example 5.6.3 with $\gamma = 0.2,\ \tau = 0.5$.

N	M	K	$\beta = 1/2$	$\beta = 1$	$\beta = 3/2$	$\beta = 2$
2	2	2	4.437×10^{-2}	3.890×10^{-2}	2.789×10^{-2}	2.021×10^{-2}
4	4	4	3.418×10^{-2}	2.518×10^{-2}	1.698×10^{-2}	1.199×10^{-2}
6	6	6	1.407×10^{-2}	9.435×10^{-3}	5.402×10^{-3}	4.942×10^{-3}
8	8	8	8.919×10^{-3}	1.119×10^{-3}	1.382×10^{-3}	3.113×10^{-4}
10	10	10	1.449×10^{-7}	8.394×10^{-8}	5.273×10^{-8}	5.143×10^{-8}

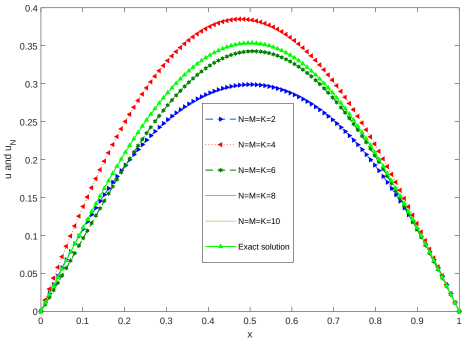


$(a)\ \beta = 2,\ z(\tau) = \tau$ $(b)\ \beta = 3/2,\ z(\tau) = \tau$

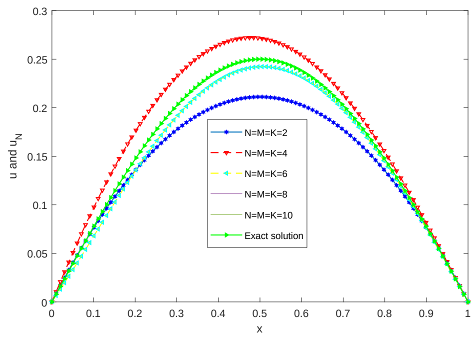


$(c)\ \beta = 1,\ z(\tau) = \tau$ $(d)\ \beta = 1/2,\ z(\tau) = \tau$

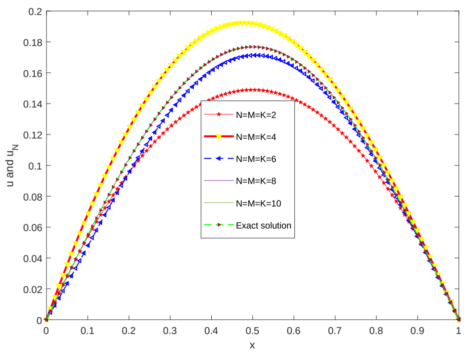
FIGURE 5.9: Plot of the AE function at different values of β with $\mathcal{N} = \mathcal{M} = \mathcal{K} = 10,\ w(\tau) = 1$ and $\gamma = 0.2$ for Example 5.6.3.



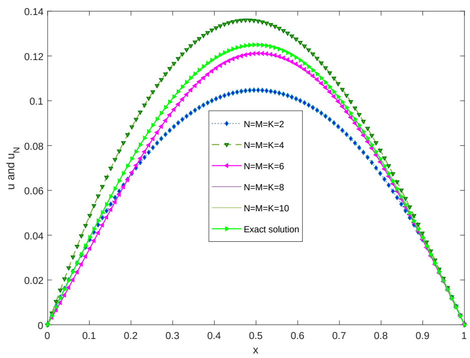
$(a)\ \beta = 0.5$



$(b)\ \beta = 1$



$(c)\ \beta = 1.5$

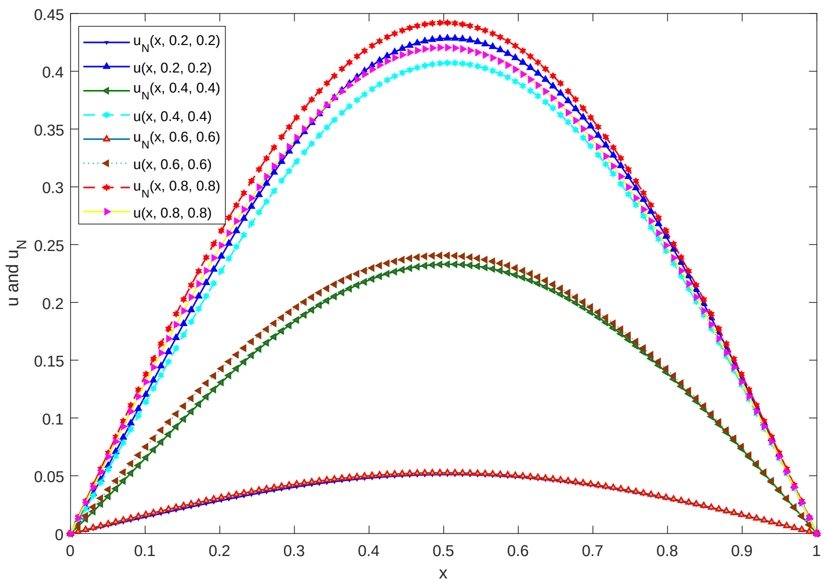


$(d)\ \beta = 2$

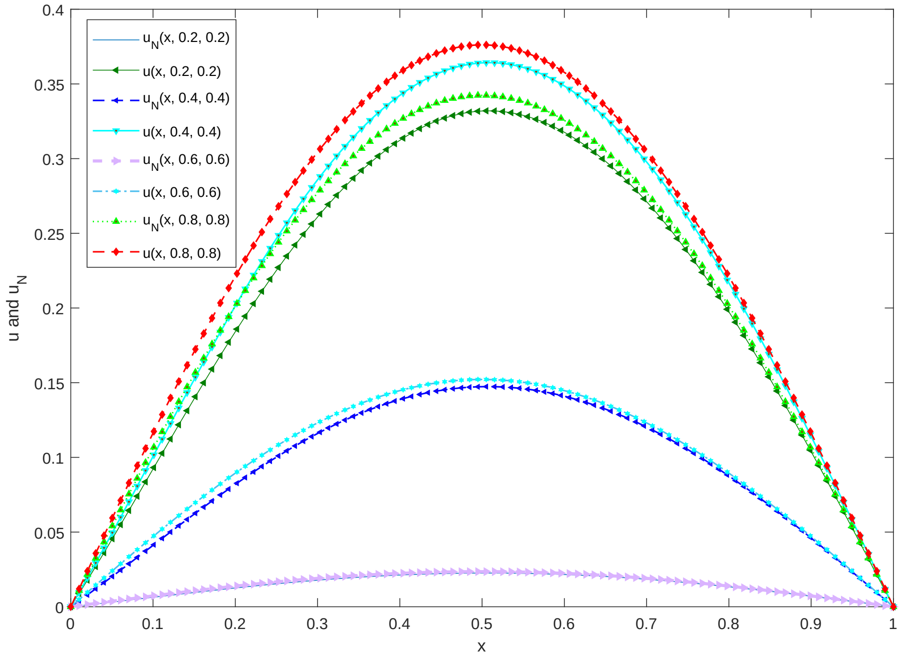
FIGURE 5.10: The exact and the approximate solution for different value of $\mathcal{N} = \mathcal{M} = \mathcal{K}$ and $\beta = 0.5, 1, 1.5$ and 2 respectively with x axis of Example 5.6.3.

TABLE 5.4: AE of the Example 5.6.3 with $\gamma = 0.2$, $\mathcal{N} = \mathcal{M} = \mathcal{K} = 10$ and various values of β.

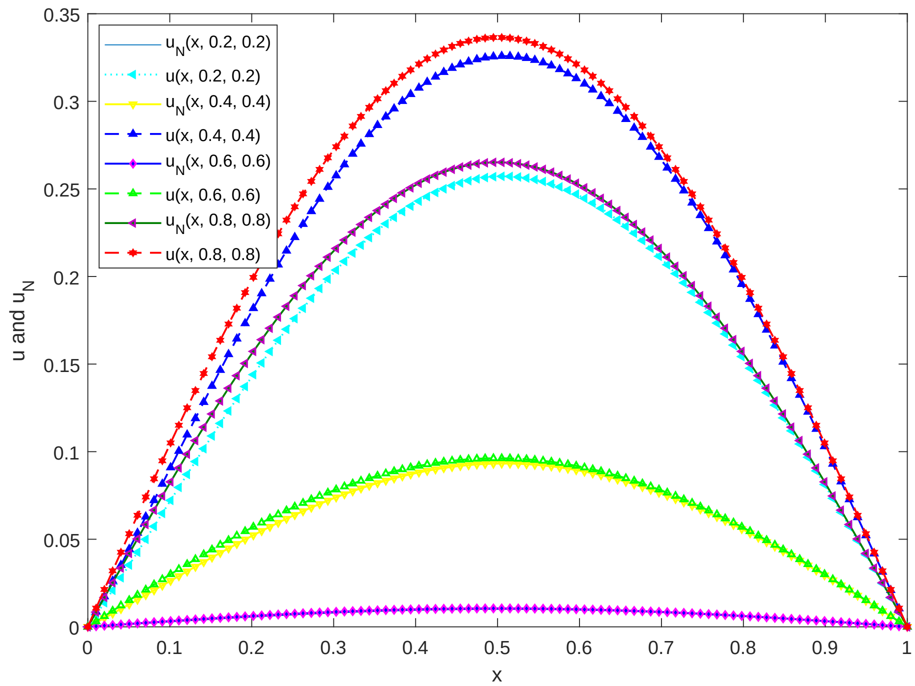
Grid points	$\beta = 1/2$	$\beta = 1$	$\beta = 3/2$	$\beta = 2$
(0.1,0.1)	3.456×10^{-7}	3.358×10^{-7}	3.352×10^{-7}	3.357×10^{-7}
(0.2,0.2)	4.452×10^{-7}	3.989×10^{-7}	3.976×10^{-7}	3.981×10^{-7}
(0.3,0.3)	5.231×10^{-7}	4.167×10^{-7}	4.131×10^{-7}	4.151×10^{-7}
(0.4,0.4)	5.622×10^{-7}	3.876×10^{-7}	3.812×10^{-7}	3.857×10^{-7}
(0.5,0.5)	5.338×10^{-7}	3.097×10^{-7}	3.0077×10^{-7}	3.078×10^{-7}
(0.6,0.6)	4.543×10^{-7}	2.213×10^{-7}	2.116×10^{-7}	2.197×10^{-7}
(0.7,0.7)	3.297×10^{-7}	1.363×10^{-7}	1.279×10^{-7}	1.353×10^{-7}
(0.8,0.8)	1.841×10^{-7}	6.659×10^{-8}	6.127×10^{-8}	6.604×10^{-8}
(0.9,0.9)	5.594×10^{-8}	1.823×10^{-8}	1.644×10^{-8}	1.805×10^{-8}



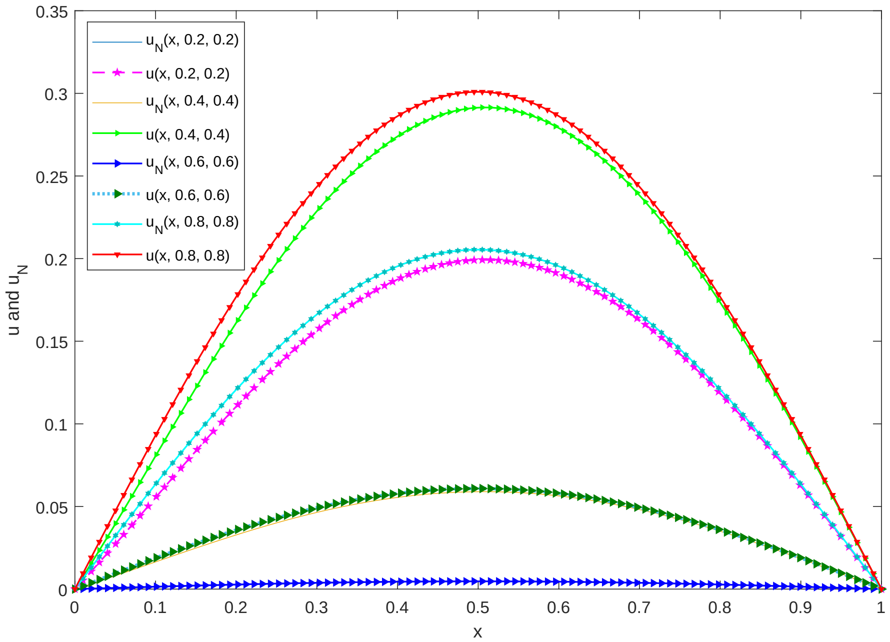
(a) $\beta = 0.5$



$(b)\ \beta = 1$

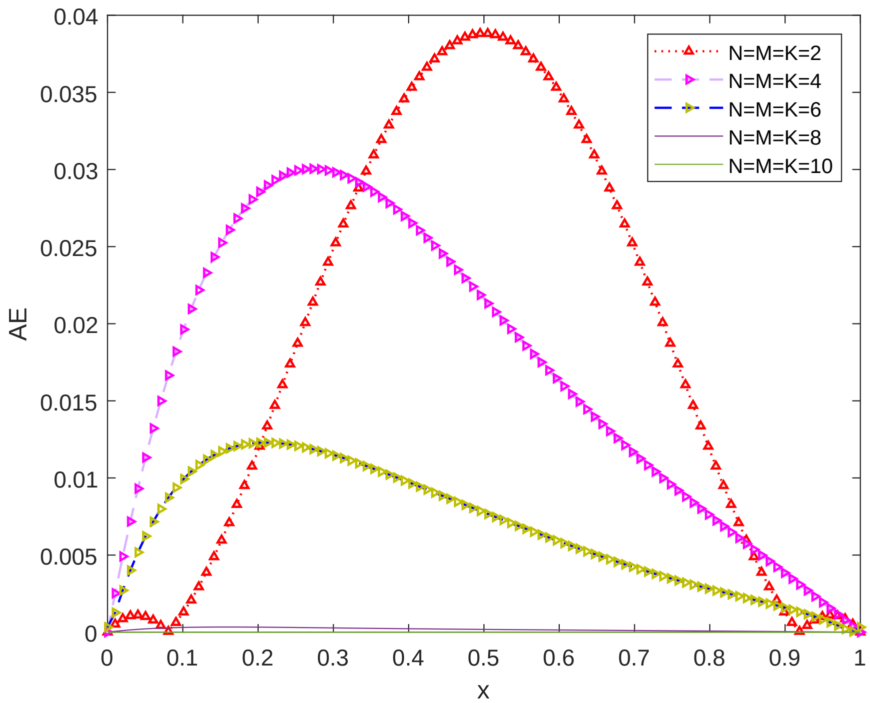


$(c)\ \beta = 1.5$

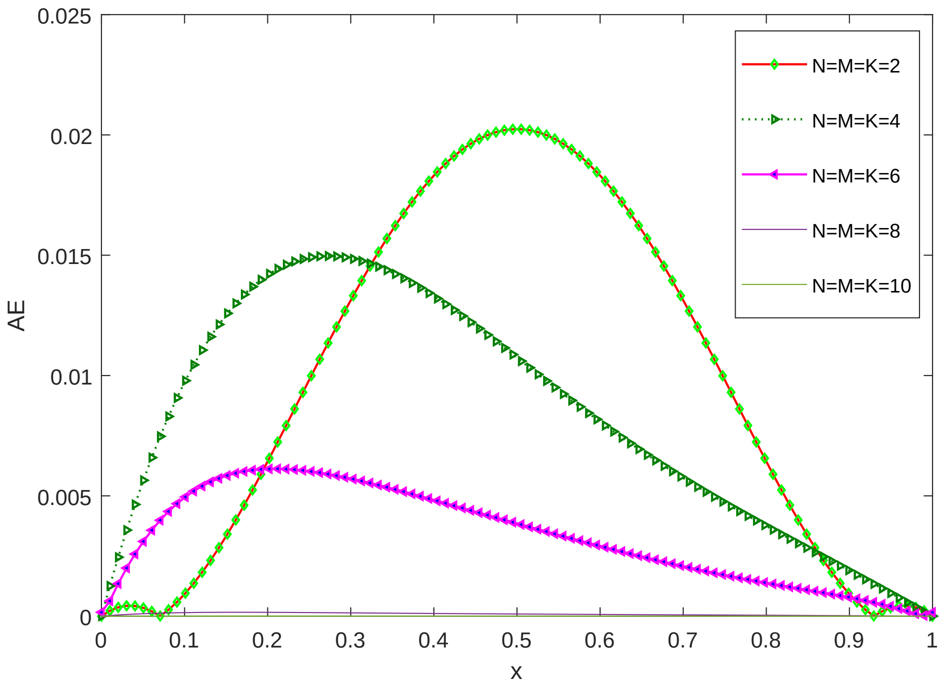


$(d)\ \beta = 2$

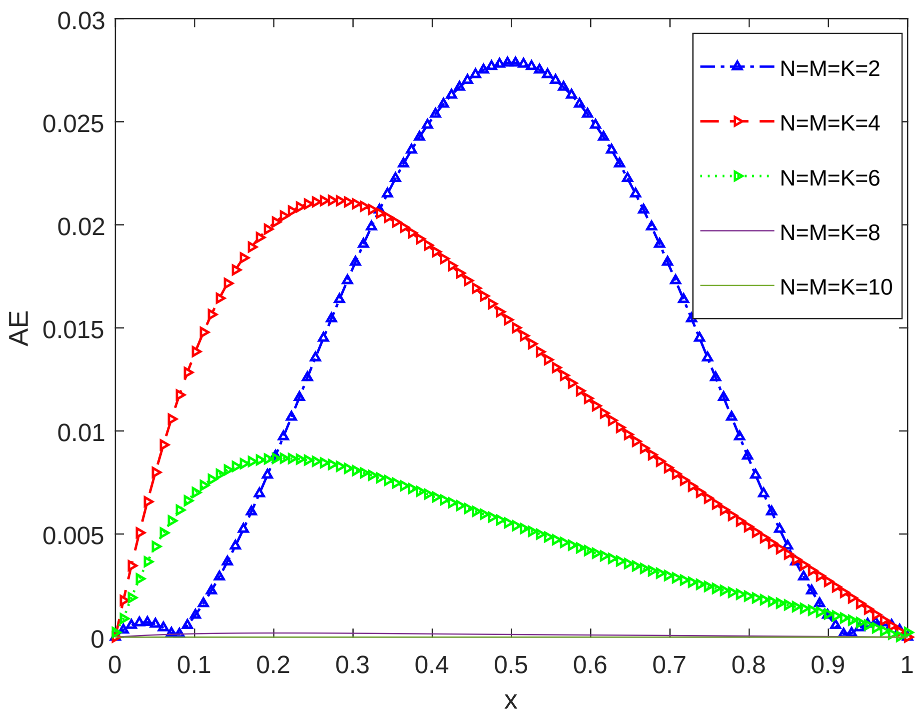
FIGURE 5.11: Plot of the exact and the approximate solutions at a particular point for different value of $\beta = 0.5, 1, 1.5$ and 2 respectively for Example 5.6.3.



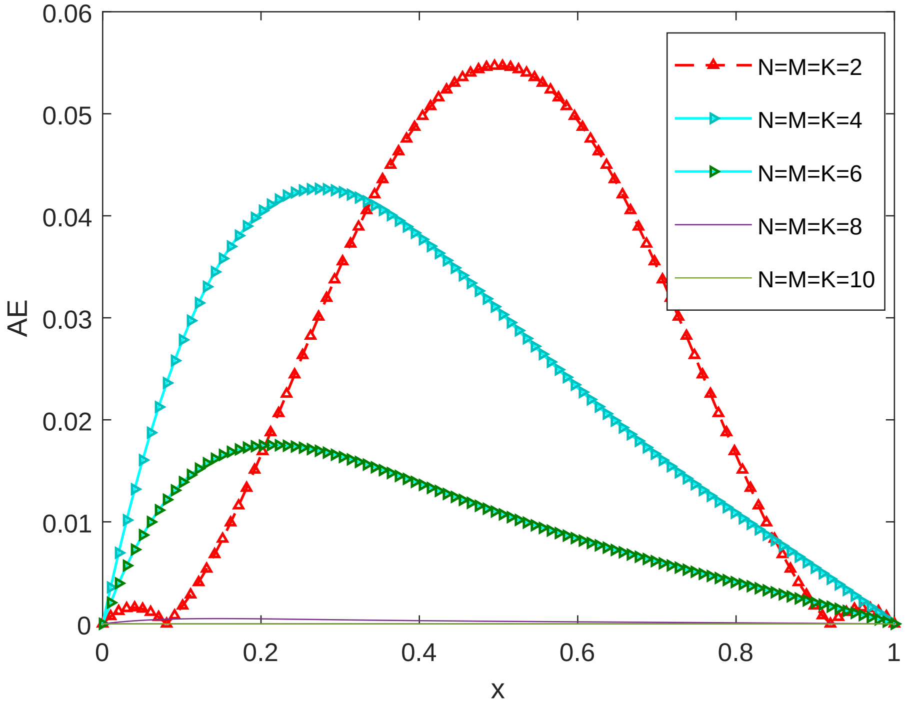
$(a)\ \beta = 0.5$



$(b)\ \beta = 1$



$(c)\ \beta = 1.5$



$(d)\ \beta = 2$

FIGURE 5.12: The comparison of AE at different values of $\beta = 1, 2, 1.5$ and 0.5 respectively with various value of $\mathcal{N} = \mathcal{M} = \mathcal{K}$ and $\gamma = 0.2$ for Example 5.6.3.

5.7 Conclusion

In this chapter, we implement a new collocation method based on the Legendre polynomials for solving GFADE with initial and boundary conditions. Due to the fact that GFD considered in this chapter involves scale and weight functions, we further study the effects of such functions on the dynamical behaviors of numerical solutions. Surprisingly, we find that an increasing scale function shifted the solution in upwards and a decreasing choice of scale function shifted the solution towards the origin. Whereas, increasing weight function contracts and decreasing weight function stretches the numerical solution. The error estimate and convergence results of the proposed collocation method are established theoretically. For checking the validity of numerical simulation, we consider three different typical examples containing both one and two-dimensional GFADE. It is concluded that our proposed method achieved good accuracy. The numerical evaluated order of convergence supports the theoretical analysis in the previous sections.

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