Evolution of Some Mathematical Models of Atmospheric Vortices

Exploration of Steady and Unsteady States

Thesis submitted in partial fulfillment for the award of the degree of

Doctor of Philosophy

by

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Overall conclusions and further scope of research

Overall conclusions

The phenomena discussed in this thesis are atmospheric vortices, specifically dust devils and tropical cyclones. The results explored are embodied in Chapters $3 - 7$.

Chapter 3 presents a significant advancement in the mathematical modelling of steady-state whirlwinds, with a particular focus on dust devils. This work builds upon the earlier model proposed by Vyas and Majdalani (2006), who assumed an azimuthal velocity of the form $v \propto \frac{1}{r}$. Their formulation has singularity at the vortex centre and fails to incorporate vertical coordinate dependence of azimuthal velocity. In contrast, the model developed in this chapter introduces a more realistic azimuthal velocity profile that depends explicitly on both the radial and axial coordinates. This new formulation remains finite at the centre of the vortex and is radially bounded, thereby addressing the key limitations of the earlier model. A significant modification in our formulation is the presence of a separability constant k, which is essential in determining the vortex strength. For regulating the intensity and dispersion of azimuthal velocity, this parameter enables the model to represent dust devils of different intensities. According to the analysis, the azimuthal velocity peaks close to $r = \frac{1}{\sqrt{2}}$ and then taper off smoothly in both directions, towards the vortex core and the outer borders. Additionally, the peak velocity moves closer to the vortex axis as k increases, suggesting a more concentrated rotating structure.

Chapter 4 presents an analytical model for tropical cyclones that incorporates viscous effects—marking a significant advancement over the inviscid framework of Cecil and Majdalani (2022). This model accounts for key physical features including vertical wind structure, realistic boundary conditions, and cyclostrophic balance. The main objective is to understand how eye size influences cyclone intensity through its effect on radial, axial, and azimuthal velocities, as well as pressure distribution.

The flow is modelled for steady, incompressible, and axisymmetric, with physically consistent boundary conditions such as zero axial velocity at the surface and zero radial velocity at the vortex centre and beyond the eye-wall. The vertical velocity follows a sinusoidal form based on experimental insights, from which the remaining components are systematically derived. Pressure is obtained by integrating the axial momentum equation and applying cyclostrophic balance using the Vatistas profile at the surface.

A key contribution is the incorporation of the eye size parameter α and Reynolds number Re into the formulation. As α increases, azimuthal velocity peaks more sharply and falls rapidly in the rain-band, consistent with observational data. Larger eyes lead to more intense but localized rotational cores, while viscosity—quantified via Re—modulates the sharpness and distribution of velocity gradients. The pressure field increases radially outward from the eye, with steeper gradients for larger eyes and lower altitudes, highlighting the link between eye size, pressure drop, and storm intensity.

Chapter 5 presents a generalized analytical model for steady-state atmospheric vortices, incorporating viscous effects. Building upon the radial velocity structure proposed by Onishchenko et al. (2021), this work advances vortex modelling by deriving all three velocity components—radial, axial, and azimuthal—as functions of both radial and axial coordinates, ensuring they remain bounded throughout the domain. A key novelty of this model lies in its use of Rankine and Burgers velocity profiles as boundary conditions to enforce cyclostrophic balance at the ground level, thereby yielding a more realistic representation of vortex dynamics.

Unlike prior models that often assume unbounded or oversimplified velocity fields, the proposed framework introduces exponential localization in both the radial and vertical directions. The incorporation of physically consistent boundary conditions allows the model to capture the complete flow structure, including the converging radial inflow, central updraft, and rotational core, as observed in real atmospheric vortices. Furthermore, by varying the characteristic parameters such as the α and vortex sharpness parameter K, the model can represent a range of vortex intensities and morphologies—from dust devils to hurricanes.

Our analysis demonstrates that the azimuthal velocity exhibits distinct behaviour depending on the boundary condition employed. When Rankine's model is used, the velocity peaks sharply near the core and increases with height, whereas Burgers' condition results in smoother transitions and non-monotonic radial profiles. The model also reveals how radial pressure distributions vary with altitude and vortex size, providing insights into the mechanisms that drive vortex formation and intensification.

In Chapter 6 we present an analytical model for atmospheric vortices that incorporates all three velocity components—radial, axial, and azimuthal—along with a pressure field derived from the momentum equations. The model modifies the stream function to include a central low-pressure core and applies a viscous correction to the azimuthal velocity, enforcing the no-slip condition at the ground. Unlike classical models by Rankine, Burgers, or Vatistas, which assume infinite azimuthal velocity decay, our formulation confines rotational motion within a finite radial domain, capturing the compact structure of dust devils more realistically.

The model reveals height-dependent inflow-outflow transitions in radial velocity, strong near-surface axial updrafts that decay with height, and azimuthal velocity fields that peak near the core and weaken vertically. Parametric control via a shape factor and

Reynolds number allows for modelling a range of vortex intensities, from weak convective vortices to intense dust devils. Higher Reynolds numbers produce sharper, more localized velocity peaks, while variations in the low-pressure core size affect both strength and spread of the flow.

Chapter 7 introduces a time-dependent analytical model that covers the full life cycle of dust devils, from rapid formation to gradual decay. By incorporating time, this work extends beyond traditional steady-state models and provides a unified framework for analysing transient vortex behaviour. The model yields expressions for radial, axial and azimuthal velocities and pressure depending on time, offering better understanding of the mechanisms driving dust devil intensification and decay.

During the intensification phase ($t \in [0, 1]$), all three velocity components—radial, vertical, and azimuthal—increase with time, but each exhibits a distinct spatial pattern. The radial velocity, zero at the vortex centre, increases outward to a maximum near $r = 1/\sqrt{2}$ before declining beyond, reflecting the strengthening of inward flow. In contrast, the vertical velocity is maximum at the axis and decreases outward, representing a strong central updraft sustained by a low-pressure core. The azimuthal velocity, initially zero at the centre, rises with radius and peaks at $r = 1$ then starts to decrease.

As the vortex enters the decay phase ($t > 1$), all velocity components decrease gradually. The weakening starts at the periphery and moves inward, illustrating an outward-propagating dissipation. Interestingly, there is a continuous core circulation as the rotating component decays more slowly than the inflow. The vortex core maintains its maximum vertical velocity and zero radial velocity throughout its existence.

Future work

The analytical models developed in this thesis provide a foundational understanding of the dynamics of atmospheric vortices; however, there are still a number of unknown areas that may be investigated further. The consideration of thermodynamic effects, such as buoyancy, heat transfer, and latent heat release, which are crucial in the development and intensification of real-world vortices, particularly tropical cyclones, may be an interesting extension. By incorporating these aspects, models could become more physically complete and the current framework would be more applicable to climate research and weather forecasting.

Analytical modelling integrated with machine learning is an interesting and present-day trend for future research. Analytical models may be refined by means of data-driven approach by accounting fot complex patterns from observational data and high-resolution simulations that are difficult to capture analytically. For example, satellite data, reanalysis datasets, or Doppler radar observations can be used to train machine learning models to estimate model parameters like eddy viscosity or core size. Moreover, artificial models built using neural networks or Gaussian processes may rapidly approximate vortex activity for real-time forecasting and early warning systems. Hybrid modelling approaches —combining physical insight from analytical solutions with the adaptability of machine learning algorithms— are expected to be powerful in improving predictive capabilities. To explore genesis, intensification, and decay under different environmental conditions, for example, the time development of vortex properties could be modelled using temporal convolutional networks or recurrent neural networks. likewise unsupervised learning techniques might help in the classification of vortex type or the finding of hidden structures in huge datasets, improving our understanding of vortex classification and lifecycle patterns.

Combining mathematical models with machine learning enables more accurate forecasting, creating broader perspective in meteorology, climate science, and disaster preparedness.

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