Appendix A

A.1 Proof of Lemma 1.4

Proof: .

(i) Let $\mathbf{A} = [\underline{a}, \overline{a}]$, $\mathbf{B} = [\underline{b}, \overline{b}]$, $\mathbf{C} = [\underline{c}, \overline{c}]$, and $\mathbf{D} = [\underline{d}, \overline{d}]$. Since $\mathbf{A} \oplus \mathbf{B} \preceq \mathbf{C} \oplus \mathbf{D}$,

$$\begin{aligned} & \underline{a} + \underline{b} \ \preceq \ \underline{c} + \underline{d} \text{ and } \overline{a} + \overline{b} \ \preceq \ \overline{c} + \overline{d} \\ \Longrightarrow \ & \underline{a} - \underline{c} \ \preceq \ \underline{d} - \underline{b} \text{ and } \overline{a} - \overline{c} \ \preceq \ \overline{d} - \overline{b} \\ \Longrightarrow \ & \min\{\underline{a} - \underline{c}, \overline{a} - \overline{c}\} \ \preceq \ \min\{\underline{d} - \underline{b}, \overline{d} - \overline{b}\}, \text{ and} \\ & \max\{\underline{a} - \underline{c}, \overline{a} - \overline{c}\} \ \preceq \ \max\{\underline{d} - \underline{b}, \overline{d} - \overline{b}\}. \end{aligned} \tag{A.1}$$

Hence, $\mathbf{A} \ominus_{gH} \mathbf{C} \ \preceq \ \mathbf{D} \ominus_{gH} \mathbf{B}$.

(ii) Let $\mathbf{A} = [\underline{a}, \overline{a}]$, and $\mathbf{B} = [\underline{b}, \overline{b}]$. Then,

$$\begin{aligned} \mathbf{A} \preceq \mathbf{B} \oplus [\epsilon, \epsilon] \iff \underline{a} \leq \underline{b} + \epsilon \text{ and } \overline{a} \leq \overline{b} + \epsilon \quad & \iff \quad \underline{a} - \underline{b} \leq \epsilon \text{ and } \overline{a} - \overline{b} \leq \epsilon \\ & \iff \quad \mathbf{A} \ominus_{gH} \mathbf{B} \preceq [\epsilon, \epsilon]. \end{aligned}$$

(iii) Let $\mathbf{A} = [\underline{a}, \overline{a}]$, and $\mathbf{B} = [\underline{b}, \overline{b}]$. Then,

$$\begin{aligned} & (\mathbf{A} \ominus_{gH} \mathbf{B}) \oplus ([\epsilon, \epsilon] \ominus_{gH} [\delta, \delta]) \\ = \ & [\min\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}, \max\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\}] \oplus [\epsilon - \delta, \epsilon - \delta] \\ = \ & [\min\{(\underline{a} + \epsilon) - (\underline{b} + \delta), (\overline{a} + \epsilon) - (\overline{b} + \delta)\}, \max\{(\underline{a} + \epsilon) - (\underline{b} + \delta), (\overline{a} + \epsilon) - (\overline{b} + \delta)\}] \\ = \ & (\mathbf{A} \oplus [\epsilon, \epsilon]) \ominus_{gH} (\mathbf{B} \oplus [\delta, \delta]). \end{aligned}$$

(iv) Let $\mathbf{A} = [\underline{a}, \overline{a}]$, $\mathbf{B} = [\underline{b}, \overline{b}]$, and $\mathbf{C} = [\underline{c}, \overline{c}]$. Since $\mathbf{A} \preceq \mathbf{B} \oplus \mathbf{C}$ and $\mathbf{C} \preceq \mathbf{0}$, then

$$\begin{aligned} & \underline{a} \leq \underline{b} + \underline{c}, \ \overline{a} \leq \overline{b} + \overline{c} \text{ and } \underline{c} \leq 0, \ \overline{c} \leq 0 \\ \Longrightarrow \quad & \underline{a} \leq \underline{b} \text{ and } \overline{a} \leq \overline{b} \ \Longrightarrow \ \mathbf{A} \preceq \mathbf{B}. \end{aligned}$$

□

A.2 Proof of Lemma 1.7

Proof: Let $\mathbf{B}_1 = \inf\limits_{y \in \mathcal{S}_1}(y)$, $\mathbf{B}_2 = \inf\limits_{y \in \mathcal{S}_2}(y)$, and $\mathbf{B} = \inf\limits_{y \in \mathcal{S}}(y)$.

(i) From Definition 1.5 and Definition 1.19, $\mathbf{B}_2$ is a lower bound of $\mathcal{S}_2$ and $\mathcal{S}_1 \subseteq \mathcal{S}_2$, therefore

$$\begin{aligned} & \mathbf{B}_2 \preceq \inf_{y \in \mathcal{S}_1} \mathbf{F}(y) \\ \implies & \inf_{y \in \mathcal{S}_2} \mathbf{F}(y) \preceq \inf_{y \in \mathcal{S}_1} \mathbf{F}(y). \end{aligned}$$

(ii) Part ((ii)) can be proved in the similar manner as ((i)).

(iii) From Definition 1.5 and Definition 1.19, for each $y \in \mathcal{S}$ and for every $\delta \geq 0$, we get

$$\mathbf{B} \preceq \mathbf{F}(y) \implies \delta \odot \mathbf{B} \preceq \delta \odot \mathbf{F}(y) \implies \delta \odot \mathbf{B} \preceq \inf_{y \in \mathcal{S}} (\delta \odot \mathbf{F}). \tag{A.2}$$

Since $\mathbf{B}$ is an infimum of $\mathbf{F}$, then for given $\epsilon > 0$ and $\delta > 0$, we have

$$\begin{aligned} & \mathbf{F}(y_1) \prec \mathbf{B} \oplus \left[\tfrac{\epsilon}{\delta}, \tfrac{\epsilon}{\delta}\right] \text{ for some } y_1 \in \mathcal{S} \\ \implies & \delta \odot \mathbf{F}(y_1) \prec \delta \odot \left(\mathbf{B} \oplus \left[\tfrac{\epsilon}{\delta}, \tfrac{\epsilon}{\delta}\right]\right) \\ \implies & \delta \odot \mathbf{F}(y_1) \prec (\delta \odot \mathbf{B}) \oplus [\epsilon, \epsilon]. \end{aligned}$$

Due to arbitrariness of ϵ, any interval $\mathbf{C} \in I(\mathbb{R})$ such that $\delta \odot \mathbf{B} \prec \mathbf{C}$, cannot be a lower bound of $\delta \odot \mathbf{B}$. Therefore,

$$\inf_{y \in \mathcal{S}} (\delta \odot \mathbf{F}) \preceq \delta \odot \mathbf{B}. \tag{A.3}$$

From (A.2) and (A.3), we obtain $\inf\limits_{y \in \mathcal{S}} (\delta \odot \mathbf{F}) = \delta \odot \inf\limits_{y \in \mathcal{S}} \mathbf{F}$.

(iv) Part (iv) can be proved in the similar manner as (iii).

□

Appendix B

B.1 Proof of Lemma 2.1

Proof: Let $\mathbf{B}_1 = \{\boldsymbol{\Gamma}(\widehat{\mathbf{X}}) : \widehat{\mathbf{X}} \in \mathbf{S}_1\}$, $\mathbf{B}_2 = \{\boldsymbol{\Gamma}(\widehat{\mathbf{X}}) : \widehat{\mathbf{X}} \in \mathbf{S}_2\}$, and $\mathbf{B} = \{\boldsymbol{\Gamma}(\widehat{\mathbf{X}}) : \widehat{\mathbf{X}} \in \mathbf{S}\}$.

(i) From Definition 2.4, we have

$$\inf_{\mathbf{S}_2} \boldsymbol{\Gamma} = \inf\{\boldsymbol{\Gamma}(\widehat{\mathbf{X}}) : \widehat{\mathbf{X}} \in \mathbf{S}_2\} = \inf \mathbf{B}_2 = \overline{\mathbf{M}}, \text{ where } \overline{\mathbf{M}} \in I(\mathbb{R}).$$

Since $\overline{\mathbf{M}}$ is a lower bound of $\mathbf{B}_2$ and $\mathbf{S}_1 \subseteq \mathbf{S}_2$, $\overline{\mathbf{M}}$ is also a lower bound of $\mathbf{B}_1$. This implies that

$$\overline{\mathbf{M}} \preceq \inf_{\mathbf{S}_1} \boldsymbol{\Gamma} \implies \inf_{\mathbf{S}_2} \boldsymbol{\Gamma} \preceq \inf_{\mathbf{S}_1} \boldsymbol{\Gamma}.$$

(ii) From Definition 2.4, we have

$$\sup_{\mathbf{S}_2} \boldsymbol{\Gamma} = \sup\{\boldsymbol{\Gamma}(\widehat{\mathbf{X}}) : \widehat{\mathbf{X}} \in \mathbf{S}_2\} = \sup \mathbf{B}_2 = \overline{\mathbf{N}}, \text{ where } \overline{\mathbf{N}} \in I(\mathbb{R}).$$

Since $\overline{\mathbf{N}}$ is an upper bound of $\mathbf{B}_2$ and $\mathbf{S}_1 \subseteq \mathbf{S}_2$, therefore $\overline{\mathbf{N}}$ is also an upper bound of $\mathbf{B}_1$. This implies that

$$\sup_{\mathbf{S}_1} \boldsymbol{\Gamma} \preceq \overline{\mathbf{M}} \implies \sup_{\mathbf{S}_1} \boldsymbol{\Gamma} \preceq \sup_{\mathbf{S}_2} \boldsymbol{\Gamma}.$$

(iii) From Definition 2.4, we have

$$\inf_{\mathbf{S}} \boldsymbol{\Gamma} = \inf\{\boldsymbol{\Gamma}(\widehat{\mathbf{X}}) : \widehat{\mathbf{X}} \in \mathbf{S}\} = \inf \mathbf{B} = \overline{\mathbf{M}}.$$

Thus, for each $\widehat{\mathbf{X}} \in \mathbf{S}$ and for every $\beta \geq 0$, we get

$$\overline{\mathbf{M}} \preceq \mathbf{\Gamma}(\widehat{\mathbf{X}}) \implies \beta \odot \overline{\mathbf{M}} \preceq \beta \odot \mathbf{\Gamma}(\widehat{\mathbf{X}}) \implies \beta \odot \overline{\mathbf{M}} \preceq \inf_{\mathbf{S}} (\beta \odot \mathbf{\Gamma}). \tag{B.1}$$

Since $\overline{\mathbf{M}}$ is an infimum of $\mathbf{B}$, for given $\epsilon > 0$ and $\beta > 0$, we have

$$\begin{aligned} & \mathbf{\Gamma}(\widehat{\mathbf{X}}_1) \prec \overline{\mathbf{M}} \oplus \left[\tfrac{\epsilon}{\beta}, \tfrac{\epsilon}{\beta}\right] \text{ for some } \widehat{\mathbf{X}}_1 \in \mathbf{S} \\ \implies \quad & \beta \odot \mathbf{\Gamma}(\widehat{\mathbf{X}}_1) \prec \beta \odot \left(\overline{\mathbf{M}} \oplus \left[\tfrac{\epsilon}{\beta}, \tfrac{\epsilon}{\beta}\right]\right) \\ \implies \quad & \beta \odot \mathbf{\Gamma}(\widehat{\mathbf{X}}_1) \prec (\beta \odot \overline{\mathbf{M}}) \oplus [\epsilon, \epsilon]. \end{aligned}$$

Due to arbitrariness of ϵ, any interval $\mathbf{C} \in I(\mathbb{R})$ such that $\beta \odot \overline{\mathbf{M}} \prec \mathbf{C}$ cannot be a lower bound of $\beta \odot \mathbf{B}$. Therefore,

$$\inf_{\mathbf{S}}(\beta \odot \mathbf{\Gamma}) \preceq \beta \odot \overline{\mathbf{M}}. \tag{B.2}$$

From (B.1) and (B.2), we obtain $\inf_{\mathbf{S}} (\beta \odot \mathbf{\Gamma}) = \beta \odot \inf_{\mathbf{S}} \mathbf{\Gamma}$.

(iv) From Definition 2.4, we have

$$\sup_{\mathbf{S}} \mathbf{\Gamma} = \sup\{\mathbf{\Gamma}(\widehat{\mathbf{X}}) : \widehat{\mathbf{X}} \in \mathbf{S}\} = \sup \mathbf{B} = \overline{\mathbf{N}}.$$

Thus, for each $\widehat{\mathbf{X}} \in \mathbf{S}$ and $\beta \geq 0$, we get

$$\mathbf{\Gamma}(\widehat{\mathbf{X}}) \preceq \overline{\mathbf{N}} \implies \beta \odot \mathbf{\Gamma}(\widehat{\mathbf{X}}) \preceq \beta \odot \overline{\mathbf{N}} \implies \sup_{\mathbf{S}} (\beta \odot \mathbf{\Gamma}) \preceq \beta \odot \overline{\mathbf{N}}. \tag{B.3}$$

Since $\overline{\mathbf{N}}$ is a supremum of $\mathbf{B}$, for given $\epsilon > 0$ and $\beta > 0$, we have

$$\begin{aligned} & \overline{\mathbf{N}} \ominus_{gH} \left[\tfrac{\epsilon}{\beta}, \tfrac{\epsilon}{\beta}\right] \prec \mathbf{\Gamma}(\widehat{\mathbf{X}}_1) \text{ for some } \widehat{\mathbf{X}}_1 \in \mathbf{S} \\ \implies \quad & \beta \odot \left(\overline{\mathbf{N}} \ominus_{gH} \left[\tfrac{\epsilon}{\beta}, \tfrac{\epsilon}{\beta}\right]\right) \prec \beta \odot \mathbf{\Gamma}(\widehat{\mathbf{X}}_1) \\ \implies \quad & \beta \odot \overline{\mathbf{N}} \ominus_{gH} [\epsilon, \epsilon] \prec \beta \odot \mathbf{\Gamma}(\widehat{\mathbf{X}}_1). \end{aligned}$$

Due to arbitrariness of ϵ, any interval $\mathbf{C} \in I(\mathbb{R})$ such that $\mathbf{C} \prec \beta \odot \overline{\mathbf{N}}$ cannot be an upper bound of $\beta \odot \mathbf{B}$. Therefore,

$$\beta \odot \overline{\mathbf{N}} \preceq \sup_{\mathbf{S}}(\beta \odot \mathbf{\Gamma}). \tag{B.4}$$

In view of (B.3) and (B.4), we obtain $\sup\limits_{\mathbf{S}} (\beta \odot \mathbf{\Gamma}) \;=\; \beta \odot \sup\limits_{\mathbf{S}} \mathbf{\Gamma}$.

□

B.2 Proof of Lemma 2.2

Proof: (i) From Definitions 1.5 and 2.4, for each $\widehat{\mathbf{X}} \in \mathbf{S}$ we have

$$\begin{aligned} & \inf_{\mathbf{S}} \mathbf{\Gamma}_1 \preceq \mathbf{\Gamma}_1(\widehat{\mathbf{X}}) \text{ and } \inf_{\mathbf{S}} \mathbf{\Gamma}_2 \preceq \mathbf{\Gamma}_2(\widehat{\mathbf{X}}) \\ \Longrightarrow \quad & \inf_{\mathbf{S}} \mathbf{\Gamma}_1 \oplus \inf_{\mathbf{S}} \mathbf{\Gamma}_2 \preceq \mathbf{\Gamma}_1(\widehat{\mathbf{X}}) \oplus \mathbf{\Gamma}_2(\widehat{\mathbf{X}}) \text{ from Lemma 2.5 of [7].} \end{aligned}$$

Since $\inf\limits_{\mathbf{S}} (\mathbf{\Gamma}_1 \oplus \mathbf{\Gamma}_2)$ is the infimum of $\mathbf{\Gamma}_1(\widehat{\mathbf{X}}) \oplus \mathbf{\Gamma}_2(\widehat{\mathbf{X}})$ for each $\widehat{\mathbf{X}} \in \mathbf{S}$,

$$\inf_{\mathbf{S}} \mathbf{\Gamma}_1 \oplus \inf_{\mathbf{S}} \mathbf{\Gamma}_2 \preceq \inf_{\mathbf{S}} (\mathbf{\Gamma}_1 \oplus \mathbf{\Gamma}_2).$$

(ii) From Definitions 1.6 and 2.4, for each $\widehat{\mathbf{X}} \in \mathbf{S}$ we have

$$\begin{aligned} & \mathbf{\Gamma}_1(\widehat{\mathbf{X}}) \preceq \sup_{\mathbf{S}} \mathbf{\Gamma}_1 \text{ and } \mathbf{\Gamma}_2(\widehat{\mathbf{X}}) \preceq \sup_{\mathbf{S}} \mathbf{\Gamma} \\ \Longrightarrow \quad & \mathbf{\Gamma}_1(\widehat{\mathbf{X}}) \oplus \mathbf{\Gamma}_2(\widehat{\mathbf{X}}) \preceq \sup_{\mathbf{S}} \mathbf{\Gamma}_1 \oplus \sup_{\mathbf{S}} \mathbf{\Gamma}_2 \text{ from Lemma 2.5 of [7].} \end{aligned}$$

Since $\sup\limits_{\mathbf{S}} (\mathbf{\Gamma}_1 \oplus \mathbf{\Gamma}_2)$ is the supremum of $\mathbf{\Gamma}_1(\widehat{\mathbf{X}}) \oplus \mathbf{F}_2(\widehat{\mathbf{X}})$ for each $\widehat{\mathbf{X}} \in \mathbf{S}$, we have

$$\sup_{\mathbf{S}} (\mathbf{\Gamma}_1 \oplus \mathbf{\Gamma}_2) \preceq \sup_{\mathbf{S}} \mathbf{\Gamma}_1 \oplus \sup_{\mathbf{S}} \mathbf{\Gamma}_2.$$

□

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LIST OF PUBLICATIONS

1. **Anshika** and D. Ghosh. Interval-valued value function and its application in interval optimization problems. Computational and Applied Mathematics, 41(4), 1-26, 2022.

2. **Anshika**, D. Ghosh, R. Mesiar, H.R. Yao, and R.S. Chauhan. Generalized-Hukuhara subdifferential analysis and its application in nonconvex composite interval optimization problems. Information Sciences, 622, 771-793, 2023.

3. **Anshika**, K. Kumar, and D. Ghosh. Generalized-Hukuhara global subdifferentiability in interval optimization problems. In: Chiranjibe Jana, Madhumangal Pal, Ghulam Muhiuddin, and Peide Liu (Eds.), Fuzzy Optimization, Decision-Making and Operations Research: Theory and Applications, pp. 135-160, Cham: Springer, 2023.

4. S. Ghosh, D. Ghosh, and **Anshika**. Normal and tangent cones for set of intervals and their application in optimization with functions of interval variables. Soft Computing, 27(15), 10737-10758, 2023.

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6. K. Kumar, **Anshika**, and D. Ghosh. Generalized-Hukuhara subdifferentiability for convex interval-valued functions and its applications in nonsmooth interval optimization. In: Tanmoy Som, Oscar Castillo, Anoop Kumar Tiwari, and Shivam Shreevastava (Eds.), Fuzzy, Rough and Intuitionistic Fuzzy Set Approaches for Data Handling: Theory and Applications, pp. 237-256, Springer Nature Singapore, 2023.

7. D. Ghosh, **Anshika**, Q.H. Ansari, and X. Zhao. Newton method for set optimization problems with set-valued mapping being finitely many vector-valued

functions. (Review Submitted to Optimization)

8. D. Ghosh, **Anshika**, J.C. Yao, and X. Zhao. Quasi-Newton method for set optimization problems with set-valued mapping given by finitely many vector-valued functions. (Review Submitted to Numerical Functional Analysis and Optimization)