Chapter 1

Introduction

1.1 Fractional Calculus

Fractional calculus (FC) is an emerging field of mathematics that deals with arbitrary order derivatives and integrals. It has wide applications and applies to almost all branches of applied science. In 1823, Niels Hernik Abel [1] was first to study the application of FC in solving the tautochrone problem. Since the 19th century, the theory of FC developed rapidly, mostly as a foundation for a number of applied disciplines, including fractional geometry, fractional differential equations (FDE) and fractional dynamics. Later on, many mathematicians wrote numerous books and papers on fractional calculus [2, 3, 4, 5, 6]. In the last few decades the FC gained remarkable attention not only in mathematics but also in physics, biology, statistics, finance, engineering, viscoelasticity, and many more [7, 8, 9, 10, 11, 12, 13]. Numerous real-world applications of FC can be found in [14, 15].

1.2 Definitions and Properties of Fractional Integrals and Derivatives

This section comprises the fundamental definitions and properties of various fractional derivatives and integrals. The detailed definitions can be found in reference books [3, 4, 5, 16].

1.2.1 Fractional Integrals and Derivatives

Definition 1.1. The left-sided and right-sided Riemann-Liouville (R-L) fractional integrals of order $\alpha \in (0,1)$ are defined as

$$ {}_aI_x^{\alpha}\Phi(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1}\Phi(s)ds, \quad x > a, \tag{1.1}$$

$$ {}_xI_b^{\alpha}\Phi(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1}\Phi(s)ds, \quad x < b. \tag{1.2}$$

respectively. Here, $\Gamma(\alpha)$ is the Euler's gamma function.

Definition 1.2. The left-sided and right-sided Riemann-Liouville (R-L) fractional derivatives of order $\alpha \in (0,1)$ are defined as

$$ {}_aD_x^{\alpha}\Phi(x) = D({}_aI_x^{1-\alpha}\Phi(x)) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-s)^{-\alpha}\Phi(s)ds, \qquad x > a, \tag{1.3}$$

$$ {}_xD_b^{\alpha}\Phi(x) = -D({}_xI_b^{1-\alpha}\Phi(x)) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (s-x)^{-\alpha}\Phi(s)ds, \quad x < b. \tag{1.4}$$

respectively.

Definition 1.3. The left-sided and right-sided Caputo fractional derivatives (CFDs) of order $\alpha \in (0,1)$ are defined as

$$ {}_a^C D_x^\alpha \Phi(x) = {}_a I_x^{1-\alpha} D\Phi(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-s)^{-\alpha} \Phi'(s) ds, \qquad x > a, \tag{1.5} $$

$$ {}_x^C D_b^\alpha \Phi(x) = -{}_x I_b^{1-\alpha} D\Phi(x) = \frac{-1}{\Gamma(1-\alpha)} \int_x^b (s-x)^{-\alpha} \Phi'(s) ds, \quad x < b. \tag{1.6} $$

1.2.1.1 Properties of Fractional Integrals and Derivatives

(i) For $\alpha_1, \alpha_2 > 0$, the fractional integral operators ${}_a I_x^\alpha$ and ${}_x I_b^\alpha$ satisfy the semi-group property.

$$ {}_a I_x^{\alpha_1} \, {}_a I_x^{\alpha_2} \Phi(x) = {}_a I_x^{\alpha_2} \, {}_a I_x^{\alpha_1} \Phi(x) = {}_a I_x^{\alpha_1+\alpha_2} \Phi(x) \tag{1.7} $$

$$ {}_x I_b^{\alpha_1} \, {}_x I_b^{\alpha_2} \Phi(x) = {}_x I_b^{\alpha_2} \, {}_x I_b^{\alpha_1} \Phi(x) = {}_x I_b^{\alpha_1+\alpha_2} \Phi(x) \tag{1.8} $$

for all $\Phi \in L^p(a,b), 1 < p \leq \infty$.

(ii) Let $\Phi \in AC[a,b]$ and $g \in L^p(a,b), 1 \leq p \leq \infty$ then for $0 < \alpha < 1$, the formula for integration by parts is

$$ \int_a^b \Phi(x) {}_a D_x^\alpha \Psi(x) dx = \int_a^b \Psi(x) {}_x^C D_b^\alpha \Phi(x) dx + \Phi(x) {}_a I_x^{1-\alpha} \Psi(x) |_{x=a}^{x=b}. \tag{1.9} $$

1.2.2 Generalized Fractional Integrals and Derivatives

Definition 1.4. For $\alpha > 0$, the left and right generalized fractional integrals of a function $\Phi(x)$ are defined as

$$ {}_{a+} I_{[\xi,w]}^\alpha \Phi(x) = \frac{[w(x)]^{-1}}{\Gamma(\alpha)} \int_a^x \frac{w(s)\xi'(s)\Phi(s)}{[\xi(x)-\xi(s)]^{1-\alpha}} ds, \qquad x > a, \tag{1.10} $$

$$ {}_{b-} I_{[\xi,w]}^\alpha \Phi(x) = \frac{w(x)}{\Gamma(\alpha)} \int_x^b \frac{[w(s)]^{-1}\xi'(s)\Phi(s)}{[\xi(s)-\xi(x)]^{1-\alpha}} ds, \qquad b > x, \tag{1.11} $$

respectively, where $\xi(x)$ and $w(x)$ are scale and weight functions.

Definition 1.5. The left and right generalized derivatives of integer order $m \geq 1$ of a function $\Phi(x)$ are defined as

$$D^{m}_{[\xi,w,L]}\Phi(x) = [w(x)]^{-1}\left[\left(\frac{1}{\xi'(x)}D_x\right)^{m}(w(x)\Phi(x))\right]. \tag{1.12}$$

$$D^{m}_{[\xi,w,R]}\Phi(x) = w(x)\left[\left(\frac{-1}{\xi'(x)}D_x\right)^{m}[(w(x)]^{-1}\Phi(x))\right]. \tag{1.13}$$

where D_x is the integer order derivative with respect to x.

Definition 1.6. The left and right Caputo generalized fractional derivatives (CGFD) of order $\alpha > 0$ of a function $\Phi(x)$ are defined as

$${}^{C}_{a+}D^{\alpha}_{[\xi,w]}\Phi(x) = ({}_{a+}I^{m-\alpha}_{[\xi,w]}D^{m}_{[\xi,w,L]}\Phi)(x), \quad m-1 < \alpha < m. \tag{1.14}$$

$${}^{C}_{b-}D^{\alpha}_{[\xi,w]}\Phi(x) = ({}_{b-}I^{m-\alpha}_{[\xi,w]}D^{m}_{[\xi,w,R]}\Phi)(x), \quad m-1 < \alpha < m. \tag{1.15}$$

Definition 1.7. The left and right R-L generalized fractional derivatives of order $\alpha > 0$ of a function $\Phi(x)$ are defined as

$${}^{RL}_{a+}D^{\alpha}_{[\xi,w]}\Phi(x) = (D^{m}_{[\xi,w,L]}\ {}_{a+}I^{m-\alpha}_{[\xi,w]}\Phi)(x), \quad m-1 < \alpha < m. \tag{1.16}$$

$${}^{RL}_{b-}D^{\alpha}_{[\xi,w]}\Phi(x) = (D^{m}_{[\xi,w,R]}\ {}_{b-}I^{m-\alpha}_{[\xi,w]}\Phi)(x), \quad m-1 < \alpha < m. \tag{1.17}$$

Definition 1.8. The generalized fractional integral operator K^{α}_{P} of a function $\Phi(x)$ is defined as [17]:

$$K^{\alpha}_{P}\Phi(x) = p\int_{a}^{x} k_{\alpha}(x,t)\Phi(t)dt + q\int_{x}^{b} k_{\alpha}(t,x)\Phi(t)dt,\ \alpha > 0, \tag{1.18}$$

Eq.(1.18) can be rewritten in convolution form as follows:

$$K_P^\alpha \Phi(x) = p\big(\Phi * k_1^\alpha\big)(x) + q\big(\Phi * k_2^\alpha\big)(x), \tag{1.19}$$

where

$$k_1^\alpha(x) = \begin{cases} k_\alpha(x) & , x > a, \\ 0 & , \text{otherwise}, \end{cases} \tag{1.20}$$

and

$$k_2^\alpha(x) = \begin{cases} k_\alpha(-x) & , x < b, \\ 0 & , \text{otherwise}, \end{cases} \tag{1.21}$$

where p and q are two real numbers, $P = (a, b; p, q)$ is a parameter set and $a < x < b$, k_α is a kernel which depend on α and satisfies the condition of integrability and singularity as mentioned in [18]. The integral operator K_P^α is known a K-operator. For $\alpha = 0$, we replace K_P^α by an identity operators the generalization of definition given in reference [19] for Riemann-Liouville fractional integral.

Now, we discuss two more operators named as B-operator and A-operator [17] which are defined in terms of K-operator as follows:

$$B_P^\alpha \Phi(x) = K_P^{n-\alpha} D^n \Phi(x), \tag{1.22}$$

$$A_P^\alpha \Phi(x) = D^n K_P^{n-\alpha} \Phi(x), \tag{1.23}$$

where n is an integer, $n - 1 < \alpha < n$ and $P = (a, b; p, q)$. We call B_P^α (A_P^α) as B-operator (A-operator) of order α and $B_P^\alpha \Phi(x)$ ($A_P^\alpha \Phi(x)$) as B-operation (A-operation) of function $\Phi(x)$ of order α.

Note that when $p = 0$ and $q = 1$ ($p = 1$ and $q = 0$) and $k_\alpha(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}$ is a power kernel, the operators K_P^α , B_P^α and A_P^α reduce to the right (left) Riemann-Liouville fractional integral, the right (left) Caputo derivative and the right (left)

Riemann-Liouville fractional derivative respectively. As a result, it is obvious that operators K_P^α, B_P^α and A_P^α are more generalized than Riemann-Liouville and Caputo derivatives and integrals.

1.2.2.1 Properties of Generalized Fractional Integrals and Derivatives

Property 1: The integration by parts formula for generalized fractional derivatives are given as

$$\begin{aligned}\int_a^b \xi'(x)\Phi(x){}_{a+}^{RL}D_{[\xi,w]}^{\alpha}\Psi(x)dx &= \int_a^b \xi'(x)\Psi(x){}_{b-}^{C}D_{[\xi,w]}^{\alpha}\Phi(x)dx \\ &\quad + \sum_{k=0}^{m-1}[(D_{[\xi,w,R]}^{k}\Phi(x)){}_{a+}^{RL}D_{[\xi,w]}^{\alpha-1-k}\Psi(x)]\Big|_a^b, \end{aligned} \tag{1.24}$$

$$\begin{aligned}\int_a^b \xi'(x)\Phi(x){}_{b-}^{RL}D_{[\xi,w]}^{\alpha}\Psi(x)dx &= \int_a^b \xi'(x)\Psi(x){}_{a+}^{C}D_{[\xi,w]}^{\alpha}\Phi(x)dx \\ &\quad - \sum_{k=0}^{m-1}[(D_{[\xi,w,L]}^{k}\Phi(x)){}_{a+}^{RL}D_{[\xi,w]}^{\alpha-1-k}\Psi(x)]\Big|_a^b. \end{aligned} \tag{1.25}$$

Property 2: The operators A_P^α and B_P^α satisfy the following formulas for integration by parts for $0 < \alpha < 1$ [17],

$$\int_a^b \Psi(x)A_P^{\alpha}\Phi(x)dx = -\int_a^b \Phi(x)B_{P^*}^{\alpha}\Psi(x)dx + \Psi(x)A_P^{\alpha-1}\Phi(x)\Big|_a^b, \tag{1.26}$$

$$\int_a^b \Psi(x)B_P^{\alpha}\Phi(x)dx = -\int_a^b \Phi(x)A_{P^*}^{\alpha}\Psi(x)dx + (A_P^{\alpha-1}\Psi(x))\Phi(x)\Big|_a^b, \tag{1.27}$$

where $\Phi(x)$ and $\Psi(x)$ are sufficiently smooth functions, $P = (a, b; p, q)$ and $P^* = (a, b; q, p)$.

1.2.3 Tempered Fractional Integrals and Derivatives

Here, we discuss some basic definitions and fundamental properties of tempered fractional integrals and derivatives. However, we refer the readers to references [20, 21] for a more detailed study.

Definition 1.9. Let $\alpha \in (m-1, m)$ and $\tau \geq 0$ be a fixed parameter. The left and right R-L tempered fractional integrals (TFIs) of a function $\Phi(x)$ of order α are defined as

$$ {}_aI_x^{\alpha,\tau}\Phi(x) = (e^{-\tau x}\ {}_aI_x^{\alpha}\ e^{\tau x})\Phi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{e^{-\tau(x-s)}\Phi(s)}{(x-s)^{1-\alpha}}ds,\ x > a, \tag{1.28}$$

$$ {}_xI_b^{\alpha,\tau}\Phi(x) = (e^{\tau x}\ {}_xI_b^{\alpha} e^{-\tau x})\Phi(x) = \frac{1}{\Gamma(\alpha)}\int_x^b \frac{e^{-\tau(s-x)}\Phi(s)}{(s-x)^{1-\alpha}}ds,\ x < b. \tag{1.29}$$

Definition 1.10. Let $\alpha \in (m-1, m)$ and $\tau \geq 0$ be a fixed parameter. The left and right R-L tempered fractional derivatives (TFDs) of a function $\Phi(x)$ of order α are defined as

$$ {}_aD_x^{\alpha,\tau}\Phi(x) = (e^{-\tau x}\ {}_aD_x^{\alpha} e^{\tau x})\Phi(x) = \frac{e^{-\tau x}}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_a^x \frac{e^{\tau s}\Phi(s)}{(x-s)^{1-m+\alpha}}ds,\ x > a, \tag{1.30}$$

$$ {}_xD_b^{\alpha,\tau}\Phi(x) = (e^{\tau x}\ {}_xD_b^{\alpha} e^{-\tau x})\Phi(x) = \frac{(-1)^m e^{\tau x}}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_x^b \frac{e^{-\tau s}\Phi(s)}{(s-x)^{1-m+\alpha}}ds,\ x < b. \tag{1.31}$$

Definition 1.11. Let $\alpha \in (m-1, m)$ and $\tau \geq 0$ be a fixed parameter. The left and right Caputo TFDs a function $\Phi(x)$ of order α are defined as

$$ {}_a^cD_x^{\alpha,\tau}\Phi(x) = (e^{-\tau x}\ {}_a^cD_x^{\alpha} e^{\tau x})f(x) = \frac{e^{-\tau x}}{\Gamma(m-\alpha)}\int_a^x \frac{[e^{\tau s}\Phi(s)]^{(m)}}{(x-s)^{1-m+\alpha}}ds,\ x > a, \tag{1.32}$$

$$ {}_x^cD_b^{\alpha,\tau}\Phi(x) = (e^{\tau x}\ {}_x^cD_b^{\alpha} e^{\tau x})f(x) = \frac{(-1)^m e^{\tau x}}{\Gamma(m-\alpha)}\int_x^b \frac{[e^{-\tau s}\Phi(s)]^{(m)}}{(s-x)^{1-m+\alpha}}ds,\ x < b. \tag{1.33}$$

Property 3: Let $\alpha \in (m-1, m)$ and $\tau \geq 0$ be a fixed parameter, for $\Phi(x) \in AC^m[a,b]$, we have

$$ {}_aI_x^{\alpha,\tau}\ {}_a^cD_x^{\alpha,\tau}\Phi(x) = \Phi(x) - \sum_{k=0}^{m-1} e^{-\tau(x-a)}\frac{(-1)^k}{k!}(x-a)^k\Phi^k(a), \tag{1.34}$$

$$ {}_xI_b^{\alpha,\tau}\ {}_x^cD_b^{\alpha,\tau}\Phi(x) = \Phi(x) - \sum_{k=0}^{m-1} e^{\tau(x-b)}\frac{(-1)^k}{k!}(b-x)^k\Phi^k(b). \tag{1.35}$$

Note: For $\tau = 0$, these definitions 1.9, 1.10 and 1.11 reduces to Riemann-Liouville fractional integrals, derivatives, and Caputo fractional derivatives respectively.

Definition 1.12. The modified power kernel [22] is defined as

$$k_\alpha(x) = \frac{1}{\Gamma(\alpha)[m + (1-m)(x)^{1-\alpha}]}, \tag{1.36}$$

which leads to the power kernel for $m = 0$.

Definition 1.13. The three parameter Mittag-Leffler function $E^{\gamma}_{\rho,\alpha}$, introduced by Prabhakar [23]

$$E^{\gamma}_{\rho,\alpha}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)z^n}{\Gamma(\gamma)\Gamma(\rho n+\alpha)n!}, \tag{1.37}$$

$(\rho, \alpha, \gamma \in \mathbb{C}; \mathrm{Re}(\rho), \mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > 0)$.

Definition 1.14. The Prabhakar kernel [24, 25] is defined as

$$k_\alpha(x) = (x)^{\alpha-1}E^{\gamma}_{\rho,\alpha}(\omega(x)^\rho). \tag{1.38}$$

Property 4: Let $\rho, \alpha, \gamma \in \mathbb{C}$ such that $\mathrm{Re}(\rho), \mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > 0$. Then

$$\int_0^t z^{\alpha-1}E^{\gamma}_{\rho,\alpha}(wz^\rho)dz = t^\alpha E^{\gamma}_{\rho,\alpha+1}(wt^\rho). \tag{1.39}$$

Theorem 1.2.1. **Poincare-Writinger inequality [26, 27]:** Assume that $1 \leq p \leq \infty$. For every function $\Phi(x)$ in the sobolev space $L_p[a,b]$, we have

$$\|\Phi(x)\|_{L^p} \leq C|\Delta\Phi(x)|_{L^p}. \tag{1.40}$$

Theorem 1.2.2. **Fractional Poincare-Friedrichs [28]:**For every function $\Phi(x) \in L_p[a,b]$, $1 \leq p \leq \infty$, we have

$$\|\Phi(x)\|_{L^2} \leq C|D^\alpha\Phi(x)|_{L^2}. \tag{1.41}$$

where D^α is the fractional order derivative of any type.

1.3 The Sturm-Liouville Problem

During 1836-37, Sturm and Liouville proposed a new theory in mathematical analysis known as SL theory. This SL theory deals with linear second order differential equation subjected to boundary conditions (BCs). The standard form of the Sturm-Liouville (SL) differential equation is given as

$$-\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)y(x) = \lambda r(x)y(x), x \in [a,b], \tag{1.42}$$

with imposed BCs

$$c_1y(a) + c_2y'(a) = 0, (c_1^2 + c_2^2 > 0), \tag{1.43}$$

$$d_1y(b) + d_2y'(b) = 0, (d_1^2 + d_2^2 > 0), \tag{1.44}$$

where $p(x), p'(x), q(x)$ and $r(x)$ are continuous functions with $p(x), r(x) > 0$ for $x \in [a, b]$. This SLP given by Eqs.(1.42)-(1.44) is classified as a regular SLP. When $p(x)$ is zero at end points of interval $[a, b]$, the problem is called a singular SLP. The value of λ for which Eqs.(1.42)-(1.44) have a non trivial solution is known as an eigenvalue, and the associated solution is known as an eigenfunction. These kind of equations are frequently used in both classical physics and quantum physics to describe processes in which system transmits some energy while boundary value is held constants. It is observed and realized that SL theory has many applications in real world. In last several years, this theory has been thoroughly researched. For an in-depth study of Sturm-Liouville theory, we refer to refs. [29, 30].

1.4 The Fractional Sturm-Liouville Problem

SLPs play a key role in numerous fields including mathematics, science, and engineering. In the last decade, researchers have focused on a generalization of classical order SLPs to the fractional one. The fractional order SLP is defined in the following form:

$$D^{\alpha}(p(x)D^{\alpha}y(x)) + q(x)y(x) = \lambda r(x)y(x), x \in [a, b], \tag{1.45}$$

where $\alpha \in (n-1, n), n \in \mathbb{N}$ and the derivative D^{α} is the fractional order derivative. Various forms of FSLPs can be defined using different types of fractional derivatives and boundary conditions and some of these types of FSLPs are studied in this thesis.

1.5 Literature Review

This section explores the past and present studies on SLPs, FSLPs, and their applications.

1.5.1 The Literature Review on Sturm-Lioville Problem

In 1836-37, Sturm and Liouville developed a new theory and later it was known as Sturm-Lioville theory. They published various research articles on SLPs. Firstly, Sturm solved the partial differential equations by transforming them into the SLP [31]. Then, Sturm and Liouville examined the properties of eigenvalues and eigenfunctions [32, 33]. The Sturm-Liouville theory has an important role in mathematics, science and engineering, and also has a large number of applications. Several articles on SLP and books on application of SL theory can be found in literature and here we cite few of them [34, 35, 36]. Nowadays, researchers have focused on extending the theories of classical SLPs to fractional order SLPs. Here, we provide a brief literature review on FSLPs in the upcoming section.

1.5.2 The Literature Review on Fractional Sturm-Liouville Problem

Over past years, scientist and researchers studied about FSLPs and began to develop numerical methods for solving FSLPs [37, 38, 39]. In [40, 41, 42], Klimek and Agrawal developed some theoretical results of FSLP and investigated the properties of FSLPs such as eigenvalues are real and eigenfunctions forms an orthogonal set. In [43], Rivero et al. presented the FSLP in operator form. Further, Zayernouri and Karniadakis [44] studied two types of FSLPs, including regular and singular

FSLPs, to demonstrate that eigenvalues are real and corresponding eigenfunctions are orthogonal. In [45], authors proposed a numerical algorithm for FSLP in integral form. Further, Klimek et al. [46] demonstrated infinite eigenvalues for FSLP of order $0 < \alpha < 1/2$ and proven the orthogonality of eigenfunctions. In [47], authors defined FSLP in terms of Riesz derivative and proposed a matrix method for its solution. Li and Qi [48] considered the fractional eigenvalue problems and showed that FSLPs have countable real eigenvalues. Then, authors in [49] presented FSLP in unbounded domains and studied its spectral properties. The existence and uniqueness of solution for FSLP is discussed in [50]. Further, Derakhshan and Ansari presented a numerical approximation of the Prabhakar fractional Sturm-Liouville problems [51]. Recently, researchers in [52] defined different types of fractional equations formed by composition of Caputo/Riemann-Liouville derivative with different BCs and demonstrate the nature of eigenvalues. In [53], authors solved FSLP using fractional differential transform method. In a recent study, Klimek et al. [54] used the mid-point rectangular rule to numerically solve the fractional Sturm-Liouville problem with homogeneous Neumann boundary conditions. Moreover, few other methods for solving FSLPs include fractional series method [55], Laplace transform method [56], finite element method [57] and fractional order Legendre tau method [58] etc.

1.6 Motivation

The fractional calculus is the extension of classical calculus which contains integrals and derivatives of arbitrary orders. In the last few decades, many applications of fractional calculus have been seen in different fields of science, physics, biology, economics, engineering and finance, etc. In the past decade, fractional calculus has been recognized among one of the suitable approaches for describing long-memory processes [59]. Fractional calculus provides essential framework for modeling various physical processes involving memory effects such as those which arise in complex or viscoelastic media. A lot of work has been done on Riemann-Liouville and Caputo fractional derivatives but recently authors have presented new definitions for fractional derivatives/ integrals like tempered fractional derivatives/integrals, generalized fractional derivatives, Hilfer derivatives etc. [7, 21, 60, 61] and also used it in solving complicated real-world problems. The fractional derivatives and integral with respect to another functions are discussed by Kilbas [16]. Further, Agrawal proposed new definitions of these integrals and derivatives using weight and scale functions known as generalized/weighted fractional derivative and integrals. Further details of weighted fractional operators can be found in [60, 61].

1.7 Problem Statement and Thesis Objectives

In this thesis, we present the numerical algorithm for solving GFSLPs and TFSLPs defined in terms of B-operator, genralized fractional derivatives and tempered fractional derivatives. Several research papers has been published on FSLPs but there is still a room of contribution to these problems by generalizing them. The pointwise objectives of this thesis are summarized as follows:

(i) To develop a numerical algorithm to solve the generalized fractional Sturm-Liouville problem defined in terms of B-operator.

(ii) To study an approximation and convergence of generalized fractional Sturm-Liouville problem via integral form.

(iii) To develop a numerical algorithm on non-uniform grid points for solving GFSLP.

(iv) To investigate the eigenvalues and eigenfunctions of a higher order TFSLP using integral form.

1.8 Outline of the Thesis

The structure of the thesis is outlined as follows:

Chapter 1: This chapter introduces the thesis, including definitions of fractional integrals and derivatives along with their properties. Further, literature reviews on fractional calculus, SLPs and, FSLPs are presented. Additionally, the motivation for the considered problem of this thesis is also discussed.

Chapter 2: Chapter 2 describes a numerical scheme for the GFSLP with mixed boundary conditions. The GFSLP is defined in terms of a B-operator consisting of an integral operator with a kernel and a differential operator. One of the main features of the B-operator is that for different kernels, it leads to different SLPs, and thus the same formulation can be used to discuss different SLPs. We prove the well-posedness of the proposed GFSLP. Further, the approximated eigenvalues of GFSLP are obtained for two different kernels namely a modified power kernel and Prabhakar kernel in the B-operator. We obtain real eigenvalues and corresponding orthogonal eigenfunctions. The theoretical and numerical convergence orders of eigenvalues and eigenvectors are also discussed. Further, the numerically obtained eigenvalues and eigenfunctions are used to construct an approximate solution of the one-dimensional fractional diffusion equation defined in a bounded domain.

Chapter 3: Chapter 3 aims to present a numerical algorithm for solving the generalized fractional Sturm-Liouville differential equation. We define the GFSLP in terms of Caputo generalized fractional derivatives. Firstly, the GFSLP is transformed into an integral form using a weighted Laplace transform. Next, we consider the GFSLP under different cases of boundary conditions. Further, we approximate the generalized fractional integral using Lagrange interpolating polynomial and form an eigenvalue problem for different types of boundary conditions. The bound for the truncation error in the approximation is proved. Finally, the test cases of GFSLPs are considered for numerical simulations of the proposed numerical scheme and examine the theoretical claims.

Chapter 4: This chapter presents a numerical algorithm for the FSLP defined in terms of Caputo generalized fractional derivative. Firstly, the well-posedness of the considered FSLP is discussed. Further, we have divided the provided interval into non-uniform node points and varied the weight and scale functions to compute the estimated eigenvalues and corresponding eigenfunctions. The resulting eigenvalues are real and eigenfunctions are orthogonal. Furthermore, we found the bound for the solution and discussed the convergence order numerically. Further, approximated eigenvalues and eigenfunctions are used to form an approximate solution of the fractional diffusion equation with generalized fractional derivatives.

Chapter 5: In this chapter, a TFSLP with mixed boundary conditions is considered. Firstly, the TFSLP is transformed into its corresponding integral form. Next, to present the numerical solution of the corresponding integral form, we used linear interpolation to determine the approximation of tempered integral. The discretized version of integral form is presented for two cases: $\alpha \in (0, 1]$ and $\alpha \in (1, 2]$. In the final part of this chapter, some examples of numerical solutions of the considered equation are shown. Also, the convergence orders of the discussed numerical scheme are presented.

Chapter 6: Finally, the thesis is concluded in this chapter and remarks on future work are given.

****