CHAPTER 5

FIXED-TIME PINNING IMPULSIVE SYNCHRONIZATION OF COUPLED NEURAL NETWORKS WITH MIXED DELAY

5.1 Introduction[1]

The dynamics and topology structure of coupled networks are notably complex [161]. When a network is unable to achieve self-synchronization to the ideal trajectory, it gives rise to the challenge of addressing network synchronization through the control method [162]. In recent years, there has been a significant focus on developing effective approaches, such as the Lyapunov direct method and principal stability function, for the study of synchronization in coupled neural networks [163, 164]. Earlier studies primarily concentrated on achieving synchronization in continuous-time networks without time delay [165, 166]. However, the inherent limitations of signal transmission speed and time-delay phenomena exist in electronic circuits and neuron models [167]. These delays typically manifest in various aspects of the system, including the state of the system, signal measurement, and transmission. Specifically, coupling delays pertain to communication delays, while node delays refer to computation time; collectively, these are known as system delays [168]. System delays can lead to system instability and

[1]The content of this sub chapter is under review in an international journal

poor performance [169]. For instance, in financial systems, the presence of historical transaction data alongside real-time information significantly influences the decision-making processes of traders [170]. Time delays are a common occurrence in numerous evolutionary processes [122], and they play an active role in the fields of communications and information technology [105]. Consequently, there is a compelling need for a comprehensive analysis of system delays.

Recently, the controlled synchronization of coupled neural networks with time delays has been the subject of growing interest. This interest has arisen due to the realization that in certain situations, coupled neural networks may fail to attain synchronization through a coupling mechanism, or there may be specific criteria for achieving synchronous behaviour. Within this scenario, achieving synchronization with a desired trajectory is a significant concern and has been effectively formulated as "leader-following synchronization." This concept involves determining controlled nodes, often referred to as "pinning control," and it represented a key technical challenge in this field. The primary goal of this work is to achieve pinning synchronization in the coupled neural networks through impulsive control. Impulsive control proves to be a practical approach for modeling abrupt changes of disturbances at some specific moments, which can either go against or make the synchronization of the systems in various fields, including medicine, biology and electronics. Compared to continuous control methods, impulsive control offers energy-saving advantages by enabling systems to possess discontinuous inputs. This disturbance can be characterized by impulsive effects which are distinguished between desynchronizing impulses and synchronizing impulses, based on their impact on the synchronization. Several studies have explored the synchronization of neural networks in the presence of impulsive effects as reported in [171, 172, 173, 174, 175]. Further, the impulsive transient depends not only on the current state but also on the historical state of the system. Such impulsive effects are called delayed impulses. The impact of delayed impulses on network synchronization has received great attention in recent years. In [176], the synchronization problem for uncertain coupled neural networks with time-varying delays of unknown bound via delayed impulsive control has been investigated and Liu et al. [170] discussed the synchronization of linear dynamical networks on time scales based on a pinning control via delayed impulsive effects. Moreover, all the above work have considered the impulsive synchronization problem in infinite time, and the impulsive synchronization problem for coupled neural networks with mixed delay in fixed-time has not yet been discussed.

This chapter is mainly concerned with fixed-time synchronization of the coupled neural networks with mixed delay via pinning impulsive control. The pinning impulsive control mechanism is designed to merely control partial nodes, but not all nodes, which not only saves resources but also improves communication efficiency. Then, based on the appropriate Lyapunov function and average impulsive interval, a new sufficient condition of the fixed-time synchronization criteria for coupled neural networks through the pinning impulsive control is derived. Meanwhile, the upper bound of the settling time is computed which is independent of the initial condition but depends on the parameters of impulsive sequence and continuous-time subsystems. Finally, the time-delayed Chua system is taken to show the effectiveness and feasibility of our proposed theoretical results.

5.2 Problem Formulation and Preliminaries

Consider the following coupled neural networks with M identical nodes defined as

$$\dot{x}_p(t) = Ax_p(t)+R(x_p(t), x_p(t-z_1(t)))+h_1\sum_{q=1}^{M} c_{pq}\Upsilon_1 x_q(t)+h_2\sum_{q=1}^{M} d_{pq}\Upsilon_2 x_q(t-z_2(t))+u_p(t), \tag{5.2.1}$$

where $x_p(t) = (x_{p1}, x_{p2}, \dots, x_{pq}) \in \mathbb{R}^n$ denote the pth state of the vector. $R(.)$ be the continuous vector-field function. $z_1(t)$ and $z_2(t)$ are time delays and satisfy $0 \le z_1(t) \le$

z_1, $0 \leq z_2(t) \leq z_2$, $0 \leq \dot{z}_1(t) \leq \hat{z}_1 < 1$, $0 \leq \dot{z}_2(t) \leq \hat{z}_2 < 1$. Let A be a n-dimensional matrix. h_1, $h_2 > 0$, Υ_1, $\Upsilon_2 \in \mathbb{R}^{n\times n}$ be the coupling strength and inner coupling matrices, respectively. $u_p(t)$ is the control input, which will be defined later on. c_{pq} and d_{pq} be the connection weights from the node q to p and satisfy following relations:

$$c_{pq} \geq 0, p \neq q, c_{pq} = -\sum_{q=1,q\neq p}^{M} c_{pq},$$
$$d_{pq} \geq 0, p \neq q, d_{pq} = -\sum_{q=1,q\neq p} d_{pq}.$$

Consider the following isolated system, where the parameters are the same as system (5.2.1):

$$\dot{s}(t) = As(t) + R(s(t), s(t - z_1(t))), \tag{5.2.2}$$

where $s(t) = [s_1(t), s_2(t), \cdots, s_n(t)]^T \in \mathbb{R}^n$.

The synchronization error state can be defined as $e_p(t) = x_p(t) - s(t)$, where $e_p(t) = [e_p^1(t), e_p^2(t), \cdots, e_p^n(t)]^T \in \mathbb{R}^n$, $p = 1, 2, \ldots, M$. Then we have the following error neural network:

$$\dot{e}_p(t) = Ae_p(t)+\tilde{R}(e_p(t), e_p(t-z_1(t)))+h_1\sum_{q=1}^{M} c_{pq}\Upsilon_1 e_q(t)+h_2\sum_{q=1}^{M} d_{pq}\Upsilon_2 e_q(t-z_2(t))+u_p(t), \tag{5.2.3}$$

where $\tilde{R}(e_p(t), e_p(t - z_1(t))) = R(x_p(t), x_p(t - z_1(t))) - R(s(t), s(t - z_1(t)))$. The initial condition of system (5.2.1) is $x_p(t) = \phi_p(t) \in C([-\max\{z_1, z_2\}, 0], \mathbb{R}^n)$.

To achieve fixed-time synchronization of the system (5.2.3), the controller should be constructed as follows:

$$u_p(t) = u_{p1}(t) + u_{p2}(t), \tag{5.2.4}$$

in which

$u_{p1} = \sum_{l=1}^{+\infty} r_l e_p(t)\delta(t - t_l),$

$u_{p2} = -k_1 e_p(t) - k_2 \text{sign}(e_p(t))|e_p(t)|^{\alpha_1} + k_3 \text{sign}(e_p(t))|e_p(t)|^{\alpha_2},$

where $0 < \alpha_1 < 1$, $\alpha_2 > 1$, positive constants k_1, k_2, k_3 are control gains. sign(.) be the signum function. r_l represents the impulsive strength that satisfies $|r_l| < 1$, and $\delta(.)$ is the Dirac delta function. Suppose that $\{t_l, l \in \mathbb{N}\}$ is the strictly increasing impulsive time sequence with $\lim_{l\to\infty} t_l = +\infty$.

Remark 5.2.1. The designed controller presented in system (5.2.3) is divided into two components as follows: the first part $u_{p1}(t)$ reflects the introduction of impulsive effects in the system (5.2.1), which is ideal for impulsive control. To get fixed-time synchronization under the impulses, the second part $u_{p2}(t)$ is crucial and relates to the comparison system.

Remark 5.2.2. This chapter proposes the pinning impulsive control protocol to assist networks in achieving synchronization in fixed-time. In other words, at each control instant, only a part of the nodes are under control. Therefore, this type of control can reduce the control costs. Furthermore, the pinned nodes are chosen based on their error values of the nodes rather than being picked at random, which has more practical and beneficial applications.

The error coupled neural networks can be defined as:

$$\begin{cases} \dot{e}_p(t) = Ae_p(t) + \tilde{R}(e_p(t), e_p(t - z_1(t))) + h_1 \sum_{q=1}^{M} c_{pq} \Upsilon_1 e_q(t) \\ \qquad +h_2 \sum_{q=1}^{M} d_{pq} \Upsilon_2 e_q(t - z_2(t)) + u_{p2}, \qquad t \neq t_l, \\ \Delta e_p(t_l) = e_p(t_l^+) - e_p(t_l^-) = r_l e_p(t_l^-),\ t = t_l,\ p \in \Omega(t_l), \\ \Delta e_p(t_l) = 0, p \notin \Omega(t_l), \end{cases} \tag{5.2.5}$$

The control set $\Omega(t_l)$ represents the set of nodes that need to be pinned at $t = t_l$. Depending on their size, we can rearrange the synchronization errors for the pinning control. Then, $||e_{p_1}(t)|| \geq ||e_{p_2}(t)|| \geq \cdots \geq ||e_{p_v}(t)|| \geq ||e_{p_{v+1}}(t)|| \geq \cdots \geq ||e_{p_M}(t)||$, where $p_v \in \{1, 2, \ldots\}$, $v = 1, 2, \ldots, M$. Adding the first v nodes to the set of controlled nodes $\Omega(t_l)$ allows us to control only nodes with large error values. Further, suppose that $e_p(t)$ is right continuous at time $t = t_l$, i.e., $e_p(t_l^+) = \lim_{t \to t_l^+} e_p(t_l) = e_p(t_l)$ and $\Delta e_p(t_l) = e_p(t_l^+) - e_p(t_l^-) = (1 + r_l)e_p(t_l^-)$.

Assumption 5.2.1. For any $a(t)$, $b(t) \in \mathbb{R}^n$, the nonlinear function satisfies

$$(R(a(t), a(t - z_1(t))) - R(b(t), b(t - z_1(t))))^T (R(a(t), a(t - z_1(t))) - R(b(t), b(t - z_1(t)))$$
$$\leq m_1(a(t) - b(t))^T (a(t) - b(t)) + m_2(a(t - g_1(t)) - b(t - g_1(t)))^T (a(t - g_1(t)) - b(t - g_1(t))).$$

5.3 Fixed-Time Synchronization

In this section, we derive fixed-time synchronization criteria for the system (5.2.1) in the presence of synchronizing and desynchronizing impulses through the designed pinning impulsive controller (5.2.4).

Theorem 5.3.1. Assume that Assumption 5.2.1 holds, and the impulsive sequence $\zeta = \{t_l, l \in \mathbb{N}\}$ has an average impulsive interval τ_a with the elasticity number N_0. If there exists a control gain k_1 such that the following inequality holds

$$k_1 \geq \frac{1}{2}\lambda_{max}(A+A^T)+\frac{1}{2}+m_1+h_1\lambda_{max}(C\otimes\Upsilon_1)+\frac{h_2^2}{4}\lambda_{max}(DD^T\otimes\Upsilon_2\Upsilon_2^T)+\frac{m_2}{(1-\hat{z}_1)}+\frac{1}{4(1-\hat{z}_2)},$$

then under the pinning impulsive controller (5.2.4), we have

(i) when $0 < \theta < 1$, the error coupled neural networks (5.2.5) achieve fixed-time synchronization and settling-time is given by

$$\mathrm{T} = \frac{\ln\left[1 + \frac{\theta^{\mathrm{N}_0(1-\beta_0)}(\hat{\mathrm{k}}_1 - \frac{\ln\theta}{\tau_\mathrm{a}})}{\hat{\mathrm{k}}_3}\right]}{(\hat{\mathrm{k}}_1 - \frac{\ln\theta}{\tau_\mathrm{a}})(\beta_0 - 1)} + \frac{\ln\left[\frac{\hat{\mathrm{k}}_2\theta^{\mathrm{N}_0(1-\alpha_0)}}{\theta^{-\mathrm{N}_0(1-\alpha_0)}(\hat{\mathrm{k}}_1 - \frac{\ln\theta}{\tau_\mathrm{a}}) + \hat{\mathrm{k}}_2\theta^{\mathrm{N}_0(1-\alpha_0)}}\right]}{(\frac{\ln\theta}{\tau_\mathrm{a}} - \hat{\mathrm{k}}_1)(1-\alpha_0)}.$$

(ii) when $\theta = 1$, the error coupled neural networks (5.2.5) achieve fixed-time synchronization and settling-time is given by

$$\mathrm{T}' = \frac{\ln\left[1 + \frac{\hat{\mathrm{k}}_1}{\hat{\mathrm{k}}_3}\right]}{\hat{\mathrm{k}}_1(\beta_0 - 1)} - \frac{\ln\left[\frac{\hat{\mathrm{k}}_2}{\hat{\mathrm{k}}_1 + \hat{\mathrm{k}}_2}\right]}{\hat{\mathrm{k}}_2(1-\alpha_0)}.$$

(iii) when $\theta > 1$ and $\hat{k}_1 > \frac{\ln\theta}{\tau_a}$, the error coupled neural networks (5.2.5) achieve fixed-time synchronization and settling-time is given by

$$\mathrm{T}'' = \frac{\ln\left[1 + \frac{(\hat{k}_1 - \frac{\ln\theta}{\tau_a})}{\hat{k}_3\theta^{N_0(1-\beta_0)}}\right]}{(\hat{k}_1 - \frac{\ln\theta}{\tau_a})(\beta_0 - 1)} + \frac{\ln\left[\frac{\hat{k}_2\theta^{-N_0(1-\alpha_0)}}{\theta^{N_0(1-\alpha_0)}(\hat{k}_1 - \frac{\ln\theta}{\tau_a}) + \hat{k}_2\theta^{-N_0(1-\alpha_0)}}\right]}{(\frac{\ln\theta}{\tau_a} - \hat{k}_1)(1-\alpha_0)},$$

where $\theta = (1 + r_l)^2 + \frac{M-v}{M}$.

Proof. Consider the following Lyapunov function as

$$W(t) = \frac{1}{2}\sum_{p=1}^{M} e_p^T(t)e_p(t) + \frac{m_2}{(1-\hat{z}_1)}\sum_{p=1}^{M}\int_{t-z_1(t)}^{t} e_p^T(s)e_p(s)ds + \frac{1}{4(1-\hat{z}_2)}\sum_{p=1}^{M}\int_{t-g_2(t)}^{t} e_p^T(s)e_p(s)ds. \quad (5.3.1)$$

Differentiating $W(t)$ along with the error coupled neural network (5.2.5) when $t \neq t_l$, we get

$$
\begin{aligned}
\dot{W}(t) =& \sum_{p=1}^{M} e_p^T(t)(\frac{A+A^T}{2})e_p(t) + \sum_{p=1}^{M} e_p^T(t)(\tilde{R}(e_p(t), e_p(t-z_1(t))) \\
&+ h_1 \sum_{q=1}^{M} c_{pq}\Upsilon_1 e_q(t) + h_2 \sum_{q=1}^{M} d_{pq}\Upsilon_2 e_q(t-z_2(t)) - k_1 e_p(t) \\
&- k_2 \text{sign}(e_p(t))|e_p(t)|^{\alpha_1} + k_3 \text{sign}(e_p(t))|e_p(t)|^{\alpha_2}) \\
&+ \frac{m_2}{(1-\hat{z}_1)} \sum_{p=1}^{M} e_p^T(t)e_p(t) + \frac{1}{4(1-\hat{z}_2)} \sum_{p=1}^{M} e_p^T(t)e_p(t) \\
&- m_2 \sum_{p=1}^{M} e_p^T(t-z_1(t))e_p(t-z_1(t)) - \frac{1}{4}\sum_{p=1}^{M} e_p^T(t-z_2(t))e_p(t-z_2(t)). \qquad (5.3.2)
\end{aligned}
$$

Based on Assumption 5.2.1, it can be obtained that

$$
\begin{aligned}
\dot{W}(t) \leq& \frac{1}{2}\lambda_{max}(A+A^T)e^T(t)e(t) + \frac{1}{2}e^T(t)e(t) + m_1 e^T(t)e(t) \\
&+ m_2 e^T(t-z_1(t))e(t-z_1(t)) + h_1 e^T(t)(C \otimes \Upsilon_1)e(t) \\
&+ \frac{h_2^2}{4}e^T(t)(DD^T \otimes \Upsilon_2\Upsilon_2^T)e(t) + \frac{1}{4}e^T(t-z_2(t))e(t-z_2(t)) \\
&- k_1 e^T(t)e(t) - k_2 e^T(t)|e(t)|^{\alpha_1} - k_2 e^T(t)|e(t)|^{\alpha_2} \\
&+ \frac{m_2}{(1-\hat{z}_1)}e^T(t)e(t) + \frac{1}{4(1-\hat{z}_2)}e^T(t)e(t) \\
&- m_2 e^T(t-z_1(t))e(t-z_1(t)) - \frac{1}{4}e^T(t-z_2(t))e(t-z_2(t)). \qquad (5.3.3)
\end{aligned}
$$

From Lemma 3.2.1, it can be followed that

$$
\begin{aligned}
\dot{W}(t) \leq& (\frac{1}{2}\lambda_{max}(A+A^T) + \frac{1}{2} + m_1 + h_1\lambda_{max}(C \otimes \Upsilon_1) + \frac{h_2^2}{4}\lambda_{max}(DD^T \otimes \Upsilon_2\Upsilon_2^T) \\
&+ \frac{m_2}{(1-\hat{z}_1)} + \frac{1}{4(1-\hat{z}_2)} - k_1)W(t) - k_2(W(t))^{\frac{1+\alpha_1}{2}} + k_3(M)^{1-\alpha_2}(W(t))^{\frac{1+\alpha_2}{2}}.
\end{aligned}
$$

Let $\hat{k}_1 = k_1 - \frac{1}{2}\lambda_{max}(A+A^T) - \frac{1}{2} - m_1 - h_1\lambda_{max}(C\otimes\Upsilon_1) - \frac{h_2^2}{4}\lambda_{max}(DD^T\otimes\Upsilon_2\Upsilon_2^T) - \frac{m_2}{(1-\hat{z}_1)} - \frac{1}{4(1-\hat{z}_2)}$, $\hat{k}_2 = k_2$, $\hat{k}_3 = k_3(M)^{1-\alpha_2}$, $\alpha_0 = \frac{1+\alpha_1}{2}$ and $\beta_0 = \frac{1+\alpha_2}{2}$ in the above expression, so that we have

$$\dot{W}(t) \leq -\hat{k}_1 W(t) - \hat{k}_2 (W(t))^{\alpha_0} - \hat{k}_3 (W(t))^{\beta_0}. \tag{5.3.4}$$

On the other hand, when $t = t_l$, we have from Eq. (5.3.1) that

$$\begin{aligned} W(t_l^+) &= \frac{1}{2}\sum_{p=1}^{M} e_p^T(t_l^+)e_p(t_l^+) + \frac{m_2}{(1-\hat{z}_1)}\sum_{p=1}^{M}\int_{t_l - z_1(t)}^{t_l} e_p^T(s)e_p(s)ds \\ &+ \frac{1}{4(1-\hat{z}_2)}\sum_{p=1}^{M}\int_{t_l - g_2(t)}^{t_l} e_p^T(s)e_p(s)ds \\ &= \frac{1}{2}\sum_{p\in\Omega(t_l)} e_p^T(t_l^+)e_p(t_l^+) + \frac{m_2}{(1-\hat{z}_1)}\sum_{p\in\Omega(t_l)}\int_{t_l - z_1(t)}^{t_l} e_p^T(s)e_p(s)ds \\ &+ \frac{1}{4(1-\hat{z}_2)}\sum_{p\in\Omega(t_l)}^{M}\int_{t_l - g_2(t)}^{t_l} e_p^T(s)e_p(s)ds \\ &+ \frac{1}{2}\sum_{p\notin\Omega(t_l)} e_p^T(t_l^+)e_p(t_l^+) + \frac{m_2}{(1-\hat{z}_1)}\sum_{p\notin\Omega(t_l)}\int_{t_l - z_1(t)}^{t_l} e_p^T(s)e_p(s)ds \\ &+ \frac{1}{4(1-\hat{z}_2)}\sum_{p\notin\Omega(t_l)}^{M}\int_{t_l - g_2(t)}^{t_l} e_p^T(s)e_p(s)ds. \end{aligned} \tag{5.3.5}$$

According to the pinning control, we have

$$\sum_{p\notin\Omega(t_l)}\int_{t_l - z_1(t)}^{t_l} e_p^T(s)e_p(s)ds \leq \frac{M-v}{M}\sum_{p=1}^{M}\int_{t_l - z_1(t)}^{t_l} e_p^T(s)e_p(s)ds,$$

$$\sum_{p\notin\Omega(t_l)}^{M}\int_{t_l - g_2(t)}^{t_l} e_p^T(s)e_p(s)ds \leq \frac{M-v}{M}\sum_{p=1}^{M}\int_{t_l - z_2(t)}^{t_l} e_p^T(s)e_p(s)ds,$$

$$\sum_{p\notin\Omega(t_l)} e_p^T(t_l^-)e_p(t_l^-) \leq \frac{M-v}{M}\sum_{p=1}^{M} e_p^T(t_l^-)e_p(t_l^-).$$

Then from Eq. (5.3.5), we get

$$\begin{aligned}
W(t_l^+) \leq& \frac{(1+r_l)^2}{2} \sum_{p\in\Omega(t_l)} e_p^T(t_l^-)e_p(t_l^-) \\
&+ \frac{(1+r_l)^2 m_2}{(1-\hat{z}_1)} \sum_{p\in\Omega(t_l)} \int_{t_l - z_1(t)}^{t_l} e_p^T(s)e_p(s)ds \\
&+ \frac{(1+r_l)^2}{4(1-\hat{z}_2)} \sum_{p\in\Omega(t_l)}^{M} \int_{t_l - g_2(t)}^{t_l} e_p^T(s)e_p(s)ds \\
&+ \frac{M-v}{2M} \sum_{p=1}^{M} e_p^T(t_l^-)e_p(t_l^-) \\
&+ \frac{m_2(M-v)}{M(1-\hat{z}_1)} \sum_{p=1}^{M} \int_{t_l - z_1(t)}^{t_l} e_p^T(s)e_p(s)ds \\
&+ \frac{M-v}{4M(1-\hat{z}_2)} \sum_{p=1}^{M} \int_{t_l - z_2(t)}^{t_l} e_p^T(s)e_p(s)ds,
\end{aligned}$$

$$\begin{aligned}
W(t_l^+) \leq& ((1+r_l)^2 + \frac{M-v}{M})(\frac{1}{2} \sum_{p=1}^{M} e_p^T(t_l^-)e_p(t_l^-) \\
&+ \frac{m_2}{(1-\hat{z}_1)} \sum_{p=1}^{M} \int_{t_l - z_1(t)}^{t_l} e_p^T(s)e_p(s)ds \\
&+ \frac{1}{4(1-\hat{z}_2)} \sum_{p=1}^{M} \int_{t_l - z_2(t)}^{t_l} e_p^T(s)e_p(s)ds) \\
=& \theta W(t_l^-),
\end{aligned} \tag{5.3.6}$$

where $\theta = (1+r_l)^2 + \frac{M-v}{M}$.

Now considering the following comparison system as

$$\begin{cases} \dot{\varphi}(t) = \begin{cases} -\hat{k}_1\varphi(t) - \hat{k}_2(\varphi(t))^{\alpha_0}, 0 < \varphi(t) < 1, & t \neq t_l, \\ -\hat{k}_1\varphi(t) - \hat{k}_3(\varphi(t))^{\beta_0}, \varphi(t) \geq 1, & t \neq t_l, \\ 0, \varphi(t) = 0, t \neq t_l, \end{cases} \\ \varphi(t_l) = \theta\varphi(t_l^-), t = t_l, \\ \varphi(0) = \varphi_0 = W(0). \end{cases} \tag{5.3.7}$$

In view of Eq. (5.3.4) and Eq. (5.3.6), a conclusion can be drawn that $0 \leq W(t) \leq \varphi(t)$, $t \geq 0$. Further, if there exists $\mathrm{T} > 0$ such that $\varphi(t) \equiv 0$, for $t \geq \mathrm{T}$, then $W(t) \equiv 0$, for $t \geq \mathrm{T}$. Hence the stability of Eq. (5.3.7) implies that stability of zero solution of system (5.2.5). Therefore, the fixed-time synchronization of the system (5.2.5) can be achieved.

From Eq. (5.3.7), the problem is divided into following two parts: if there exist T_1 and T_2 such that $\varphi(t)$ tends to 0 at time T_1 and from 1 to 0 at time T_2. Thus $\varphi(t)$ tends to 0 in the fixed-time $\mathrm{T}_1 + \mathrm{T}_2$. In order to get T_1 and T_2, we can divide the proof into following three cases: $0 < \theta < 1$, $\theta = 1$ and $\theta > 1$.

(i) Case 1: $0 < \theta < 1$.

When $\varphi(t) \geq 1$, assume that $\pi(t) = \varphi^{1-\beta_0}(t)$. Therefore we can get $\pi(t) \to 1$ when $\varphi(t) \to 1$ and $\pi(t) \to 0$ when $\varphi(t) \to +\infty$. Then, we have

$$\begin{cases} \dot{\pi}(t) = \hat{k}_1(\beta_0 - 1)\pi(t) + \hat{k}_3(\beta_0 - 1), t \neq t_l, \quad 0 < \pi(t) \leq 1, \\ \pi(t_l) = \theta_1\pi(t_l^-), t = t_l, \\ \pi(0) = \varphi_0^{1-\beta_0}, \end{cases} \tag{5.3.8}$$

where $\theta_1 = \theta^{1-\beta_0} \in [1, \infty)$, we can derive from Eq. (5.3.8)

$$\pi(t) = e^{\hat{k}_1(\beta_0-1)t}\theta_1^{N_\varsigma(t,0)}\pi(0) + \hat{k}_3(\beta_0-1)\int_0^t e^{\hat{k}_1(\beta_0-1)(t-s)}\theta_1^{N_\varsigma(t,s)}ds. \tag{5.3.9}$$

Since $\pi(0) = e^{\hat{k}_1(\beta_0-1)t}\theta_1^{N_\varsigma(t,0)}\pi(0) = \pi_0 < 1$, $\lim_{t\to+\infty}\pi(t) = +\infty$ and $\pi(t)$ is monotonically increasing when $t \geq 0$, there exists T_1 such that $\lim_{t\to\mathrm{T}_1}\pi(t) = 1$ and $0 < \pi(t) < 1$ for $0 < t < \mathrm{T}_1$. From Eq. (5.3.9), one can obtain

$$e^{\hat{k}_1(\beta_0-1)t}\theta_1^{N_\varsigma(t,0)}\pi(0) + \hat{k}_3(\beta_0-1)\int_0^t e^{\hat{k}_1(\beta_0-1)(t-s)}\theta_1^{N_\varsigma(t,s)}ds = 1. \tag{5.3.10}$$

Therefore, we find

$$\hat{k}_3(\beta_0-1)\int_0^t e^{\hat{k}_1(\beta_0-1)(t-s)}\theta_1^{N_\varsigma(t,s)}ds \leq 1. \tag{5.3.11}$$

From Definition 2.1.2.4 one can get $\theta_1^{\frac{t-s}{\tau_a}-N_0} \leq \theta_1^{N_\varsigma(t,s)}$, the above inequality directly follows that

$$t \leq \frac{\ln\left[1 + \frac{\theta_1^{N_0}(\hat{k}_1(\beta_0-1)+\frac{\ln\theta_1}{\tau_a})}{\hat{k}_3(\beta_0-1)}\right]}{(\hat{k}_1(\beta_0-1)+\frac{\ln\theta_1}{\tau_a})} = \mathrm{T}_1.$$

Substituting $\theta_1 = \theta^{1-\beta_0}$ in the above inequality, we have

$$\mathrm{T}_1 = \frac{\ln\left[1 + \frac{\theta^{N_0(1-\beta_0)}(\hat{k}_1-\frac{\ln\theta}{\tau_a})}{\hat{k}_3}\right]}{(\hat{k}_1-\frac{\ln\theta}{\tau_a})(\beta_0-1)}. \tag{5.3.12}$$

When $t \to \mathrm{T}_1$, we can obtain $\pi(t) \to 1$ which implies $\varphi(t) \to 1$.

Next, we have to estimate T_2 such that $\varphi(t)$ tends to 0 from 1. For this, it is assumed that $\pi(t) = \varphi^{1-\alpha_0}(t)$, when $0 < \varphi(t) < 1$. It can be found that $\pi(t) \to 1$ when $\varphi(t) \to 1$

and $\pi(t) \to 0$ when $\varphi(t) \to 1$, thus we have

$$\begin{cases} \dot{\pi}(t) = -\hat{k}_1(1-\alpha_0)\pi(t) - \hat{k}_2(1-\alpha_0)\pi(t), t \neq t_l, \quad 0 \leq \pi(t) < 1, \\ \pi(t_l) = \theta_2 \pi(t_l^-), t = t_l, \\ \pi(\mathrm{T}_1) = \varphi^{1-\alpha_0}(\mathrm{T}_1) = 1, \end{cases} \tag{5.3.13}$$

where $\theta_2 = \theta^{1-\alpha_0} \in (0,1)$. Similar to Eq. (5.3.9), we have from Eq. (5.3.13) that

$$\pi(t) = e^{-\hat{k}_1(1-\alpha_0)(t-\mathrm{T}_1)} \theta_2^{N_\zeta(t,\mathrm{T}_1)} \pi(\mathrm{T}_1) - \hat{k}_2(1-\alpha_0) \int\limits_{\mathrm{T}_1}^{t} e^{-\hat{k}_1(1-\alpha_0)(t-s)} \theta_2^{N_\zeta(t,s)} ds. \tag{5.3.14}$$

When $0 < \theta_2 < 1$, it follows from Definition 2.1.2.4 that $\theta_2^{\frac{t-s}{\tau_a}+N_0} \leq \theta_2^{N_\zeta(t,s)} \leq \theta_2^{\frac{t-s}{\tau_a}-N_0}$. Hence, by using $\pi(\mathrm{T}_1) = 1$, we can obtain from above equation that

$$\begin{aligned} \pi(t) &= e^{-\hat{k}_1(1-\alpha_0)(t-\mathrm{T}_1)} \theta_2^{N_\zeta(t,\mathrm{T}_1)} - \hat{k}_2(1-\alpha_0) \int\limits_{\mathrm{T}_1}^{t} e^{-\hat{k}_1(1-\alpha_0)(t-s)} \theta_2^{N_\zeta(t,\mathrm{T}_1)} ds \\ &\leq e^{-\hat{k}_1(1-\alpha_0)(t-\mathrm{T}_1)} \theta_2^{\frac{t-\mathrm{T}_1}{\tau_a}-N_0} - \hat{k}_2(1-\alpha_0) \int\limits_{\mathrm{T}_1}^{t} e^{-\hat{k}_1(1-\alpha_0)(t-s)} \theta_2^{\frac{t-s}{\tau_a}+N_0} ds \\ &= e^{-(\hat{k}_1(1-\alpha_0)-\frac{\ln\theta_2}{\tau_a})(t-\mathrm{T}_1)} \left[\theta_2^{-N_0} + \frac{\hat{k}_2 \theta_2^{N_0}(1-\alpha_0)}{(\hat{k}_1(1-\alpha_0)-\frac{\ln\theta_2}{\tau_a})} \right] - \frac{\hat{k}_2 \theta_2^{N_0}(1-\alpha_0)}{(\hat{k}_1(1-\alpha_0)-\frac{\ln\theta_2}{\tau_a})}. \end{aligned} \tag{5.3.15}$$

Let $M(t) = e^{-(\hat{k}_1(1-\alpha_0)-\frac{\ln\theta_2}{\tau_a})(t-\mathrm{T}_1)} \left[\theta_2^{-N_0} + \frac{\hat{k}_2 \theta_2^{N_0}(1-\alpha_0)}{(\hat{k}_1(1-\alpha_0)-\frac{\ln\theta_2}{\tau_a})} \right] - \frac{\hat{k}_2 \theta_2^{N_0}(1-\alpha_0)}{(\hat{k}_1(1-\alpha_0)-\frac{\ln\theta_2}{\tau_a})}$, then $M(\mathrm{T}_1) = \theta_2^{-N_0}$, $M(+\infty) = -\frac{\hat{k}_2 \theta_2^{N_0}(1-\alpha_0)}{(\hat{k}_1(1-\alpha_0)-\frac{\ln\theta_2}{\tau_a})} < 0$ and $\dot{M}(t) < 0$. Therefore by the concept of calculus, there exists a unique T_2 such that $M(\mathrm{T}_2) = 0$. Then from Eq. (5.3.15) we have

$$t - \mathrm{T}_1 = \frac{\ln\left[\frac{\hat{k}_2 \theta_2^{N_0}(1-\alpha_0)}{\theta_2^{-N_0}(\hat{k}_1(1-\alpha_0)-\frac{\ln\theta_2}{\tau_a})+\hat{k}_2\theta_2^{N_0}(1-\alpha_0)} \right]}{(\frac{\ln\theta_2}{\tau_a} - \hat{k}_1(1-\alpha_0))} = \mathrm{T}_2.$$

Substituting $\theta_2 = \theta^{1-\alpha_0}$ into the above inequality, we get

$$\mathrm{T}_2 = \frac{\ln\left[\frac{\hat{k}_2\theta^{N_0(1-\alpha_0)}}{\theta^{-N_0(1-\alpha_0)}(\hat{k}_1-\frac{\ln\theta}{\tau_a})+\hat{k}_2\theta^{N_0(1-\alpha_0)}}\right]}{(\frac{\ln\theta}{\tau_a}-\hat{k}_1)(1-\alpha_0)}. \tag{5.3.16}$$

Hence, when $0 < \theta < 1$, we have $\varphi(t) \equiv 0$ for the fixed-time T given by

$$\mathrm{T} = \mathrm{T}_1 + \mathrm{T}_2 = \frac{\ln\left[1 + \frac{\theta^{\mathrm{N}_0(1-\beta_0)}(\hat{\mathrm{k}}_1-\frac{\ln\theta}{\tau_\mathrm{a}})}{\hat{\mathrm{k}}_3}\right]}{(\hat{\mathrm{k}}_1 - \frac{\ln\theta}{\tau_\mathrm{a}})(\beta_0-1)} + \frac{\ln\left[\frac{\hat{\mathrm{k}}_2\theta^{\mathrm{N}_0(1-\alpha_0)}}{\theta^{-\mathrm{N}_0(1-\alpha_0)}(\hat{\mathrm{k}}_1-\frac{\ln\theta}{\tau_\mathrm{a}})+\hat{\mathrm{k}}_2\theta^{\mathrm{N}_0(1-\alpha_0)}}\right]}{(\frac{\ln\theta}{\tau_\mathrm{a}}-\hat{\mathrm{k}}_1)(1-\alpha_0)}. \tag{5.3.17}$$

(ii) Case 2: $\theta = 1$.

It is evident to note that $\theta_1 = 1$ and $\theta_2 = 1$. Applying similar approach as given above, Eq. (5.3.9) becomes

$$\pi(t) = e^{\hat{k}_1(\beta_0-1)t}\pi(0) + \hat{k}_3(\beta_0-1)\int_0^t e^{\hat{k}_1(\beta_0-1)(t-s)}ds.$$

Then, we have

$$\hat{k}_3(\beta_0-1)\int_0^t e^{\hat{k}_1(\beta_0-1)(t-s)}ds \le 1,$$

$$t \le \frac{\ln\left[1+\frac{\hat{k}_1}{\hat{k}_3}\right]}{\hat{k}_1(\beta_0-1)} = \mathrm{T}_1^{'}.$$

Since $\pi(\mathrm{T}_1^{'}) = 1$ and $\pi(\mathrm{T}_2^{'}) = 0$, therefore it follows from (5.3.14) that

$$\pi(t) = e^{-\hat{k}_1(1-\alpha_0)(t-\mathrm{T}_1)} - \hat{k}_2(1-\alpha_0)\int_{\mathrm{T}_1}^t e^{-\hat{k}_1(1-\alpha_0)(t-s)}ds,$$

$$\pi(T_2^{'}) = e^{-\hat{k}_1(1-\alpha_0)(t-\mathrm{T}_1)} - \hat{k}_2(1-\alpha_0)\int_{\mathrm{T}_1}^t e^{-\hat{k}_1(1-\alpha_0)(t-s)}ds = 0.$$

Solving above inequality, we can get

$$\mathrm{T}_2^{'} = \frac{\ln\left[\frac{\hat{k}_2}{\hat{k}_1+\hat{k}_2}\right]}{-\hat{k}_2(1-\alpha_0)}.$$

Therefore, when $\theta = 1$, it can be concluded that $\varphi(t) \equiv 0$ for the fixed-time $\mathrm{T}^{'}$ given by

$$\mathrm{T}^{'} = \mathrm{T}_1^{'} + \mathrm{T}_2^{'} = \frac{\ln\left[1+\frac{\hat{k}_1}{\hat{k}_3}\right]}{\hat{k}_1(\beta_0-1)} - \frac{\ln\left[\frac{\hat{k}_2}{\hat{k}_1+\hat{k}_2}\right]}{\hat{k}_2(1-\alpha_0)}. \tag{5.3.18}$$

(iii) Case 3: $\theta > 1$.

When $\varphi(t) \geq 1$, assume $\pi(t) = \varphi^{1-\beta_0}(t)$. Therefore we can get $\pi(t) \to 1$ when $\varphi(t) \to 1$ and $\pi(t) \to 0$ when $\varphi(t) \to +\infty$. Then, we have

$$\begin{cases} \dot{\pi}(t) = \hat{k}_1(\beta_0-1)\pi(t) + \hat{k}_3(\beta_0-1), t \neq t_l, \quad 0 < \pi(t) \leq 1, \\ \pi(t_l) = \theta_1\pi(t_l^-),\ t = t_l, \\ \pi(0) = \varphi_0^{1-\beta_0}, \end{cases} \tag{5.3.19}$$

where $\theta_1 = \theta^{1-\beta_0} \in (0,1)$, we can derive from Eq. (5.3.19) that

$$\pi(t) = e^{\hat{k}_1(\beta_0-1)t}\theta_1^{N_\zeta(t,0)}\pi(0) + \hat{k}_3(\beta_0-1)\int_0^t e^{\hat{k}_1(\beta_0-1)(t-s)}\theta_1^{N_\zeta(t,s)}ds. \tag{5.3.20}$$

From Eq. (5.3.20) and $0 < \theta_1 < 1$, there exists $\mathrm{T}_1^{''}$ such that $\lim_{t\to\mathrm{T}_1^{''}}\pi(t) = 1$ and $0 < \pi(t) < 1$ for $0 < t < \mathrm{T}_1^{''}$. Also from Definition 2.1.2.4, we can have

$$\hat{k}_3(\beta_0-1)\int_0^t e^{\hat{k}_1(\beta_0-1)(t-s)}\theta_1^{N_\zeta(t,s)}ds \leq 1,$$

$$i.e.,\ \hat{k}_3(\beta_0-1)\int_0^t e^{\hat{k}_1(\beta_0-1)(t-s)}\theta_1^{\frac{t-s}{\tau_a}+N_0}ds \leq 1.$$

After solving the above inequality, we obtain that

$$t \leq \frac{\ln\left[1 + \frac{(\hat{k}_1(\beta_0-1)+\frac{\ln\theta_1}{\tau_a})}{\hat{k}_3(\beta_0-1)\theta_1^{N_0}}\right]}{(\hat{k}_1(\beta_0-1)+\frac{\ln\theta_1}{\tau_a})}. \tag{5.3.21}$$

Substituting $\theta_1 = \theta^{1-\beta_0}$ in Eq. (5.3.21), we have

$$\mathrm{T}_1'' = \frac{\ln\left[1 + \frac{(\hat{k}_1-\frac{\ln\theta}{\tau_a})}{\hat{k}_3\theta^{N_0(1-\beta_0)}}\right]}{(\hat{k}_1-\frac{\ln\theta}{\tau_a})(\beta_0-1)}. \tag{5.3.22}$$

When $t \to \mathrm{T}_1''$, then $\pi(t) \to 1$ which implies $\varphi(t) \to 1$. Next, to estimate the T_2'' such that $\varphi(t)$ tends to 0 from 1, we assume that $\pi(t) = \varphi^{1-\alpha_0}(t)$, when $0 < \varphi(t) < 1$. It can therefore be obtained that $\pi(t) \to 1$ when $\varphi(t) \to 1$ and $\pi(t) \to 0$ when $\varphi(t) \to 0$. Hence, we have

$$\begin{cases} \dot{\pi}(t) = -\hat{k}_1(1-\alpha_0)\pi(t) - \hat{k}_2(1-\alpha_0)\pi(t), t \neq t_l, \quad 0 \leq \pi(t) < 1, \\ \pi(t_l) = \theta_2\pi(t_l^-),\ t = t_l, \\ \pi(\mathrm{T}_1'') = \varphi^{1-\alpha_0}(\mathrm{T}_1'') = 1, \end{cases} \tag{5.3.23}$$

where $\theta_2 = \theta^{1-\alpha_0} \in (1, \infty)$. Similar to Eq. (5.3.20), we find from Eq. (5.3.23) that

$$\pi(t) = e^{-\hat{k}_1(1-\alpha_0)(t-\mathrm{T}_1'')}\theta_1^{N_\zeta(t,\mathrm{T}_1'')}\pi(\mathrm{T}_1'') - \hat{k}_2(1-\alpha_0)\int_{\mathrm{T}_1''}^{t} e^{-\hat{k}_1(1-\alpha_0)(t-s)}\theta_1^{N_\zeta(t,s)}ds. \tag{5.3.24}$$

When $1 < \theta_2 < \infty$, from the Definition 2.1.2.4 it follows that $\theta_2^{\frac{t-s}{\tau_a}-N_0} \leq \theta_2^{N_\zeta(t,s)} \leq \theta_2^{\frac{t-s}{\tau_a}+N_0}$. Hence, from Eq. (5.3.24), and since $\pi(\mathrm{T}_1'') = 1$, we have

$$\begin{aligned}\pi(t) &= e^{-\hat{k}_1(1-\alpha_0)(t-\mathrm{T}_1)}\theta_2^{N_\varsigma(t,\mathrm{T}_1^{''})} - \hat{k}_2(1-\alpha_0)\int\limits_{\mathrm{T}_1^{''}}^{t} e^{-\hat{k}_1(1-\alpha_0)(t-s)}\theta_2^{N_\varsigma(t,s)}ds \\ &\le e^{-\hat{k}_1(1-\alpha_0)(t-\mathrm{T}_1)}\theta_2^{\frac{t-\mathrm{T}_1^{''}}{\tau_a}+N_0} - \hat{k}_2(1-\alpha_0)\int\limits_{\mathrm{T}_1^{''}}^{t} e^{-\hat{k}_1(1-\alpha_0)(t-s)}\theta_2^{\frac{t-s}{\tau_a}-N_0}ds \\ &= e^{-(\hat{k}_1(1-\alpha_0)-\frac{\ln\theta_2}{\tau_a})(t-\mathrm{T}_1^{''})}\left[\theta_2^{N_0} + \frac{\hat{k}_2\theta_2^{-N_0}(1-\alpha_0)}{(\hat{k}_1(1-\alpha_0)-\frac{\ln\theta_2}{\tau_a})}\right] - \frac{\hat{k}_2\theta_2^{-N_0}(1-\alpha_0)}{(\hat{k}_1(1-\alpha_0)-\frac{\ln\theta_2}{\tau_a})}.\end{aligned} \tag{5.3.25}$$

Now, $M(\mathrm{T}''_1) = \theta_2^{N_0}$, $M(+\infty) = -\frac{\hat{k}_2\theta_2^{-N_0}(1-\alpha_0)}{(\hat{k}_1(1-\alpha_0)-\frac{\ln\theta_2}{\tau_a})} < 0$ and $\dot{M}(t) < 0$. Therefore by the concept of calculus, there exists a unique $\mathrm{T}_2^{''}$ such that $M(\mathrm{T}_2^{''}) = 0$. Then from Eq. (5.3.15), we have

$$t - \mathrm{T}_1^{''} = \frac{\ln\left[\frac{\hat{k}_2\theta_2^{-N_0}(1-\alpha_0)}{\theta_2^{N_0}(\hat{k}_1(1-\alpha_0)-\frac{\ln\theta_2}{\tau_a})+\hat{k}_2\theta_2^{-N_0}(1-\alpha_0)}\right]}{(\frac{\ln\theta_2}{\tau_a} - \hat{k}_1(1-\alpha_0))} = \mathrm{T}_2^{''}.$$

Substituting $\theta_2 = \theta^{1-\alpha_0}$ into the above inequality, we have

$$\mathrm{T}_2^{''} = \frac{\ln\left[\frac{\hat{k}_2\theta^{-N_0(1-\alpha_0)}}{\theta^{N_0(1-\alpha_0)}(\hat{k}_1-\frac{\ln\theta}{\tau_a})+\hat{k}_2\theta^{-N_0(1-\alpha_0)}}\right]}{(\frac{\ln\theta}{\tau_a} - \hat{k}_1)(1-\alpha_0)}.$$

Hence, when $0 < \theta < 1$, we have $\varphi(t) \equiv 0$ for the fixed-time $\mathrm{T}^{''}$ given by

$$T^{''} = \mathrm{T}_1^{''} + \mathrm{T}_2^{''} = \frac{\ln\left[1 + \frac{(\hat{k}_1-\frac{\ln\theta}{\tau_a})}{\hat{k}_3\theta^{N_0(1-\beta_0)}}\right]}{(\hat{k}_1 - \frac{\ln\theta}{\tau_a})(\beta_0 - 1)} + \frac{\ln\left[\frac{\hat{k}_2\theta^{-N_0(1-\alpha_0)}}{\theta^{N_0(1-\alpha_0)}(\hat{k}_1-\frac{\ln\theta}{\tau_a})+\hat{k}_2\theta^{-N_0(1-\alpha_0)}}\right]}{(\frac{\ln\theta}{\tau_a} - \hat{k}_1)(1-\alpha_0)}. \tag{5.3.26}$$

From the Definition 3.2.2, we can conclude that the error coupled neural networks (5.2.5) can achieve fixed-time synchronization in above all cases. Therefore the proof of Theorem 5.3.1 is completed.

Remark 5.3.1. The sufficient conditions are discussed to achieve synchronization in fixed-time for a coupled neural network (5.2.1) with mixed delays for synchronizing and desynchronizing impulses. The settling-time is independent of the initial state. However, it is related to the controller parameters, impulsive strength, average impulsive interval and pinning rule. Consequently, it differs from finite-time synchronization.

Remark 5.3.2. Theorem 5.3.1 shows that we have established a sufficient condition $\hat{k}_1 > \frac{\ln\theta}{\tau_a}$ under which the error coupled neural network (5.2.4) with mixed delays can achieve fixed-time synchronization when impulses are desynchronizing, i.e., $\theta > 1$.

Remark 5.3.3. In [177], the authors have proposed the results of delayed complex dynamical systems through pinning impulsive control. To achieve fixed-time synchronization, they have taken synchronizing impulses into consideration. It also states that fixed-time synchronization would not be possible for desynchronizing impulses. Nonetheless, we have found results for both synchronizing and desynchronizing impulses through the pinning rule in this chapter to achieve fixed-time synchronization.

To verify the applicability of the pinning impulsive control presented in this work, let us expand the finding results for the general coupled neural networks with delayed impulses. Initially, we examine the following general coupled neural networks without mixed delays

$$\dot{x}_p(t) = Ax_p(t) + R(x_p(t)) + h_1 \sum_{q=1}^{M} c_{pq}\Upsilon_1 x_q(t) + u_p(t), p = 1, 2, \ldots, M. \tag{5.3.27}$$

Therefore, under the pinning impulsive controller (5.2.4), the corresponding error coupled neural networks can be described as follows:

$$\begin{cases} \dot{e}_p(t) = Ae_p(t) + \tilde{R}(e_p(t)) + h_1 \sum_{q=1}^{M} c_{pq} \Upsilon_1 e_q(t) + u_{p2}, \quad t \neq t_l, \\ \Delta e_p(t_l) = e_p(t_l^+) - e_p(t_l^-) = r_l e_p(t^-), t = t_l, p \in \Omega(t_l), \\ \Delta e_p(t_l) = 0, p \in \Omega(t_l). \end{cases} \tag{5.3.28}$$

To obtain the sufficient condition for error coupled neural networks (5.3.28) under the pinning impulsive controller (5.2.4), we proposed the following theorem.

Theorem 5.3.2. Assume that Assumption 5.2.1 holds, the impulsive sequence $\zeta = \{t_l, l \in \mathbb{N}\}$ has an average impulsive interval τ_a with the elasticity number N_0. If there exists a control gain k_1 such that the following inequality holds:

$$k_1 \geq \frac{1}{2}\lambda_{max}(A + A^T) + \frac{1}{2} + m_1 + h_1 \lambda_{max}(C \otimes \Upsilon_1),$$

then under the pinning impulsive controller (5.2.4), we have

(i) when $0 < \theta < 1$, the error coupled neural networks (5.3.28) achieve fixed-time synchronization and settling-time is given by

$$\mathrm{T} = \frac{\ln\left[1 + \frac{\theta^{\mathrm{N}_0(1-\beta_0)}(\hat{\mathrm{k}}_1 - \frac{\ln\theta}{\tau_\mathrm{a}})}{\hat{\mathrm{k}}_3}\right]}{(\hat{\mathrm{k}}_1 - \frac{\ln\theta}{\tau_\mathrm{a}})(\beta_0 - 1)} + \frac{\ln\left[\frac{\hat{\mathrm{k}}_2\theta^{\mathrm{N}_0(1-\alpha_0)}}{\theta^{-\mathrm{N}_0(1-\alpha_0)}(\hat{\mathrm{k}}_1 - \frac{\ln\theta}{\tau_\mathrm{a}}) + \hat{\mathrm{k}}_2\theta^{\mathrm{N}_0(1-\alpha_0)}}\right]}{(\frac{\ln\theta}{\tau_\mathrm{a}} - \hat{\mathrm{k}}_1)(1-\alpha_0)}.$$

(ii) when $\theta = 1$, the error coupled neural networks (5.3.28) achieve fixed-time synchronization and settling-time is given by

$$\mathrm{T}' = \frac{\ln\left[1 + \frac{\hat{\mathrm{k}}_1}{\hat{\mathrm{k}}_3}\right]}{\hat{\mathrm{k}}_1(\beta_0 - 1)} - \frac{\ln\left[\frac{\hat{\mathrm{k}}_2}{\hat{\mathrm{k}}_1 + \hat{\mathrm{k}}_2}\right]}{\hat{\mathrm{k}}_2(1-\alpha_0)}.$$

(iii) when $\theta > 1$ and $\hat{k}_1 > \frac{\ln\theta}{\tau_a}$, the error coupled neural networks (5.3.28) achieve fixed-time synchronization and settling-time is given by

$$\mathrm{T}'' = \frac{\ln\left[1 + \frac{(\hat{k}_1 - \frac{\ln\theta}{\tau_a})}{\hat{k}_3\theta^{N_0(1-\beta_0)}}\right]}{(\hat{k}_1 - \frac{\ln\theta}{\tau_a})(\beta_0 - 1)} + \frac{\ln\left[\frac{\hat{k}_2\theta^{-N_0(1-\alpha_0)}}{\theta^{N_0(1-\alpha_0)}(\hat{k}_1 - \frac{\ln\theta}{\tau_a}) + \hat{k}_2\theta^{-N_0(1-\alpha_0)}}\right]}{(\frac{\ln\theta}{\tau_a} - \hat{k}_1)(1-\alpha_0)}.$$

Proof. Let us consider the following Lyapunov function as

$$W_1(t) = \frac{1}{2}\sum_{p=1}^{M} e_p^T(t)e_p(t). \tag{5.3.29}$$

Now differentiating $W_1(t)$ along with the error coupled neural network (5.3.28) when $t \neq t_l$, we get

$$\begin{aligned}\dot{W}_1(t) = &\sum_{p=1}^{M} e_p^T(t)(\frac{A + A^T}{2})e_p(t) + \sum_{p=1}^{M} e_p^T(t)(\tilde{R}(e_p(t)) + h_1\sum_{q=1}^{M} c_{pq}\Upsilon_1 e_q(t) - k_1 e_p(t) \\ &- k_2\mathrm{sign}(e_p(t))|e_p(t)|^{\alpha_1} + k_3\mathrm{sign}(e_p(t))|e_p(t)|^{\alpha_2}).\end{aligned} \tag{5.3.30}$$

Based on Assumption 5.2.1, it can be obtained that

$$\begin{aligned}\dot{W}_1(t) \leq &\frac{1}{2}\lambda_{max}(A + A^T)e^T(t)e(t) + \frac{1}{2}e^T(t)e(t) + m_1 e^T(t)e(t) + h_1 e^T(t)(C \otimes \Upsilon_1)e(t) \\ &- k_1 e^T(t)e(t) - k_2 e^T(t)|e(t)|^{\alpha_1} - k_2 e^T(t)|e(t)|^{\alpha_2}.\end{aligned} \tag{5.3.31}$$

From Lemma 3.2.1, we obtain

$$\begin{aligned}\dot{W}_1(t) \leq &(\frac{1}{2}\lambda_{max}(A + A^T) + \frac{1}{2} + m_1 + h_1\lambda_{max}(C \otimes \Upsilon_1) \\ &- k_1)W(t) - k_2(W(t))^{\frac{1+\alpha_1}{2}} + k_3(M)^{1-\alpha_2}(W(t))^{\frac{1+\alpha_2}{2}}.\end{aligned}$$

Let $\hat{k}_1 = k_1 - \frac{1}{2}\lambda_{max}(A + A^T) - \frac{1}{2} - m_1 - h_1\lambda_{max}(C \otimes \Upsilon_1)$, $\hat{k}_2 = k_2$, $\hat{k}_3 = k_3(M)^{1-\alpha_2}$,

$\alpha_0 = \frac{1+\alpha_1}{2}$ and $\beta_0 = \frac{1+\alpha_2}{2}$ in the above expression, we have

$$\dot{W}_1(t) \leq -\hat{k}_1 W(t) - \hat{k}_2 (W(t))^{\alpha_0} - \hat{k}_3 (W(t))^{\beta_0}. \tag{5.3.32}$$

On the other hand, when $t = t_l$, from Eq. (5.3.29), we have

$$\begin{aligned} W_1(t_l^+) &= \frac{1}{2}\sum_{p=1}^{M} e_p^T(t_l^+)e_p(t_l^+) \\ &= \frac{1}{2}\sum_{p\in\Omega(t_l)} e_p^T(t_l^+)e_p(t_l^+) + \frac{1}{2}\sum_{p\notin\Omega(t_l)} e_p^T(t_l^+)e_p(t_l^+) \\ &= \frac{(1+r_l)^2}{2}\sum_{p\in\Omega(t_l)} e_p^T(t_l^-)e_p(t_l^-) + \frac{1}{2}\sum_{p\notin\Omega(t_l)} e_p^T(t_l^-)e_p(t_l^-). \end{aligned} \tag{5.3.33}$$

From pinning control, we have

$$\sum_{p\notin\Omega(t_l)} e_p^T(t_l^-)e_p(t_l^-) \leq \frac{M-v}{M}\sum_{p=1}^{M} e_p^T(t_l^-)e_p(t_l^-).$$

Then from Eq. (5.3.5), we get

$$\begin{aligned} W_1(t_l^+) &\leq \frac{(1+r_l)^2}{2}\sum_{p\in\Omega(t_l)} e_p^T(t_l^-)e_p(t_l^-) + \frac{M-v}{M}\sum_{p=1}^{M} e_p^T(t_l^-)e_p(t_l^-) \\ &\leq ((1+r_l)^2 + \frac{M-v}{M})(\frac{1}{2}\sum_{p=1}^{M} e_p^T(t_l^-)e_p(t_l^-)) \\ &= \theta W(t_l^-). \end{aligned} \tag{5.3.34}$$

The sufficient criteria for fixed-time synchronization can be directly obtained from Eq. (5.3.7). These are already covered in Theorem 5.3.1 and therefore, it is not needed to repeat the same here. Thus, fixed-time synchronization can achieved for the error coupled neural network (5.3.28).

In many practical systems, the impulses describe a phenomenon where the impulsive transient depends not only on their current states but also on historical system states. Thus, a drive is taken to extend the proposed results to a coupled neural network with delayed impulses. For system (5.2.1), the delayed impulsive control designs are constructed as

$$u_p(t) = u_{p1}(t) + u_{p2}(t), \tag{5.3.35}$$

in which

$u_{p1} = \sum_{l=1}^{+\infty}[r_l e_p(t) + s_l e_p(t-\tau_l)]\delta(t-t_l)$,

$u_{p2} = -k_1 e_p(t) - k_2 \text{sign}(e_p(t))|e_p(t)|^{\alpha_1} + k_3 \text{sign}(e_p(t))|e_p(t)|^{\alpha_2}$,

where $0 < \alpha_1 < 1$, $\alpha_2 > 1$, positive constants k_1, k_2, k_3 are control gains. sign(.) be the signum function. r_l and s_l represents the impulsive strength that satisfy $|r_l| < 1$ and $s_l < 1$, and $\delta(.)$ is the Dirac delta function. τ_l represents the time delay at time t_l and there exists a positive constant ℓ such that $0 \le \tau_l \le \ell$. The time sequence $\{t_l - \tau_l\}$ satisfies $t_l - \ell > t_{l-1}$. Then, the error coupled neural networks (5.2.5) can be rewritten as

$$\begin{cases} \dot{e}_p(t) = Ae_p(t) + \tilde{R}(e_p(t), e_p(t - z_1(t))) + h_1 \sum_{q=1}^{M} c_{pq}\Upsilon_1 e_q(t) \\ \qquad +h_2 \sum_{q=1}^{M} d_{pq}\Upsilon_2 e_q(t - z_2(t)) + u_{p2}, \qquad t \neq t_l, \\ \Delta e_p(t_l) = e_p(t_l^+) - e_p(t_l^-) = r_l e_p(t_l^-) + s_l e_p((t_l - \tau_l)^-), t = t_l, p \in \Omega(t_l), \\ \Delta e_p(t_l) = 0, p \notin \Omega(t_l). \end{cases} \tag{5.3.36}$$

From controller (5.3.35) and error coupled neural network (5.3.36), we can obtain the following theorem.

Theorem 5.3.3. Assume that Assumption 5.2.1 holds, then the impulsive sequence $\zeta = \{t_l, l \in \mathbb{N}\}$ has an average impulsive interval τ_a with the elasticity number N_0. If there exists a control gain k_1 such that the following inequality holds

$$k_1 \ge \frac{1}{2}\lambda_{max}(A+A^T)+\frac{1}{2}+m_1+h_1\lambda_{max}(C\otimes\Upsilon_1)+\frac{h_2^2}{4}\lambda_{max}(DD^T\otimes\Upsilon_2\Upsilon_2^T)+\frac{m_2}{(1-\hat{z}_1)}+\frac{1}{4(1-\hat{z}_2)},$$

then under the pinning impulsive controller (5.3.35), we have

(i) when $0 < \upsilon < 1$, the error coupled neural networks (5.3.36) achieve fixed-time synchronization and settling-time is given by

$$\mathrm{T} = \frac{\ln\left[1 + \frac{\upsilon^{\mathrm{N}_0(1-\beta_0)}(\hat{\mathrm{k}}_1 - \frac{\ln\upsilon}{\tau_\mathrm{a}})}{\hat{\mathrm{k}}_3}\right]}{(\hat{\mathrm{k}}_1 - \frac{\ln\upsilon}{\tau_\mathrm{a}})(\beta_0 - 1)} + \frac{\ln\left[\frac{\hat{\mathrm{k}}_2\upsilon^{\mathrm{N}_0(1-\alpha_0)}}{\upsilon^{-\mathrm{N}_0(1-\alpha_0)}(\hat{\mathrm{k}}_1 - \frac{\ln\upsilon}{\tau_\mathrm{a}}) + \hat{\mathrm{k}}_2\upsilon^{\mathrm{N}_0(1-\alpha_0)}}\right]}{(\frac{\ln\upsilon}{\tau_\mathrm{a}} - \hat{\mathrm{k}}_1)(1-\alpha_0)}.$$

(ii) when $\upsilon = 1$, the error coupled neural networks (5.3.36) achieve fixed-time synchronization and settling-time is given by

$$\mathrm{T}' = \frac{\ln\left[1 + \frac{\hat{\mathrm{k}}_1}{\hat{\mathrm{k}}_3}\right]}{\hat{\mathrm{k}}_1(\beta_0 - 1)} - \frac{\ln\left[\frac{\hat{\mathrm{k}}_2}{\hat{\mathrm{k}}_1 + \hat{\mathrm{k}}_2}\right]}{\hat{\mathrm{k}}_2(1-\alpha_0)}.$$

(iii) when $\upsilon > 1$ and $\hat{k}_1 > \frac{\ln\upsilon}{\tau_a}$, the error coupled neural networks (5.3.36) achieve fixed-time synchronization and settling-time is given by

$$\mathrm{T}'' = \frac{\ln\left[1 + \frac{(\hat{k}_1 - \frac{\ln\upsilon}{\tau_a})}{\hat{k}_3\upsilon^{N_0(1-\beta_0)}}\right]}{(\hat{k}_1 - \frac{\ln\upsilon}{\tau_a})(\beta_0 - 1)} + \frac{\ln\left[\frac{\hat{k}_2\upsilon^{-N_0(1-\alpha_0)}}{\upsilon^{N_0(1-\alpha_0)}(\hat{k}_1 - \frac{\ln\upsilon}{\tau_a}) + \hat{k}_2\upsilon^{-N_0(1-\alpha_0)}}\right]}{(\frac{\ln\upsilon}{\tau_a} - \hat{k}_1)(1-\alpha_0)}.$$

Proof. Let us consider the following Lyapunov function as

$$\begin{aligned} W_2(t) &= \frac{1}{2}\sum_{p=1}^{M} e_p^T(t)e_p(t) + \frac{m_2}{(1-\hat{z}_1)}\sum_{p=1}^{M}\int_{t-z_1(t)}^{t} e_p^T(s)e_p(s)ds \\ &\quad + \frac{1}{4(1-\hat{z}_2)}\sum_{p=1}^{M}\int_{t-g_2(t)}^{t} e_p^T(s)e_p(s)ds. \end{aligned} \tag{5.3.37}$$

Now differentiating $W_2(t)$ along the error coupled neural network (5.3.36) when $t \neq t_l$, we get the same derivation as given in Theorem 5.3.1. We therefore obtain

$$\dot{W}_2(t) \leq -\hat{k}_1 W(t) - \hat{k}_2(W(t))^{\alpha_0} - \hat{k}_3(W(t))^{\beta_0}. \tag{5.3.38}$$

From Eq. (5.3.36), we can find that for $t = t_l$,

$$\begin{aligned}\sum_{p=1}^{M} e_p^T(t_l^+)e_p(t_l^+) &= \sum_{p=1}^{M}((1+r_l)e_p(t_l^-) + s_l e_p((t_l-\tau_l)^-))^T((1+r_l)e_p(t_l^-) + s_l e_p((t_l-\tau_l)^-)) \\ &= (1+r_l)^2\sum_{p=1}^{M} e_p^T(t_l^-)e_p(t_l^-) \\ &+ s_l^2 e_p^T((t_l-\tau_l)^-)e_p((t_l-\tau_l)^-) \\ &+ 2s_l(1+r_l)e_p^T(t_l^-)e_p((t_l-\tau_l)^-) \\ &\leq ((1+r_l)^2 + 2s_l^2(1+r_l)^2)\sum_{p=1}^{M} e_p^T(t_l^-)e_p(t_l^-) \\ &+ (s_l^2 + \frac{1}{2})\sum_{p=1}^{M} e_p^T((t_l-\tau_l)^-)e_p((t_l-\tau_l)^-), p \in \Omega(t_l). \end{aligned} \tag{5.3.39}$$

From pinning control, combining Eq. (5.3.37) and Eq. (5.3.39) for $t = t_l$, we obtain that

$$\begin{aligned} W_2(t_l^+) =& \frac{((1+r_l)^2 + 2s_l^2(1+r_l)^2)}{2}\sum_{p\in\Omega(t_l)} e_p^T(t_l^-)e_p(t_l^-) \\ &+ \frac{(s_l^2+\frac{1}{2})}{2}\sum_{p\in\Omega(t_l)} e_p^T((t_l-\tau_l)^-)e_p((t_l-\tau_l)^-) \\ &+ \frac{m_2((1+r_l)^2 + 2s_l^2(1+r_l)^2)}{(1-\hat{z}_1)}\sum_{p\in\Omega(t_l)}\int_{t_l-z_1(t)}^{t_l} e_p^T(s)e_p(s)ds \\ &+ \frac{m_2(s_l^2+\frac{1}{2})}{(1-\hat{z}_1)}\sum_{p\in\Omega(t_l)}\int_{t_l-z_1(t)}^{t_l-\tau_l} e_p^T(s)e_p(s)ds \\ &+ \frac{((1+r_l)^2 + 2s_l^2(1+r_l)^2)}{4(1-\hat{z}_2)}\sum_{p\in\Omega(t_l)}^{M}\int_{t_l-g_2(t)}^{t_l} e_p^T(s)e_p(s)ds \\ &+ \frac{(s_l^2+\frac{1}{2})}{4(1-\hat{z}_2)}\sum_{p\in\Omega(t_l)}^{M}\int_{t_l-g_2(t)}^{t_l-\tau_l} e_p^T(s)e_p(s)ds + \frac{M-v}{2M}\sum_{p=1}^{M} e_p^T(t_l^-)e_p(t_l^-) \\ &+ \frac{m_2(M-v)}{M(1-\hat{z}_1)}\sum_{p=1}^{M}\int_{t_l-z_1(t)}^{t_l} e_p^T(s)e_p(s)ds + \frac{M-v}{4M(1-\hat{z}_2)}\sum_{p=1}^{M}\int_{t_l-z_2(t)}^{t_l} e_p^T(s)e_p(s)ds \end{aligned}$$

$$
\begin{aligned}
i.e., W_2(t_l^+) \leq & (((1+r_l)^2 + 2s_l^2(1+r_l)^2) + \frac{M-\upsilon}{M})(\frac{1}{2}\sum_{p=1}^{M} e_p^T(t_l^-)e_p(t_l^-) \\
& + \frac{m_2}{(1-\hat{z}_1)}\sum_{p=1}^{M}\int_{t_l - z_1(t)}^{t_l} e_p^T(s)e_p(s)ds \\
& + \frac{1}{4(1-\hat{z}_2)}\sum_{p=1}^{M}\int_{t_l - z_2(t)}^{t_l} e_p^T(s)e_p(s)ds) \\
& + (s_l^2 + \frac{1}{2})(\frac{1}{2}\sum_{p=1}^{M} e_p^T((t_l - \tau_l)^-)e_p((t_l-\tau_l)^-) \\
& + \frac{m_2}{(1-\hat{z}_1)}\sum_{p=1}^{M}\int_{t_l - z_1(t)}^{t_l - \tau_l} e_p^T(s)e_p(s)ds \\
& + \frac{1}{4(1-\hat{z}_2)}\sum_{p\in\Omega(t_l)}\int_{t_l - g_2(t)}^{t_l-\tau_l} e_p^T(s)e_p(s)ds) \\
& \leq \Phi W_2(t_l^-) + \Psi W_2((t_l-\tau_l)^-.
\end{aligned}
$$

$$
W_2(t_l^+) \leq \Phi W_2(t_l^-) + \Theta W_2(t_l^-) = \upsilon W_2(t_l^-), \tag{5.3.40}
$$

where $\Phi = ((1+r_l)^2 + 2s_l^2(1+r_l)^2) + \frac{M-\upsilon}{M}$, $\Psi = s_l^2 + \frac{1}{2}$, $\upsilon = \Phi + \Theta$ and Θ satisfies $\Psi W_2((t_l-\tau_l)^- \leq \Theta W_2(t_l^-)$. From Eq. (5.3.38) and Eq. (5.3.40), we can get comparison system (5.3.7). Based on designed controller (5.3.35), the error coupled neural networks (5.3.36) can achieve fixed-time synchronization. Therefore, the proof of Theorem 5.3.3 is completed.

Remark 5.3.4. In contrast to [177], the coupled neural network with mixed delays can achieve fixed-time synchronization for both synchronizing impulses and desynchronizing impulses based on the saving in control costs. Additionally, the designed control strategy for this network can also be applied to another general neural network with delayed impulses. Consequently, the pinning impulsive control mechanism described above has a considerable deal of extensibility.

5.4 Numerical Simulations and Discussions

In this section, an application is provided to demonstrate the proposed theoretical results.

Example 5.4.1. Consider a coupled neural networks consisting of the 10 nodes with mixed delays to verify the correctness of Theorem 5.3.1.

$$\dot{x}_p(t) = f(x_p(t), x_p(t - z_1(t))) + h_1 \sum_{q=1}^{M} c_{pq} \Upsilon_1 x_q(t) + h_2 \sum_{q=1}^{M} d_{pq} \Upsilon_2 x_q(t - z_2(t)). \quad (5.4.1)$$

Select the coupling strength $h_1 = 0.5$ and $h_2 = 0.5$ respectively, the inner coupling matrix Υ_1 and Υ_2 be the identity matrix. The connection weight matrices are defined as follows:

$$[c_{pq}]_{10\times 10} = \begin{bmatrix} -6 & 1 & 0 & 1 & 0 & -1 & 2 & 0 & 2 & 1 \\ 0 & -5 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 1 \\ -2 & 1 & -6 & 0 & 1 & 2 & 0 & 2 & 0 & 2 \\ 1 & 1 & 0 & -4 & 1 & 0 & -2 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 & -3 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 & 0 & -5 & 0 & -1 & 0 & 2 \\ 2 & 0 & -1 & 2 & 0 & 2 & -6 & 0 & 0 & 1 \\ 0 & -2 & 3 & 0 & 2 & 0 & 1 & -4 & 0 & 0 \\ 1 & 0 & 3 & 0 & -1 & 1 & 0 & 0 & -4 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 & 1 & 3 & 1 & -7 \end{bmatrix}$$

$$[d_{pq}]_{10\times 10} = \begin{bmatrix} -4 & 0 & 1 & 0 & 2 & 0 & -2 & 1 & 1 & 1 \\ 0 & -5 & 0 & 1 & 0 & 2 & 1 & 0 & 0 & 1 \\ -1 & 0 & -3 & 1 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 0 & 1 & -1 & 1 & 1 \\ 1 & 0 & -1 & 0 & -2 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & -6 & 0 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & -4 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 & 1 & 0 & -1 & -4 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 2 & 0 & 0 & -5 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 & 0 & 2 & 1 & -3 \end{bmatrix}.$$

Now, consider the nonlinear term $f(.,.)$ to be the time-delayed Chua's system [178], which can be described as follows:

$$\begin{cases} \dot{x}_1(t) = \alpha(-x_1(t) + x_2(t) + m(x_1(t))), \\ \dot{x}_2(t) = x_1(t) - x_2(t) + x_3(t), \\ \dot{x}_3(t) = -\beta x_2(t) - \gamma x_3 - \beta\epsilon \sin(\sigma x_1(t - z_2(t))), \end{cases} \tag{5.4.2}$$

where $m(x_1(t)) = bx_1(t) + \frac{1}{2}(a-b)(|x_1(t)+1| - |x_1(t)-1|)$, $\alpha = 10$, $\beta = 19.53$, $\gamma = 0.1636$, $a = -1.4325$,$b = -0.7831$, $\sigma = 0.5$, $\epsilon = 0.2$. Let the initial condition be given by $x_0 = (0.15, 0.1, 0.2)^T$ and $x_p(t) = (x_{p1}(t), x_{p2}(t), x_{p3}(t))^T$, so that the system (5.4.2) can be defined as

$$\dot{x}_p(t) = Ax_p(t) + R(x_p(t), x_p(t - z_1(t))), \tag{5.4.3}$$

where $A = \begin{bmatrix} -\alpha(1+b) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & \beta & \gamma \end{bmatrix}$, $R(x_p(t), x_p(t - z_1(t))) = g_1(x_p(t)) + g_2(x_p(t))$,

$$g_1(x_p(t)) = \begin{pmatrix} m_1(x_1(t)) \\ 0 \\ 0 \end{pmatrix}, g_2(x_p(t-z_1(t))) = \begin{pmatrix} 0 \\ 0 \\ -\beta\epsilon\sin(\sigma x_{p1}(t-z_1(t))) \end{pmatrix}, m_1(x_1(t)) =$$

$\frac{1}{2}(|x_{p1}(t)+1| - |x_{p1}(t)-1|)$, $z_1(t) = 0.02$. We can verify that the nonlinear function $R(.,.)$ satisfies Assumption 5.2.1 with Lipschitz constants $m_1 = 1$ and $m_2 = 3.81$. Figure 5.4.1 shows the chaotic attractor of the system (5.4.3) with an initial value x_0.

In this example, the pinning impulsive controller (5.2.4) is considered with the pinned node $v = 4$. The time-varying delays are defined as $z_1(t) = 0.02$ and $z_2(t) = 0.2+0.05\sin t$, we have $\hat{z}_1 = 0.5$ and $\hat{z}_2 = 0.5$. Now, the control parameters are designed as $k_1 = 39$, $k_2 = 2.4$ and $k_3 = 4$ to satisfy the condition of Theorem 5.3.1. Then we can get $\hat{k}_1 = 1.5$, $\hat{k}_2 = 2.4$ and $\hat{k}_3 = 1.27$. Further, we choose $\alpha_1 = 0.5$ and $\alpha_2 = 1.5$ such that $\alpha_0 = 0.75$ and $\beta_0 = 1.25$.

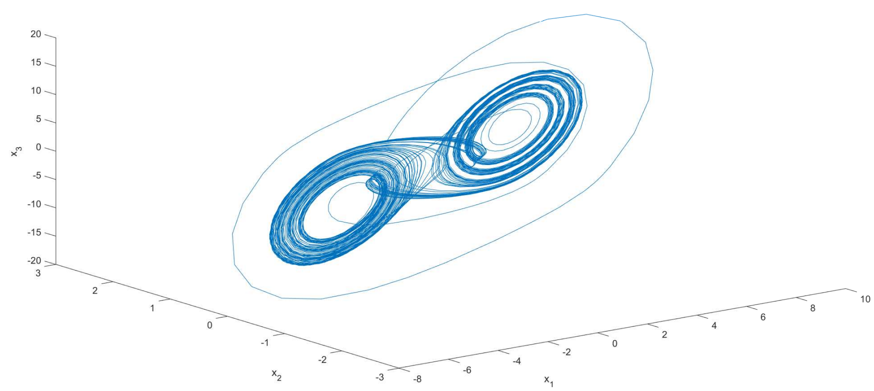

Figure 5.4.1: Chaotic behaviour of time delayed Chua's system

From [106], the impulsive sequence $\{t_l, l \in \mathbb{N}\}$ is taken, where N_0 is the elasticity number and τ_a is the average impulsive interval. By selecting $r_l = -0.5$ as the impulsive strength, we obtain $\theta = 0.85$, which suggests that the impulses are synchronizing. Figure 5.4.2, which depicts the error coupled neural network curve, depicts the results of Theorem 5.3.1 in the presence of a controller (5.2.4). Then the system (5.4.1) can realize fixed-time synchronization with the control strategy presented in this study, as shown in Figure 5.4.2. Moreover, the estimated settling-time is computed as $T = 1.581$.

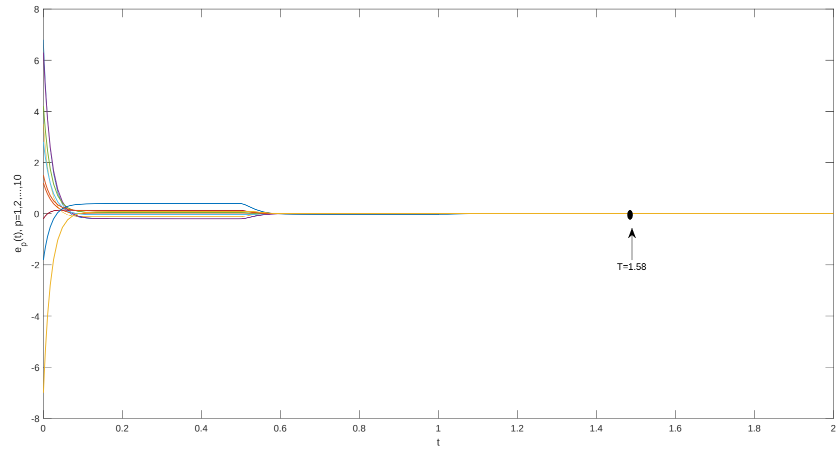

Figure 5.4.2: Error trajectories of system (5.4.1) with synchronizing impulses

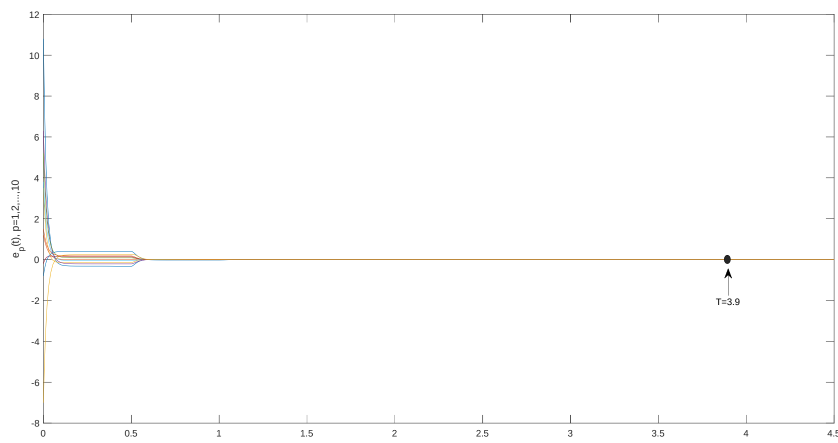

Figure 5.4.3: Error trajectories of system (5.4.1) with desynchronizing impulses

Given the elasticity number N_0 and the average impulsive interval τ_a, the impulsive sequence $\{t_l, l \in \mathbb{N}\}$ is taken from [106]. The value of $\theta = 2.85$, which corresponds to the impulsive strength $r_l = 0.5$, suggests that the impulses are desynchronizing. As illustrated in Figure 5.4.3, which depicts the error coupled neural network curve, can derive Theorem 5.3.1 under the controller (5.2.4). It can be deduced from Figure 5.4.3

that the control strategy presented in this study ensures the fixed-time synchronization for the system (5.4.1) and the estimated settling-time is given by $T = 3.9$.

5.5 Conclusion

In this chapter, the fixed-time synchronization problem for coupled neural networks with mixed-delays has been investigated. Firstly, a novel pinning impulsive controller has been developed by combining the strategies of pinning control and impulsive control. Controlling those nodes in which the error value is high, prevents the waste of control resources. Then, based on the concept of average impulsive interval and Lyapunov function, a sufficient condition for realizing fixed-time synchronization of the coupled neural networks with mixed delay under the effects of synchronizing impulses and desynchronizing impulses have been proposed, and also the obtained results have been extended to the general neural networks. Moreover, the fixed-time synchronization problem of coupled neural networks with mixed delays under delayed impulses has been analyzed and discussed, and the synchronizing criteria for achieving fixed-time synchronization has been established. Finally, the validity of the control scheme has been verified by a time-delayed Chua system.