Chapter 3

Approximation properties of a modified Jain operator

Throughout this chapter, we consider X to be the set $[0,\infty)$.

3.1 Introduction

In 1972, Jain [67] generalized the Szász operators by constructing the following positive linear operators:

$$S_m^{\alpha}(\Psi;x)=\sum_{j=0}^{\infty}\frac{mx(mx+j\alpha)^{j-1}}{j!}e^{-(mx+j\alpha)}\Psi\left(\frac{j}{m}\right),$$

where $\Psi\in\mathfrak{C}_B(X)$, $x\in X$, $m\in\mathbb{N}$ and $\alpha\in[0,1)$ with α depending only on m. Başcanbaz-Tunca et al. [29], have introduced a generalization of the above operators as

$$L_m^{\alpha}(\Psi)(x)=\sum_{j=0}^{\infty}\frac{mx(mx+1+j\alpha)_{j-1}}{2^j j!}2^{-(mx+j\alpha)}\Psi\left(\frac{j}{m}\right),\quad x\in(0,\infty).\tag{3.1}$$

Also, $L_m^{\alpha}(\Psi)(0)=\Psi(0)$.

Recently, Patel and Bodur [97] have defined a Durrmeyer-type integral generalization of the operators (3.1) as

$$D_m^{\alpha}(\Psi;x) = m\sum_{j=0}^{\infty} l_{m,j}^{\alpha}(x)\int_0^{\infty} s_{m,j}(t)\Psi(t)dt,\ x\in(0,\infty), \tag{3.2}$$

where

$$l_{m,j}^{\alpha}(x) = \frac{mx(mx+1+j\alpha)_{j-1}}{2^j j!}2^{-(mx+j\alpha)}, \tag{3.3}$$

$$s_{m,j}(t) = e^{-mt}\frac{(mt)^j}{j!}, \tag{3.4}$$

and $\Psi : X \to \mathbb{R}$ is an integrable function such that $D_m^{\alpha}(\Psi;x)$ exists.

In the recent years, Stancu-type generalizations of several operators have been proposed by many researchers. For more details, we refer the readers to [5, 7, 11, 19, 33, 39, 78, 86, 87, 89, 90]. An important advantage of this type of generalization is their modeling flexibility due to the newly introduced parameters.

Motivated by the above works, in this chapter we introduce a Stancu-type generalization of the operators (3.2) as follows:

$$A_m^{(\alpha,\beta,\gamma)}(\Psi;x) := A_m^{(\alpha,\beta,\gamma)}(\Psi)(x) = m\sum_{j=0}^{\infty} l_{m,j}^{\alpha}(x)\int_0^{\infty} s_{m,j}(t)\Psi\left(\frac{mt+\beta}{m+\gamma}\right)dt, \tag{3.5}$$

where $l_{m,j}^{\alpha}(x)$ and $s_{m,j}(t)$ are given by (3.3) and (3.4) respectively and $\Psi : X \to \mathbb{R}$ is an integrable function such that $A_m^{(\alpha,\beta,\gamma)}(\Psi;x)$ exists. Here, $x \in (0,\infty)$, $\alpha \in [0,1)$, $0 \le \beta \le \gamma$ and $A_m^{(\alpha,\beta,\gamma)}(\Psi;0) = \Psi(0)$. It can be easily seen that the operators (3.5) are positive and linear on X.

Remark 3.1. If we put $\beta = \gamma = 0$ in (3.5), then we get the operators defined in (3.2).

Throughout this chapter, the spaces $\mathfrak{C}_B(X)$, $\mathfrak{C}_B^2(X)$ and $\mathfrak{U}(X)$ are endowed with the norm:

$$\|\Psi\| = \sup\{|\Psi(x)| : x \in X\}.$$

3.2 Auxiliary Results

In this section, we shall discuss some basic lemmas that will serve as supports for the main results.

Lemma 3.2. [97] *Let $e_i(t) := t^i$ for $i = 0, 1, 2, 3$. Then, for the operators defined by* (3.2)*, we have*

$$\begin{aligned}
D_m^{\alpha}(e_0; x) =& 1,\\
D_m^{\alpha}(e_1; x) =& \frac{x}{1-\alpha} + \frac{1}{m},\\
D_m^{\alpha}(e_2; x) =& \frac{x^2}{(1-\alpha)^2} + \frac{x(3\alpha^2 - 6\alpha + 5)}{m(1-\alpha)^3} + \frac{2}{m^2},\\
D_m^{\alpha}(e_3; x) =& \frac{x^3}{(1-\alpha)^3} + \frac{6x^2(\alpha^2 - 2\alpha + 2)}{m(1-\alpha)^4} + \frac{x^2(11\alpha^4 - 44\alpha^3 + 78\alpha^2 - 62\alpha + 29)}{m^2(1-\alpha)^5} + \frac{6}{m^3},\\
D_m^{\alpha}(e_4; x) =& \frac{x^4}{(1-\alpha)^4} + \frac{2x^3(5\alpha^2 - 10\alpha + 11)}{m(1-\alpha)^5} + \frac{x^2(35\alpha^4 - 140\alpha^3 + 270\alpha^2 - 236\alpha + 131)}{m^2(1-\alpha)^6}\\
&+ \frac{2x(25\alpha^6 - 150\alpha^5 + 410\alpha^4 - 610\alpha^3 + 568\alpha^2 - 286\alpha + 103)}{m^3(1-\alpha)^7} + \frac{24}{m^4}.
\end{aligned}$$

Depending upon the above lemma, we derive following moments for the operator defined by (3.5).

Lemma 3.3. *The moments for the operator* (3.5) *are calculated to be*

$$\begin{aligned}
A_m^{(\alpha,\beta,\gamma)}(e_0; x) =& 1,\\
A_m^{(\alpha,\beta,\gamma)}(e_1; x) =& \frac{1}{(m+\gamma)}\left[\frac{mx}{1-\alpha} + (1+\beta)\right],\\
A_m^{(\alpha,\beta,\gamma)}(e_2; x) =& \frac{1}{(m+\gamma)^2}\left[\frac{m^2x^2}{(1-\alpha)^2}\right.
\end{aligned}$$

$$+\frac{mx\{\alpha^2(2\beta+3)-2\alpha(2\beta+3)+(2\beta+5)\}}{(1-\alpha)^3}+(\beta^2+2\beta+2)\bigg],$$

$$\begin{aligned}A_m^{(\alpha,\beta,\gamma)}(e_3;x)=&\frac{1}{(m+\gamma)^3}\left[\frac{m^3x^3}{(1-\alpha)^3}+\frac{3m^2x^2\{\alpha^2(\beta+2)-2\alpha(\beta+2)+(\beta+4)\}}{(1-\alpha)^4}\right.\\&+\frac{mx\{\alpha^4(3\beta^2+9\beta+11)-\alpha^3(12\beta^2+36\beta+44)+\alpha^2(18\beta^2+60\beta+78)\}}{(1-\alpha)^5}\\&\left.-\frac{mx\{\alpha(12\beta^2+48\beta+62)-(3\beta^2+15\beta+29)\}}{(1-\alpha)^5}+(\beta^3+3\beta^2+6\beta+6)\right].\end{aligned}$$

Proof. Here, we'll directly use the Lemma 3.2 and the facts:

$$\begin{aligned}A_m^{(\alpha,\beta,\gamma)}(e_0;x)=&D_m^{\alpha}(e_0;x),\\A_m^{(\alpha,\beta,\gamma)}(e_1;x)=&\frac{1}{(m+\gamma)}\left[mD_m^{\alpha}(e_1;x)+\beta D_m^{\alpha}(e_0;x)\right],\\A_m^{(\alpha,\beta,\gamma)}(e_2;x)=&\frac{1}{(m+\gamma)^2}\left[m^2D_m^{\alpha}(e_2;x)+2m\beta D_m^{\alpha}(e_1;x)+\beta^2D_m^{\alpha}(e_0;x)\right],\\A_m^{(\alpha,\beta,\gamma)}(e_3;x)=&\frac{1}{(m+\gamma)^3}\left[m^3D_m^{\alpha}(e_3;x)+3m^2\beta D_m^{\alpha}(e_2;x)+3m\beta^2D_m^{\alpha}(e_1;x)+\beta^3D_m^{\alpha}(e_0;x)\right]\end{aligned}$$

to conclude our desired results. □

As a consequence of the above lemma, in the next lemma, we define the central moments of the operators (3.5).

Lemma 3.4. *The central moments for the operators (3.5) are*

$$A_m^{(\alpha,\beta,\gamma)}((t-x);x)=\frac{1}{m+\gamma}\left[\frac{x(m\alpha-\gamma+\gamma\alpha)}{(1-\alpha)}+(\beta+1)\right],$$

$$\begin{aligned}A_m^{(\alpha,\beta,\gamma)}((t-x)^2;x)=&\frac{1}{(m+\gamma)^2}\left[\frac{x^2\{\alpha^2(m^2+2m\gamma+\gamma^2)-2\alpha\gamma(m+\gamma)+\gamma^2\}}{(1-\alpha)^2}\right.\\&+\frac{1}{(1-\alpha)^3}\{x\{2\alpha^3(m+\gamma)(1+\beta)-\alpha^2(3m+4m\beta+6\gamma+6\gamma\beta)\\&\left.+2\alpha(m\beta+3\gamma\beta+3\gamma)+(3m-2\gamma-2\gamma\beta)\}\}+(\beta^2+2\beta+2)\right]\\&=\lambda_m^{(\alpha,\beta,\gamma)}(x).\end{aligned}\tag{3.6}$$

Remark 3.5. For the operators (3.2), the third central moment is calculated to be:

$$\begin{aligned}D_m^{\alpha}((t-x)^3;x) =& \frac{x^3\alpha^3}{1-\alpha^3}+\frac{3x^2\alpha(\alpha^3-\alpha^2-\alpha+11)}{m(1-\alpha)^4}\\&+\frac{x(6\alpha^5-19\alpha^4+16\alpha^3+8\alpha^2-32\alpha+23)}{m^2(1-\alpha)^5}+\frac{6}{m^3}.\end{aligned}$$

Also, if $\{\alpha_m\}_{m\geq 1}$ be a sequence such that $\alpha_m \in [0,1)$ $\forall m$, $\lim\limits_{m\to\infty}\alpha_m = 0$ and $\lim\limits_{m\to\infty} m\alpha_m = 0$, then

$$\lim_{m\to\infty} m^2\left(D_m^{\alpha_m}((t-x)^3;x)\right) = 23x. \tag{3.7}$$

Remark 3.6. Let $\{\alpha_m\}_{m\geq 1}$ be a sequence such that $\alpha_m \in [0,1)$ $\forall m$, $\lim\limits_{m\to\infty}\alpha_m = 0$ and $\lim\limits_{m\to\infty} m\alpha_m = 0$. Then it is to observe that for the operators defined by (3.5), the central moments satisfy

$$\begin{aligned}\lim_{m\to\infty} m\left(A_m^{(\alpha_m,\beta,\gamma)}((t-x);x)\right) =& 1,\\ \lim_{m\to\infty} m\left(A_m^{(\alpha_m,\beta,\gamma)}((t-x)^2;x)\right) =& 3x.\end{aligned}$$

It is also noticed that

$$\begin{aligned}A_m^{(\alpha_m,\beta,\gamma)}((t-x)^4;x) =& \frac{m^4}{(m+\gamma)^4}\left(D_m^{\alpha_m}((t-x)^4;x)\right)\\&+\frac{4m^3(\beta-\gamma x)}{(m+\gamma)^4}\left(D_m^{\alpha_m}((t-x)^3;x)\right)\\&+\frac{6m^2(\beta-\gamma x)^2}{(m+\gamma)^4}\left(D_m^{\alpha_m}((t-x)^2;x)\right)\\&+\frac{4m(\beta-\gamma x)^3}{(m+\gamma)^4}\left(D_m^{\alpha_m}(t-x;x)\right)+\frac{(\beta-\gamma x)^4}{(m+\gamma)^4}.\end{aligned} \tag{3.8}$$

Using (3.7) and Remark 2.5 of [97] in (3.8), we have

$$\lim_{m\to\infty} m^2\left(A_m^{(\alpha_m,\beta,\gamma)}((t-x)^4;x)\right) =27x^2. \tag{3.9}$$

Remark 3.7. For $\alpha\in[0,1)$, we get the following

$$\begin{aligned}
A_m^{(\alpha,\beta,\gamma)}(t^2+1;x) =&\frac{1}{(m+\gamma)^2}\left[\frac{m^2x^2}{(1-\alpha)^2}\right.\\
&\left.+\frac{mx\{\alpha^2(2\beta+3)-2\alpha(2\beta+3)+(2\beta+5)\}}{(1-\alpha)^3}+(\beta^2+2\beta+2)\right]+1\\
=&\frac{1}{(m+\gamma)^2}\left[\frac{m^2x^2}{(1-\alpha)^2}\right.\\
&\left.+\frac{mx\{(2\beta\alpha^2-4\beta\alpha+2\beta)+(3\alpha^2-6\alpha+5)\}}{(1-\alpha)^3}+(\beta^2+2\beta+2)\right]+1\\
\le&\frac{(1+x^2)}{(m+\gamma)^2}\left[\frac{m^2x^2}{(1-\alpha)^2(1+x^2)}+\frac{mx(2\beta+5)}{(1-\alpha)^3(1+x^2)}+\frac{(\beta^2+2\beta+2)}{(1+x^2)}\right]+1\\
\le&\frac{(1+x^2)}{(m+\gamma)^2}\left[\frac{m^2}{(1-\alpha)^2}+\frac{m(2\beta+5)}{(1-\alpha)^3}+(\beta^2+2\beta+2)\right]+(1+x^2)\\
\le&(1+x^2)\left[\frac{m^2+m(2\beta+5)+(\beta^2+2\beta+2)}{(m+\gamma)^2(1-\alpha)^3}+1\right]\\
=&M_m(1+x^2)
\end{aligned}$$

where $M_m=\frac{m^2+m(2\beta+5)+(\beta^2+2\beta+2)}{(m+\gamma)^2(1-\alpha)^3}+1$.

Lemma 3.8. *If* $\Psi\in\mathfrak{C}_B(X)$, *then* $\|A_m^{(\alpha,\beta,\gamma)}(\Psi)\|\le\|\Psi\|$.

Proof. Using Lemma 3.3 and the norm defined for $\mathfrak{C}_B(X)$, the result follows immediately. □

Theorem 3.9. *Let* $\Psi\in\mathfrak{C}_B(X)$ *and* $\{\alpha_m\}_{m\ge1}$ *be a sequence such that* $\alpha_m\in[0,1)$, $\forall m$ *and* $\lim\limits_{m\to\infty}\alpha_m=0$. *Then* $\lim\limits_{m\to\infty}A_m^{(\alpha_m,\beta,\gamma)}(\Psi;x)=\Psi(x)$ *uniformly on each compact subset of* X.

Proof. We note that from Lemma 3.3

$$\lim_{m\to\infty}A_m^{(\alpha_m,\beta,\gamma)}(e_i;x)=e_i,\text{ for i=0,1,2 as }\lim_{m\to\infty}\alpha_m=0.$$

The above convergence is true in each compact subset of X. Hence the required result follows from Korovkin's theorem (see[74]). $\square$

3.3 Voronovskaja-type Asymptotic Formula

After observing the convergence of the operators (3.5), the most significant question that comes to the mind is the speed of approximation of these operators to the function Ψ. To answer this, we present a Voronovskaya-type asymptotic formula for the operators $A_m^{(\alpha_m,\beta,\gamma)}$.

Theorem 3.10. *Let Ψ be a bounded integrable function on X and Ψ'' exists at a point $x \in X$. Also, let $\{\alpha_m\}_{m\geq 1}$ be a sequence with $\alpha_m \in [0,1)$, $\forall m$ such that $\lim_{m\to\infty} \alpha_m = 0$, and $\lim_{m\to\infty} m\alpha_m = 0$. Then*

$$\lim_{m\to\infty} m\left(A_m^{(\alpha_m,\beta,\gamma)}(\Psi;x) - \Psi\right) = \Psi'(x) + \frac{3x}{2}\Psi''(x).$$

Proof. By the well-known Taylor's expansion,

$$\Psi(t) = \Psi(x) + \Psi'(x)(t-x) + \frac{1}{2}\Psi''(x)(t-x)^2 + \Lambda(t,x)(t-x)^2. \tag{3.10}$$

Here $\Lambda(t,x) \in \mathfrak{C}_B(X)$ and satisfies $\lim_{t\to x} \Lambda(t,x) = 0$. Next, we apply the operators $A_m^{(\alpha_m,\beta,\gamma)}$ and then multiply m on both sides of (3.10) to get

$$\begin{aligned} m\left(A_m^{(\alpha_n,\beta,\gamma)}(\Psi;x) - \Psi\right) =& mA_m^{(\alpha_m,\beta,\gamma)}(t-x;x)\Psi'(x) + \frac{m}{2}A_m^{(\alpha_m,\beta,\gamma)}((t-x)^2;x)\Psi''(x) \\ &+ mA_m^{(\alpha_m,\beta,\gamma)}(\Lambda(t,x)(t-x)^2;x). \end{aligned} \tag{3.11}$$

Applying Cauchy-Schwarz inequality to the third term of the right hand side of (3.11), we find

$$m\left|A_m^{(\alpha_m,\beta,\gamma)}(\Lambda(t,x)(t-x)^2;x)\right| \leq \left(m^2 A_m^{(\alpha_m,\beta,\gamma)}((t-x)^4;x)\right)^{1/2}\left(A_m^{(\alpha_m,\beta,\gamma)}(\Lambda^2(t,x);x)\right)^{1/2},$$

which yields

$$\begin{aligned}
&\lim_{m\to\infty} m\left(A_m^{(\alpha_m,\beta,\gamma)}(\Psi;x)-\Psi\right)\\
&\leq \lim_{m\to\infty} mA_m^{(\alpha_m,\beta,\gamma)}(t-x;x)\Psi'(x)+\lim_{m\to\infty}\frac{m}{2}A_m^{(\alpha_m,\beta,\gamma)}((t-x)^2;x)\Psi''(x)\\
&\quad+\lim_{m\to\infty}\left(m^2A_m^{(\alpha_m,\beta,\gamma)}((t-x)^4;x)\right)^{1/2}\lim_{m\to\infty}\left(A_m^{(\alpha_m,\beta,\gamma)}(\Lambda^2(t,x);x)\right)^{1/2}.
\end{aligned}$$

By using Remark 3.6, we have

$$\begin{aligned}
&\lim_{m\to\infty} m\left(A_m^{(\alpha_m,\beta,\gamma)}(\Psi;x)-\Psi\right)\\
&\leq \Psi'(x)+\frac{3x}{2}\Psi''(x)+3\sqrt{3}x\lim_{m\to\infty}\left(A_m^{(\alpha_m,\beta,\gamma)}(\Lambda^2(t,x);x)\right)^{1/2}.
\end{aligned} \tag{3.12}$$

Also, notice that $\Lambda^2(t,x)\in\mathfrak{C}_B(X)$ and $\Lambda^2(x,x)=0$. Hence using Theorem 3.9, we can have $\lim_{m\to\infty}\left(A_m^{(\alpha_m,\beta,\gamma)}(\Lambda^2(t,x);x)\right)^{1/2}=0$ and hence (3.12) concludes that

$$\lim_{m\to\infty} m\left(A_m^{(\alpha_m,\beta,\gamma)}(\Psi;x)-\Psi\right)=\Psi'(x)+\frac{3x}{2}\Psi''(x).$$

□

3.4 Weighted Approximation

In this section, we provide a Korovkin-type theorem for the weighted approximation of the operators (3.5). For the weight function $\sigma(x)=1+x^2$, we already have defined the spaces $B_\sigma(x)$, $\mathfrak{C}_\sigma(x)$, $\tilde{\mathfrak{C}}_\sigma(x)$, $\mathfrak{U}_\sigma(x)$ and $\tilde{\mathfrak{U}}_\sigma(x)$. For contributions of researchers on approximation of functions in such type of space we refer to [3, 12, 20, 25, 66, 104].

We have the following result from Gadjiev [57]:

Lemma 3.11. [57] *The positive linear operators J_m, $m \geq 1$, act from $\mathfrak{C}_\sigma(X)$ to $B_\sigma(X)$ if and only if*

$$|J_m(\sigma)(x)| \leq K_m \sigma(x),$$

where K_m is a positive constant.

We provide a weighted uniform approximation result for the newly defined operators as follows:

Theorem 3.12. *Let $\{\alpha_m\}_{m\geq 1}$ be a sequence with $\alpha_m \in [0,1)$, $\forall m$ such that $\lim\limits_{m\to\infty} \alpha_m = 0$. Then for each $\Psi \in \tilde{\mathfrak{C}}_\sigma(X)$,*

$$\lim_{m\to\infty} \|A_m^{(\alpha_m,\beta,\gamma)}(\Psi) - \Psi\|_\sigma = 0.$$

Proof. By Lemma 3.11 and Remark 3.7, it is easy to see that each term of the sequence of operators $\left\{A_m^{(\beta,\gamma,\alpha_m)}\right\}_{m\geq 1}$ acts from $\mathfrak{C}_\sigma(X)$ to $B_\sigma(X)$.
To prove our result, it is sufficient to show (using Gadjiev's theorem [57]) that

$$\lim_{m\to\infty} \|A_m^{(\alpha_m,\beta,\gamma)}(e_j) - e_j\|_\sigma = 0, \text{ for } j = 0, 1, 2.$$

For $j = 0$, using Lemma 3.3 we have

$$\lim_{m\to\infty} \|A_m^{(\alpha_m,\beta,\gamma)}(e_0) - e_0\|_\sigma = 0.$$

Next, we obtain

$$\begin{aligned}\|A_m^{(\alpha_m,\beta,\gamma)}(e_1)-e_1\|_\sigma &\leq \sup_{x\in X}\frac{|A_m^{(\alpha_m,\beta,\gamma)}(e_1;x)-e_1|}{1+x^2}=\sup_{x\in X}\frac{\left|\frac{1}{(m+\gamma)}\left\{\frac{mx}{1-\alpha_m}+(1+\beta)\right\}-x\right|}{1+x^2}\\ &=\sup_{x\in X}\left|\frac{m\alpha_m x-\gamma x+\gamma\alpha_m x}{(m+\gamma)(1-\alpha_m)(1+x^2)}+\frac{(1+\beta)}{(m+\gamma)(1+x^2)}\right|\\ &\leq \sup_{x\in X}\left[\frac{m\alpha_m+\gamma+\gamma\alpha_m}{(m+\gamma)(1-\alpha_m)}+\frac{(1+\beta)}{(m+\gamma)}\right].\end{aligned}$$

The above implies that

$$\lim_{m\to\infty}\|A_m^{(\alpha_m,\beta,\gamma)}(e_1)-e_1\|_\sigma=0.$$

Also,

$$\begin{aligned}&\|A_m^{(\alpha_m,\beta,\gamma)}(e_2)-e_2\|_\sigma\\ &\leq \sup_{x\in X}\frac{|A_m^{(\alpha_m,\beta,\gamma)}(e_2)-e_2|}{1+x^2}\\ &=\sup_{x\in X}\left|\frac{-x^2\{m^2(\alpha_m^2-2\alpha_m)+(\gamma^2+2m\gamma)(1-\alpha_m)^2\}}{(m+\gamma)^2(1-\alpha_m)^2(1+x^2)}\right.\\ &\quad\left.+\frac{1}{(m+\gamma)^2}\left[\frac{mx\{\alpha_m^2(2\beta+3)-2\alpha_m(2\beta+3)+(2\beta+5)\}}{(1-\alpha_m)^3(1+x^2)}+\frac{(\beta^2+2\beta+2)}{(1+x^2)}\right]\right|\\ &\leq \sup_{x\in X}\left[\frac{m^2(\alpha_m^2+2\alpha_m)+(\gamma^2+2m\gamma)(1-\alpha_m)^2}{(m+\gamma)^2(1-\alpha_m)^2}\right.\\ &\quad\left.+\frac{1}{(m+\gamma)^2}\left\{\frac{m\{\alpha_m^2(2\beta+3)-2\alpha_m(2\beta+3)+(2\beta+5)\}}{(1-\alpha_m)^3}+(\beta^2+2\beta+2)\right\}\right].\end{aligned}$$

Hence, from the above discussion, we must have

$$\lim_{m\to\infty}\|A_m^{(\alpha_m,\beta,\gamma)}(e_2)-e_2\|_\sigma=0.$$

Thus, the proof is completed. □

3.5 Quantitative Estimation

Next, we will try to find a quantitative estimation for the operators (3.5) using the weighted modulus of continuity (1.17).

We consider ζ to be the identity function on X, i,e. $\zeta(x) = x, \quad \forall x \in X$.

Using the properties of $\omega_\zeta(\Psi, \delta)$, Holhoş [66] provided the following theorem and remark:

Theorem 3.13. [66] *Let $\{J_m\}_{m\geq 1}$ be a sequence of positive linear operators mapping from $\mathfrak{C}_\sigma(X)$ into $B_\sigma(X)$ with*

$$\begin{aligned} \|J_m(\zeta^0) - \zeta^0\|_{\sigma^0} &= p_m, \\ \|J_m(\zeta) - \zeta\|_{\sigma^{1/2}} &= q_m, \\ \|J_m(\zeta^2) - \zeta^2\|_{\sigma} &= r_m, \\ \|J_m(\zeta^3) - \zeta^3\|_{\sigma^{3/2}} &= \ s_m, \end{aligned}$$

where p_m, q_m, r_m, s_m, tend to zero as m tends to infinity. Then

$$\|J_m(\Psi) - \Psi\|_{\sigma^{3/2}} \leq (7 + 4p_m + 2r_m)\omega_\zeta(\Psi; \delta_m) + \|\Psi\|_\sigma \ p_m$$

for all $\Psi \in \mathfrak{C}_\sigma(X)$, where

$$\delta_m = 2\sqrt{(p_m + 2q_m + r_m)(1 + p_m)} + p_m + 3q_m + 3r_m + s_m.$$

Remark 3.14. [66] Under the condition of the Theorem 3.13 and using the fact that $\lim\limits_{\delta_m \to 0} \omega_\zeta(\Psi; \delta_m) = 0$, we can conclude that

$$\forall \Psi \in \tilde{\mathfrak{U}}_{\sigma^{3/2}}(X), \quad \lim_{m\to\infty} \|J_m(\Psi) - (\Psi)\|_{\sigma^{3/2}} = 0.$$

Theorem 3.15. *Let $\zeta(x) = x$ and $\sigma(x) = 1 + x^2$. Let $\{\alpha_m\}_{m\geq 1}$ be a sequence with $\alpha_m \in [0, 1)$, $\forall m$ such that $\lim\limits_{m\to\infty} \alpha_m = 0$ and $\{A_m^{(\alpha_m,\beta,\gamma)}\}_{m\geq 1}$ be a sequence of positive linear operators. Then for all $\Psi \in \mathfrak{C}_\sigma(X)$,*

$$\begin{aligned}\|A_m^{(\alpha_m,\beta,\gamma)}(\Psi) - \Psi\|_{\sigma^{3/2}} \leq & \Bigg[7 + \frac{2m^2(\alpha_m^2 + 2\alpha_m) + (2\gamma^2 + 4m\gamma)(1-\alpha_m)^2}{(m+\gamma)^2(1-\alpha_m)^2} \\ & + \frac{2m\{\alpha_m^2(2\beta+3) - 2\alpha_m(2\beta+3) + (2\beta+5)\}}{(m+\gamma)^2(1-\alpha_m)^3} \\ & + \frac{(2\beta^2 + 4\beta + 4)}{(m+\gamma)^2}\Bigg]\omega_\zeta(\Psi; \delta_m)\end{aligned}$$

where,

$$\begin{aligned}\delta := \delta_m =& 2\Bigg[\frac{2m\alpha_m + 2\gamma + 2\gamma\alpha_m}{(m+\gamma)(1-\alpha_m)} + \frac{2+2\beta}{(m+\gamma)} + \frac{m^2(\alpha_m^2 + 2\alpha_m) + (\gamma^2 + 2m\gamma)(1-\alpha_m)^2}{(m+\gamma)^2(1-\alpha_m)^2} \\ & + \frac{m\{\alpha_m^2(2\beta+3) - 2\alpha_m(2\beta+3) + (2\beta+5)\}}{(m+\gamma)^2(1-\alpha_m)^3} + \frac{(\beta^2 + 2\beta + 2)}{(m+\gamma)^2}\Bigg]^{1/2} \\ & + \frac{3m\alpha_m + 3\gamma + 3\gamma\alpha_m}{(m+\gamma)(1-\alpha_m)} + \frac{3+3\beta}{(m+\gamma)} + \frac{3m^2(\alpha_m^2 + 2\alpha_m) + (3\gamma^2 + 6m\gamma)(1-\alpha_m)^2}{(m+\gamma)^2(1-\alpha_m)^2} \\ & + \frac{3m\{\alpha_m^2(2\beta+3) - 2\alpha_m(2\beta+3) + (2\beta+5)\}}{(m+\gamma)^2(1-\alpha_m)^3} + \frac{(3\beta^2 + 6\beta + 6)}{(m+\gamma)^2} \\ & + \frac{m^3(\alpha_m^3 - 3\alpha_m^2 + 3\alpha_m) + (\gamma^3 + 3m^2\gamma + 3\gamma^2 m)(1-\alpha_m)^3}{(m+\gamma)^3(1-\alpha_m)^3} \\ & + \frac{3m^2\{\alpha_m^2(\beta+2) - 2\alpha_m(\beta+2) + (\beta+4)\}}{(m+\gamma)^3(1-\alpha_m)^4} \\ & + \frac{m\{\alpha_m^4(3\beta^2 + 9\beta + 11) + \alpha_m^3(12\beta^2 + 36\beta + 44)\}}{(m+\gamma)^3(1-\alpha_m)^5} \\ & + \frac{m\{\alpha_m^2(18\beta^2 + 60\beta + 78) + \alpha_m(12\beta^2 + 48\beta + 62)\}}{(m+\gamma)^3(1-\alpha_m)^5} \\ & + \frac{m(\beta^3 + 6\beta + 6)}{(m+\gamma)^3}.\end{aligned}$$

Proof. In view of Lemma 3.3, we have the following

$$\begin{aligned}p_m =& \|A_m^{(\alpha_m,\beta,\gamma)}(\zeta^0) - (\zeta^0)\|_{\sigma^0} = 0, \\ q_m =& \|A_m^{(\alpha_m,\beta,\gamma)}(\zeta) - (\zeta)\|_{\sigma^{1/2}} \leq \frac{m\alpha_m + \gamma + \gamma\alpha_m}{(m+\gamma)(1-\alpha_m)} + \frac{1+\beta}{(m+\gamma)}, \\ r_m =& \|A_m^{(\alpha_m,\beta,\gamma)}(\zeta^2) - (\zeta^2)\|_\sigma\end{aligned}$$

$$\leq \frac{m^2(\alpha_m^2+2\alpha_m)+(\gamma^2+2m\gamma)(1-\alpha_m)^2}{(m+\gamma)^2(1-\alpha_m)^2} + \frac{1}{(m+\gamma)^2}\left[\frac{m\{\alpha_m^2(2\beta+3)-2\alpha_m(2\beta+3)+(2\beta+5)\}}{(1-\alpha_m)^3}+(\beta^2+2\beta+2)\right],$$

$$\begin{aligned} s_m =& \|A_m^{(\alpha_m,\beta,\gamma)}(\zeta^3)-(\zeta^3)\|_{\sigma^{3/2}} \\ \leq& \frac{m^3(\alpha_m^3-3\alpha_m^2+3\alpha_m)+(\gamma^3+3m^2\gamma+3\gamma^2m)(1-\alpha_m)^3}{(m+\gamma)^3(1-\alpha_m)^3} \\ +& \frac{3m^2\{\alpha_m^2(\beta+2)-2\alpha_m(\beta+2)+(\beta+4)\}}{(m+\gamma)^3(1-\alpha_m)^4} \\ +& \frac{m\{\alpha_m^4(3\beta^2+9\beta+11)+\alpha_m^3(12\beta^2+36\beta+44)\}}{(m+\gamma)^3(1-\alpha_m)^5} \\ +& \frac{m\{\alpha_m^2(18\beta^2+60\beta+78)+\alpha_m(12\beta^2+48\beta+62)\}}{(m+\gamma)^3(1-\alpha_m)^5} \\ +& \frac{m(\beta^3+6\beta+6)}{(m+\gamma)^3}. \end{aligned}$$

Since $\lim\limits_{m\to\infty}\alpha_m=0$, it is clear that p_m, q_m, r_m and s_m tends to 0 as m goes to infinity. Finally, selecting $\delta := \delta_m$, we obtain

$$\begin{aligned} \|A_m^{(\alpha_m,\beta,\gamma)}(\Psi)-\Psi\|_{\sigma^{3/2}} \leq \Bigg[7 &+ \frac{2m^2(\alpha_m^2+2\alpha_m)+(2\gamma^2+4m\gamma)(1-\alpha_m)^2}{(m+\gamma)^2(1-\alpha_m)^2} \\ &+ \frac{2m\{\alpha_m^2(2\beta+3)-2\alpha_m(2\beta+3)+(2\beta+5)\}}{(m+\gamma)^2(1-\alpha_m)^3} \\ &+ \frac{(2\beta^2+4\beta+4)}{(m+\gamma)^2}\Bigg]\omega_\zeta(\Psi;\delta_m), \end{aligned}$$

which completes the desired proof. □

Remark 3.16. Depending upon all the conditions of Theorem 3.15 and in view of Remark 3.14 we can conclude: if $\lim\limits_{\delta_m\to 0}\omega_\zeta(\Psi;\delta_m)=0$, then

$$\forall \Psi \in \tilde{\mathfrak{U}}_{\sigma^{3/2}}(X), \quad \lim_{m\to\infty}\|A_m^{(\alpha_m,\beta,\gamma)}(\Psi)-(\Psi)\|_{\sigma^{3/2}}=0.$$

3.6 Rate of Convergence

In this section, we estimate the rate of convergence of the newly defined operators (3.5) in terms of the modulus of continuity.

Theorem 3.17. *let* $\Psi \in \mathfrak{U}(X)$*,* $m \in \mathbb{N}$*,* $\alpha \in [0,1)$ *and* $x \in X$*. Then the operators* $A_m^{(\alpha,\beta,\gamma)}$ *satisfy*

$$|A_m^{(\alpha,\beta,\gamma)}(\Psi;x) - \Psi(x)| \leq 2\Omega_1(\Psi,\delta),$$

where $\delta := \sqrt{\lambda_m^{(\alpha,\beta,\gamma)}(x)}$ *and* $\lambda_m^{(\alpha,\beta,\gamma)}(x)$ *is defined in* (3.6)*.*

Proof. Using the following property of modulus of continuity

$$|\Psi(t) - \Psi(x)| \leq \Omega_1(\Psi,\delta)\left(\frac{(t-x)^2}{\delta^2} + 1\right)$$

and then applying the newly defined operators on both sides of the above inequality, we get

$$\begin{aligned}|A_m^{(\alpha,\beta,\gamma)}(\Psi;x) - \Psi(x)| &\leq A_m^{(\alpha,\beta,\gamma)}(|\Psi(t) - \Psi(x)|;x)\\ &\leq \Omega_1(\Psi,\delta)\left(1 + \frac{1}{\delta^2}A_m^{(\alpha,\beta,\gamma)}((t-x)^2 : x)\right).\end{aligned}$$

Next, we choose $\delta := \sqrt{\lambda_m^{(\alpha,\beta,\gamma)}(x)}$ to obtain

$$|A_m^{(\alpha,\beta,\gamma)}(\Psi;x) - \Psi(x)| \leq 2\Omega_1(\Psi,\delta).$$

This completes the proof. □

Theorem 3.18. *Let $\Psi \in \mathfrak{U}(X)$ and $n > 1$, Then there exists $C > 0$ such that*

$$\left|A_m^{(\alpha,\beta,\gamma)}(\Psi;x) - \Psi(x)\right| \leq C\Omega_2\left(\Psi, \sqrt{\delta_m^{(\alpha,\beta,\gamma)}(x)}\right) + \Omega_1\left(\Psi, \frac{mx}{(1-\alpha)(m+\gamma)} + \frac{(1+\beta)}{(m+\gamma)}\right),$$

where

$$\begin{aligned}\delta_m^{(\alpha,\beta,\gamma)}(x) =& \frac{1}{(m+\gamma)^2}\left[\frac{x^2\{\alpha^2(m^2+2m\gamma+\gamma^2) - 2\alpha\gamma(m+\gamma)+\gamma^2\}}{(1-\alpha)^2}\right.\\ &+\frac{2x\alpha^3(m+\gamma)(1+\beta)}{(1-\alpha)^3} - \frac{\alpha^2x(7m+8m\beta+12\gamma+12\gamma\beta)}{2(1-\alpha)^3}\\ &\left.+\frac{2\alpha x(2m\beta+6\gamma\beta+6\gamma+m)}{2(1-\alpha)^3} + \frac{3m-4\gamma-4\gamma\beta}{2(1-\alpha)^3} + \left(\beta^2+2\beta+\frac{3}{2}\right)\right] \end{aligned} \tag{3.13}$$

Proof. We introduce an auxiliary operator $\tilde{A}_m^{(\alpha,\beta,\gamma)} : \mathfrak{U}(X) \to \mathfrak{U}(X)$ as follows

$$\tilde{A}_m^{(\alpha,\beta,\gamma)}(\Psi;x) = A_m^{(\alpha,\beta,\gamma)}(\Psi;x) - \Psi\left(\frac{mx}{(1-\alpha)(m+\gamma)} + \frac{(1+\beta)}{(m+\gamma)}\right) + \Psi(x). \tag{3.14}$$

Clearly, the above defined auxiliary operator preserves linear as well as constant functions.

Let $\Phi \in \mathfrak{C}_B^2(X)$ and $x, t \in X$. Then by Taylor's expansion, we have

$$\Phi(t) = \Phi(x) + (t-x)\Phi'(x) + \int_x^t (t-z)\Phi''(z)dz. \tag{3.15}$$

Now, applying the auxiliary operators on both sides on (3.15), we obtain

$$\begin{aligned}&|\tilde{A}_m^{(\alpha,\beta,\gamma)}(\Phi;x) - \Phi(x)|\\ \leq&\tilde{A}_m^{(\alpha,\beta,\gamma)}\left(\left|\int_x^t (t-z)\Phi''(z)dz, x\right|\right)\\ \leq&A_m^{(\alpha,\beta,\gamma)}\left(\left|\int_x^t (t-z)\Phi''(z)dz, x\right|\right)\\ &+\left|\int_x^{\frac{mx}{(1-\alpha)(m+\gamma)}+\frac{(1+\beta)}{(m+\gamma)}}\left(\frac{mx}{(1-\alpha)(m+\gamma)} + \frac{(1+\beta)}{(m+\gamma)} - z\right)\Phi''(z)dz\right|\\ \leq&\frac{1}{2}\Big[A_m^{(\alpha,\beta,\gamma)}((t-x)^2;x)\|\Phi''\|\end{aligned}$$

$$
\begin{aligned}
&\quad + \left(\frac{mx}{(1-\alpha)(m+\gamma)} + \frac{(1+\beta)}{(m+\gamma)} - x\right)^2 \|\Phi''\|\Bigg] \\
=&\frac{1}{(m+\gamma)^2}\Bigg[\frac{x^2\{\alpha^2(m^2+2m\gamma+\gamma^2) - 2\alpha\gamma(m+\gamma)+\gamma^2\}}{(1-\alpha)^2} \\
&+ \frac{2x\alpha^3(m+\gamma)(1+\beta)}{(1-\alpha)^3} \\
&+ \frac{x\{-\alpha^2(7m+8m\beta+12\gamma+12\gamma\beta)+2\alpha(2m\beta+6\gamma\beta+6\gamma+m)\}}{2(1-\alpha)^3} \\
&+ \frac{x(3m-4\gamma-4\gamma\beta)}{2(1-\alpha)^3} + \left(\beta^2+2\beta+\frac{3}{2}\right)\Bigg]\|\Phi''\| \\
=& \ \delta_m^{(\alpha,\beta,\gamma)}(x)\|\Phi''\|,
\end{aligned}
$$

where $\delta_m^{(\alpha,\beta,\gamma)}(x)$ is as given in (3.13).

On the other hand by Lemma 3.8 for $\Psi \in \mathfrak{U}(X)$ and $x \in X$, we have

$$
\|A_m^{(\alpha,\beta,\gamma)}(\Psi)\| \leq \|\Psi\|.
$$

Now applying triangle inequality on (3.14), we obtain

$$
\|\tilde{A}_m^{(\alpha,\beta,\gamma)}(\Psi;x)\| \leq \|A_m^{(\alpha,\beta,\gamma)}(\Psi;x)\| + 2\|\Psi\| \ \leq \ 3\|\Psi\|.
$$

Using all the above inequalities, we can write

$$
\begin{aligned}
&|A_m^{(\alpha,\beta,\gamma)}(\Psi;x) - \Psi(x)| \\
\leq&|\tilde{A}_n^{(\alpha,\beta,\gamma)}(\Psi-\Phi;x) - (\Psi-\Phi)(x)| + |\tilde{A}_m^{(\alpha,\beta,\gamma)}(\Phi;x) - \Phi(x)| \\
&+ \left|\Psi(x) - \Psi\left(\frac{mx}{(1-\alpha)(m+\gamma)} + \frac{(1+\beta)}{(m+\gamma)}\right)\right| \\
\leq&4\|\Psi-\Phi\| + \delta_m^{(\alpha,\beta,\gamma)}(x)\|\Phi''\| + \Omega_1\left(\Psi, \frac{mx}{(1-\alpha)(m+\gamma)} + \frac{(1+\beta)}{(m+\gamma)}\right) \\
\leq&4\left\{\|\Psi-\Phi\| + \delta_m^{(\alpha,\beta,\gamma)}(x)\|\Phi''\|\right\} + \Omega_1\left(\Psi, \frac{mx}{(1-\alpha)(m+\gamma)} + \frac{(1+\beta)}{(m+\gamma)}\right).
\end{aligned}
$$

Now, taking infimum over all $\Phi \in \mathfrak{C}_B^2(X)$ on right-hand side of the above equation and using (1.9), we get the desired assertion. $\square$

We conclude this section by establishing a convergence result for the Lipschitz-type class of functions.

Theorem 3.19. *If* $\Psi \in Lip_{M,\eta}^{a,b}(X)$, *Then*

$$|A_m^{(\alpha,\beta,\gamma)}(\Psi;x)-\Psi(x)| \leq M\left(\frac{\lambda_m^{(\alpha,\beta,\gamma)}(x)}{ax^2+bx}\right)^{\eta/2}.$$

Proof. First of all, we notice that if $\eta=1$, then

$$|A_m^{(\alpha,\beta,\gamma)}(\Psi;x)-\Psi(x)| \leq A_m^{(\alpha,\beta,\gamma)}(|\Psi(t)-\Psi(x)|;x) \leq M\left[A_m^{(\alpha,\beta,\gamma)}\left(\frac{|t-x|}{\sqrt{t+ax^2+bx}};x\right)\right].$$

Using the fact: $\frac{1}{\sqrt{\frac{mt+\beta}{m+\gamma}+ax^2+bx}} \leq \frac{1}{\sqrt{ax^2+bx}}$ and the well-known Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|A_m^{(\alpha,\beta,\gamma)}(\Psi;x)-\Psi(x)| \leq& \frac{M}{\sqrt{ax^2+bx}}A_m^{(\alpha,\beta,\gamma)}\left(|t-x|;x\right)\\ \leq& \frac{M}{\sqrt{ax^2+bx}}\left(A_m^{(\alpha,\beta,\gamma)}((t-x)^2;x)\right)^{1/2}=M\left(\frac{\lambda_m^{(\alpha,\beta,\gamma)}(x)}{ax^2+bx}\right)^{1/2},\end{aligned}$$

where $\lambda_m^{(\alpha,\beta,\gamma)}(x)$ is defined by the right hand side of (3.6).

Hence the result holds for $\eta=1$.

Next we consider the case when $\eta \in (0,1)$.

Clearly,

$$|A_m^{(\alpha,\beta,\gamma)}(\Psi;x)-\Psi(x)| \leq m\sum_{j=0}^{\infty} l_{m,j}^{\alpha}(x)\int_0^{\infty} s_{m,j}(t)\left|\Psi\left(\frac{mt+\beta}{m+\gamma}\right)-\Psi(x)\right|dt.$$

Let $p = \frac{2}{\eta}$ and $q = \frac{2}{2-\eta}$ and using the Hölder inequality, we obtain

$$\begin{aligned}
|A_m^{(\alpha,\beta,\gamma)}(\Psi;x) - \Psi(x)| \leq & \left[m \sum_{j=0}^{\infty} l_{m,j}^{\alpha}(x) \int_0^{\infty} s_{m,j}(t) \left| \Psi\left(\frac{mt+\beta}{m+\gamma}\right) - \Psi(x) \right|^{2/\eta} dt \right]^{\eta/2} \\
& \times \left[m \sum_{j=0}^{\infty} l_{m,j}^{\alpha}(x) \int_0^{\infty} s_{m,j}(t) dt \right]^{(2-\eta)/2} \\
\leq & M \left[m \sum_{j=0}^{\infty} l_{m,j}^{\alpha}(x) \int_0^{\infty} s_{m,j}(t) \frac{\left(\frac{mt+\beta}{m+\gamma} - x\right)^2}{\left(\frac{mt+\beta}{m+\gamma} + ax^2 + bx\right)} dt \right]^{\eta/2} \\
\leq & \frac{M}{(ax^2+bx)^{\eta/2}} \left[A_m^{(\alpha,\beta,\gamma)}((t-x)^2 : x) \right]^{\eta/2} \\
= & M \left(\frac{\lambda_m^{(\alpha,\beta,\gamma)}(x)}{ax^2+bx} \right)^{\eta/2}.
\end{aligned}$$

This completes the proof. □

Remark 3.20. One of the crucial remark is that we have investigated the local approximation results from Theorem 3.17 to Theorem 3.19. Therefore, we indicate that $\lambda_m^{(\alpha,\beta,\gamma)}$ *and* $\delta_m^{(\alpha,\beta,\gamma)}$ tend to zero while taking $\alpha = \alpha_m, \alpha_m \in [0,1), \forall m$ *and* $\lim\limits_{m\to\infty} \alpha_m = 0$. Otherwise, these results just stand as an inequality or an estimation.

3.7 Graphical Examples

This section provides a few numerical examples for the operators (3.5) to justify their approximation properties as well as the relevance of the generalization.

In both the Figures 3.1 and 3.2, we can see that the operator $A_m^{(1,2,\alpha_m)}(\Psi;x)$ converges faster than the operator $D_m^{\alpha_m}(\Psi;x)\big($ i.e., $A_m^{(\alpha_m,0,0)}(\Psi;x)\big)$ to the considered test functions $\Psi(x) = e^{-\frac{x}{2}}(5x^8 + x^4 + 1)$ and $\Psi(x) = x^2 sin(\pi x)$ respectively. Here we assumed $m = 500$, $x \in [0,1]$ and $\{\alpha_m\}_{m\geq 1}$ is the sequence $\{\frac{1}{m^2}\}_{m\geq 1}$ so that the hypothesis $\lim\limits_{m\to\infty} \alpha_m = 0$ is satisfied.

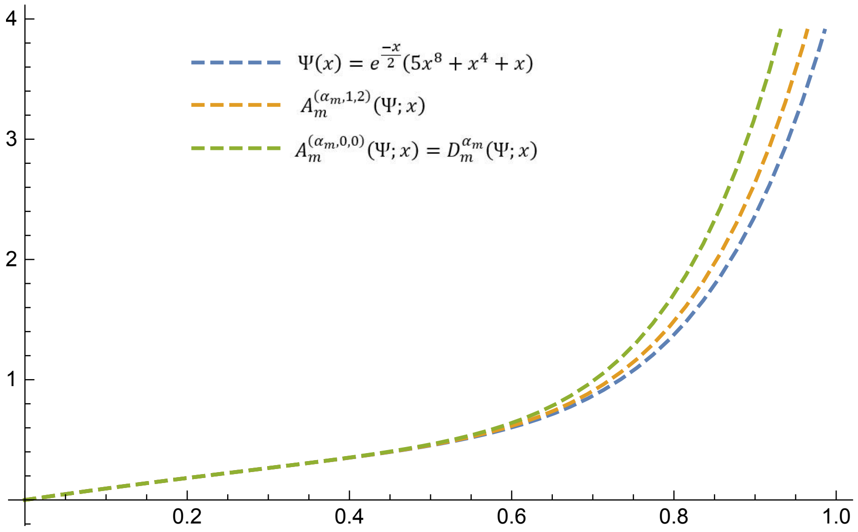

Figure 3.1: Convergence of the considered operators $A_m^{(\alpha,\beta,\gamma)}(\Psi; x)$ to the test function $\Psi(x) = e^{-\frac{x}{2}}(5x^8 + x^4 + 1)$ for $m = 500$, $x \in [0,1]$.

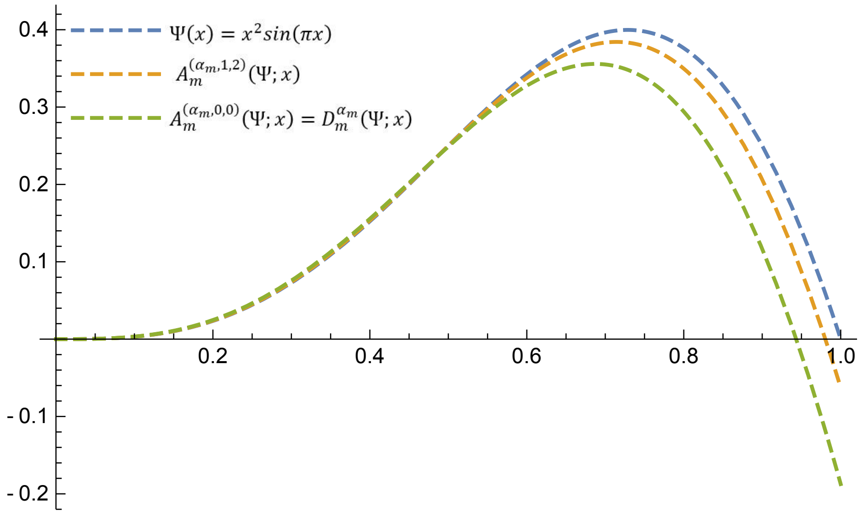

Figure 3.2: Convergence of the considered operators $A_m^{(\alpha,\beta,\gamma)}(\Psi; x)$ to the test function $\Psi(x) = x^2 \sin(\pi x)$ for $m = 500$, $x \in [0,1]$.

****