Chapter 2

Quasi projective synchronization of time varying delayed complex valued Cohen-Grossberg neural networks

2.1 Introduction

This chapter explores quasi-projective synchronization in drive-response systems within complex-valued Cohen-Grossberg neural networks (CGNNs) characterized by time-varying delays. This study aims to develop a criterion for achieving quasi-projective synchronization between two non-identical CGNNs by designing an appropriate controller and employing a direct method. In this context, the state vectors, connection weights, activation, amplification, and behaved functions of CGNNs are all complex-valued. Selecting an activation function poses a significant challenge in investigating the dynamic behavior of complex-valued CGNNs. As cited in [86], Liouville's theorem establishes that every bounded and analytic function in the complex plane must be constant. Consequently, the activation functions of complex-valued CGNNs cannot be both bounded and analytic. Thus, the critical issues for complex-valued CGNNs extend beyond choosing activation functions to consider behaved and amplification functions. This model exhibits greater complexity and flexibility than real-valued models due to its information processing in the complex domain. A noteworthy contribution of the chapter lies in estimating the bound of the synchronization error and establishing sufficient criteria for synchronization between master and response systems. The efficacy of the proposed method is validated through numerical simulations applied to a specific example. Here are some notable contributions to this research.

1. A linear feedback controller is developed to investigate projective synchronization in non-identical time-varying delayed complex-valued CGNNs.

2. Synchronization and stability conditions are established, aided by definitions, lemmas, and specific cases. The synchronization error bound is determined, and its correlation with the controller parameter is elucidated.

3. The methodology is executed in the complex domain, utilizing the direct method.

2.2 Model description and preliminaries

Complex-valued time-varying delayed CGNNs is consider as the master system as

$$\dot{\omega}(t) = -C_1[h(\omega(t)) - Af(\omega(t)) - Bg(\omega(t - \mathcal{T}_0(t))) + \hat{\mathcal{R}}(t)], \tag{2.1}$$

where $\omega(t) = (\omega_1(t), \omega_2(t), \ldots, \omega_n(t))^T \in \mathbb{C}^n$ is the state vector, $C_1 = diag(c_1, c_2, \ldots, c_n) \in \mathbb{C}^{n\times n}$ denotes the amplification gain, $A = (a_{\hat{i}j})_{n\times n} \in \mathbb{C}^{n\times n}$ is the connection weight matrix without time varying delays and $B = (b_{\hat{i}j})_{n\times n} \in \mathbb{C}^{n\times n}$ is the connection weight with time varying delays, $\mathcal{T}_0(t)$ is the transmission delay, $\hat{i}, j \in I$, $h(\omega(t)) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ where $h(\omega(t)) = (h_1(\omega_1(t)), h_2(\omega_2(t)), \ldots, h_n(\omega_n(t)))^T$ represents the vector valued behaved function, $f(\omega(t)) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $g(\omega(t - \mathcal{T}_0(t))) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ are the complex valued activation functions with and without time varying delays respectively, where $f(\omega(t)) = (f_1(\omega_1(t)), f_2(\omega_2(t)), \ldots, f_n(\omega_n(t)))^T$ and $g(\omega(t - \mathcal{T}_0(t))) = (g_1(\omega_1(t - \mathcal{T}_0(t))), g_2(\omega_2(t - \mathcal{T}_0(t))), \ldots, g_n(\omega_n(t - \mathcal{T}_0(t))))^T$, and the external input vector is $\hat{\mathcal{R}}(t) = (\hat{\mathcal{R}}_1(t), \hat{\mathcal{R}}_2(t), \ldots, \hat{\mathcal{R}}_n(t))^T \in \mathbb{C}^n$.

The following assumptions are satisfied by the complex valued activation function, behaved function and $\mathcal{T}_0(t)$.

Assumption 2.2.1. Each of behaved and activation functions $h_j(\cdot)$, $g_j(\cdot)$ and $f_j(\cdot)$ for $j \in I$ satisfies the following Lipschitz conditions

$$|h_j(x) - h_j(y)| \leq s_j|x - y|,$$
$$|f_j(x) - f_j(y)| \leq m_j|x - y|,$$
$$|g_j(x) - g_j(y)| \leq l_j|x - y|,$$

for all $x,\ y \in \mathbb{C}$.

Assumption 2.2.2. Each of activation functions $f_j(\cdot)$, $g_j(\cdot)$ and behaved function $h_j(\cdot)$, for $j \in I$ are bounded i.e., $\exists$ positive real numbers P_j, Q_j, N_j for any $x \in \mathbb{C}$ such that

$$|h_j(x)| \leq P_j,\ |f_j(x)| \leq Q_j,\ |g_j(x)| \leq N_j.$$

Assumption 2.2.3. $\mathcal{T}_0(t) \geq 0$ is a differential function with $0 \leq \mathcal{T}_0(t) \leq \delta\ \forall\ t$, where δ is a constant.

The equation that represents the response system for (2.1) is defined as

$$\dot{\tilde{\omega}}(t) = -C_2[h(\tilde{\omega}(t)) - A^{'}f(\tilde{\omega}(t)) - B^{'}g(\tilde{\omega}(t - \mathcal{T}_0(t))) + \tilde{\mathcal{R}}(t)] + W(t), \tag{2.2}$$

where $\tilde{\omega}(t) = (\tilde{\omega}_1(t), \tilde{\omega}_2(t) \ldots, \tilde{\omega}_n(t))^T \in \mathbb{C}^n$ is the state vector of (2), $A^{'}, B^{'} \in \mathbb{C}^{n\times n}$ and $\tilde{\mathcal{R}}(t) = (\tilde{\mathcal{R}}_1(t), \tilde{\mathcal{R}}_2(t), \ldots, \tilde{\mathcal{R}}_n(t))^T \in \mathbb{C}^n$ denotes the external input vector and $W(t) = (W_1(t), W_2(t) \ldots, W_n(t))^T \in \mathbb{C}^n$ is a controller designed by

$$W_j(t) = -k_j(\tilde{\omega}_j(t) - r\omega_j(t)),\ j \in I, \tag{2.3}$$

where $k_j \in \mathbb{C}$ is a proportionality constant, and the difference $\tilde{\omega}_j(t) - r\omega_j(t)$ represents the weighted difference between the response variable $\tilde{\omega}_j(t)$ and and the drive variable $\omega_j(t)$). The controller is proportional to a weighted difference of the variables of response and drive.

Definition 2.1. [87] The response system (2.2) and master system (2.1) are said to be Quasi-Projectively synchronized, if there exists a small bound $\theta > 0$ such that

$$\lim_{t\to\infty} ||\tilde{\omega}(t) - r\omega(t)|| \leq \theta,$$

where $r \in \mathbb{C}$ is a projective coefficient.

Remark 2.2.1. It's worth noting that if we substitute projective constant r with scaling function matrix in the formulation of QPS, our suggested approach can be extended to MFPS of time varying delayed complex valued chaotic neural networks. The problem becomes quasi anti-synchronisation of time-varying delayed complex valued chaotic neural networks if the projective constant r is substituted by -1.

Definition 2.2. [88] If there exist positive constants ζ, φ such that

$$|\epsilon(t)| \leq \varphi e^{-\zeta t},$$

for all $t > 0$, then $\epsilon(t)$ is global exponentially stable with convergence rate ζ.

Lemma 2.3. *[89] Let us suppose that the continuous function* $\psi(t) : [t_0 - \omega, \infty) \to [0, \infty)$ *satisfies the following inequality:*

$$\dot{\psi}(t) \leq -X_1\psi(t) + X_2\psi(t - \mathcal{T}_0(t)) + P,$$

for $t \geq t_0$*, where* $X_1 > X_2 > 0,\ P > 0,\ \mathcal{T}_0(t) \leq \xi$*, then* $\psi(t) \leq \sup\limits_{-\xi \leq s \leq 0} (\psi(s))e^{-\eta t} + \frac{P}{\eta}$*, where* $\eta > 0$ *is the unique solution of* $X_1 - X_2 e^{\eta\xi} - \eta = 0$.

Lemma 2.4. *[90] For any* $G, H \in \mathbb{C}^n$*, and* $c \in \mathbb{R}^+$*, the following inequality is true.*

$$G^T\bar{H} + \bar{G}^T H \leq c\,\|G\|^2 + \frac{1}{c}\,\|H\|^2 .$$

2.3 Main results

This chapter focuses on the quasi-projective synchronization of two non-identical CGNNs i.e., $C_1 \neq C_2$, $A' \neq A$, $B' \neq B$, and $\tilde{\mathcal{R}} \neq \hat{\mathcal{R}}$. Now, let us define the synchronization error between the master and response systems as $\epsilon_j(t) = \tilde{\omega}_j(t) -$

$r\omega_j(t)$, $j \in I$, where $r \in \mathbb{C}$ is known as projective coefficient. From equations (2.1) and (2.2), and by definition of error function, we get the error system as

$$\begin{aligned}\dot{\epsilon}(t) = &- C_2 h(\tilde{\omega}(t)) + rC_1 h(\omega(t)) + C_2 A' f(\tilde{\omega}(t)) - rC_1 A f(\omega(t)) + C_2 B' g(\tilde{\omega}(t \\ &- \mathcal{T}_0(t))) - rC_1 B g(\omega(t - \mathcal{T}_0(t))) - C_2 \tilde{\mathcal{R}} + rC_1 \hat{\mathcal{R}}(t) + W(t). \end{aligned} \tag{2.4}$$

Because of the presence of non-identical parameters and a projective coefficient r, it is evident that the equilibrium point of (2.4) is non-zero, i.e., $\epsilon \neq 0$. Consequently, it is impossible to achieve complete projective synchronization between the systems (2.1) and (2.2). However, up to a small error bound θ, the quasi-projective synchronization can be achieved via the controller.

The error system (2.4) could be rewritten by substituting the controller (2.3) in (2.4) as

$$\begin{aligned}\dot{\epsilon}(t) = &- C_2 h(\epsilon(t)) + C_2 A' f(\epsilon(t)) + C_2 B' g(\epsilon(t - \mathcal{T}_0(t)) + rC_1 h(\omega(t)) - C_2 h(r\omega(t)) \\ &+ C_2 A' f(r\omega(t)) - rC_1 A f(\omega(t)) + C_2 B' g(r\omega(t - \mathcal{T}_0(t)) - rC_1 B g(\omega(t - \mathcal{T}_0(t))) \\ &- C_2 \tilde{\mathcal{R}}(t) + rC_1 \hat{\mathcal{R}}(t) - k\epsilon(t), \end{aligned} \tag{2.5}$$

where $h(\epsilon(t)) = h(\tilde{\omega}(t)) - h(r\omega(t))$, $f(\epsilon(t)) = f(\tilde{\omega}(t)) - f(r\omega(t))$, $g(\epsilon(t - \mathcal{T}_0(t))) = g(\tilde{\omega}(t - \mathcal{T}_0(t))) - g(r\omega(t - \mathcal{T}_0(t)))$, and $k = diag\{k_1, k_2, \ldots, k_n\}$.

Theorem 2.5. *If the Assumptions 2.2.1-2.2.2 hold, then the master system (2.1) is quasi-projectively synchronized with the response system (2.2) based on the controller*

(2.3), *if there exist constants* $d_1, d_2 > 0$ *and* $\alpha_{\hat{i}} > 0$ $(\hat{i} = 1, 2, 3)$ *such that*

$$\lambda_{\min}\Big((2\operatorname{Re}\{(k)\} + \alpha_1 \|C_2\| + \frac{1}{\alpha_1}\|C_2\|\|S\|^2 - \alpha_2\|C_2A'\| - \frac{1}{\alpha_2}\|C_2A'\|\|M\|^2$$
$$- \alpha_3\|C_2B'\| - 2(\|C_1\| + \|C_2\|) - (\|C_1A\| + \|C_2A'\|) - (\|C_1B\| + \|C_2B'\|)$$
$$- d_1)I_n\Big) > 0,$$
$$\frac{1}{\alpha_3}\|C_2B'\|\|L\|^2 - d_2 < 0,$$

and

$$d_1 - d_2 > 0. \tag{2.6}$$

Where $\lambda_{\min}(.)$ *is the minimum eigenvalue and* $2\operatorname{Re}\{(k)\} = k + \bar{k}$. *Moreover, in the region* $\mathcal{D} = \Big\{\epsilon(t) \in \mathbb{C}^n : \|\epsilon(t)\|^2 \le \frac{X}{\eta}\Big\}$, *the synchronization error converges exponentially, where* $X = (\|C_2\| + r\bar{r}\|C_1\|)\|P\|^2 + (\|C_2A'\| + r\bar{r}\|C_1A\|)\|Q\|^2 + (\|C_2B'\| + r\bar{r}\|C_1B\|)\|N\|^2 + \|C_2\|\Big\|\tilde{\mathcal{R}}\Big\|^2 + r\bar{r}\|C_1\|\Big\|\hat{\mathcal{R}}\Big\|^2$. η *is the unique solution of* $d_1 - d_2e^{\eta\delta} - \eta = 0$.

Proof. Defining the Lyapunov function as

$$V(t) = \epsilon(t)^T\bar{\epsilon}(t).$$

Taking the first order time derivative and from equations (2.4), we get

$$\begin{aligned}\dot{V}(t) =& \bar{\epsilon}^T(t)[-C_2h(\epsilon(t)) + C_2A^{'}f(\epsilon(t)) + C_2B^{'}g(\epsilon(t - \mathcal{T}_0(t)) + rC_1h(\omega(t))\\ &- C_2h(r\omega(t)) + C_2A^{'}f(r\omega(t)) - rC_1Af(\omega(t)) + C_2B^{'}g(r\omega(t - \mathcal{T}_0(t))\\ &- rC_1Bg(\omega(t - \mathcal{T}_0(t))) - C_2\tilde{\mathcal{R}}(t) + rC_1\hat{\mathcal{R}}(t) - k\epsilon(t)] + \epsilon^T(t)[-\bar{C}_2\bar{h}(\epsilon(t))\\ &+ \bar{C}_2\bar{A}^{'}\bar{f}(\epsilon(t)) + \bar{C}_2\bar{B}^{'}\bar{g}(\epsilon(t - \mathcal{T}_0(t)) + \bar{r}\bar{C}_1\bar{h}(\omega(t)) - \bar{C}_2\bar{h}(r\omega(t))\end{aligned}$$

$$
\begin{aligned}
&+ \bar{C}_2\bar{A}^{'}\bar{f}(r\omega(t)) - \bar{r}\bar{C}_1\bar{A}\bar{f}(\omega(t)) + \bar{C}_2\bar{B}^{'}\bar{g}(r\omega(t-\mathcal{T}_0(t)) - \bar{r}\bar{C}_1\bar{B}\bar{g}(\omega(t \\
&- \mathcal{T}_0(t))) - \bar{C}_2\bar{\tilde{\mathcal{R}}}(t) + \bar{r}\bar{C}_1\bar{\hat{\mathcal{R}}}(t) - \bar{k}\bar{\epsilon}(t)] \\
=& -2\operatorname{Re}\{(k)\}\epsilon^T(t)\bar{\epsilon}(t) - [\bar{\epsilon}^T(t)C_2h(\epsilon(t)) + \epsilon^T(t)\bar{C}_2\bar{h}(\epsilon(t))] + [\bar{\epsilon}^T(t)C_2A^{'} \\
&\times f(\epsilon(t)) + \epsilon^T(t)\bar{C}_2\bar{A}'\bar{f}(\epsilon(t))] + [\bar{\epsilon}^T(t)C_2B^{'}g(\epsilon(t-\mathcal{T}_0(t))) + \epsilon^T(t)\bar{C}_2\bar{B}' \\
&\times \bar{g}(\epsilon(t-\mathcal{T}_0(t)))] - \bar{\epsilon}(t)C_2h(r\omega(t)) + \bar{\epsilon}^T(t)rC_1h(\omega(t)) - \epsilon^T(t)\bar{C}_2\bar{h}(r\omega(t)) \\
&+ \epsilon^T(t)\bar{r}\bar{C}_1\bar{h}(\omega(t)) + \bar{\epsilon}^T(t)C_2A^{'}f(r\omega(t)) - \bar{\epsilon}^T(t)rC_1Af(\omega(t)) + \epsilon^T(t)\bar{C}_2 \\
&\times \bar{A}'\bar{f}(r\omega(t)) - \epsilon^T(t)\bar{r}\bar{C}_1\bar{A}\bar{f}(\omega(t)) + \bar{\epsilon}^T(t)C_2B^{'}g(r\omega(t-\mathcal{T}_0(t))) \\
&- \bar{\epsilon}^T(t)rC_1Bg(\omega(t-\mathcal{T}_0(t))) + \epsilon^T(t)\bar{C}_2\bar{B}'\bar{g}(r\omega(t-\mathcal{T}_0(t))) - \epsilon^T(t) \\
&\times \bar{r}\bar{C}_1\bar{B}\bar{g}(\omega(t-\mathcal{T}_0(t))) - \bar{\epsilon}^T(t)C_2\tilde{\mathcal{R}}(t) + \bar{\epsilon}^T(t)rC_1\hat{\mathcal{R}}(t) - \epsilon^T(t)\bar{C}_2\bar{\tilde{\mathcal{R}}}(t) \\
&+ \epsilon^T(t)\bar{r}\bar{C}_1\bar{\hat{\mathcal{R}}}(t). \qquad (2.7)
\end{aligned}
$$

In view of Lemma 2.4 and Assumption 2.2.1, let $S = diag\{s_1, s_2, \ldots, s_n\}$, $M = diag\{m_1, m_2, \ldots, m_n\}$, $L = diag\{l_1, l_2, \ldots, l_n\}$, then we have

$$
\begin{aligned}
h^T(\epsilon(t))\bar{h}(\epsilon(t)) \leq& \epsilon(t)^TS^TS\bar{\epsilon}(t) \\
f^T(\epsilon(t))\bar{f}(\epsilon(t)) \leq& \epsilon(t)^TM^TM\bar{\epsilon}(t) \\
g^T(\epsilon(t-\mathcal{T}_0(t)))\bar{g}(\epsilon(t-\mathcal{T}_0(t))) \leq& \epsilon(t-\mathcal{T}_0(t))^TL^TL\bar{\epsilon}(t-\mathcal{T}_0(t)), \qquad (2.8)
\end{aligned}
$$

and then merging them with Lemma 2.4 to produce

$$
\begin{aligned}
\bar{\epsilon}(t)^TC_2h(\epsilon(t)) + \epsilon^T\bar{C}_2\bar{h}(\epsilon(t)) \leq& \alpha_1\|C_2\|\|\epsilon(t)\|^2 + \frac{1}{\alpha_1\|C_2\|}\|C_2h(\epsilon(t))\|^2 \\
\leq& \alpha_1\|C_2\|\|\epsilon(t)\|^2 + \frac{1}{\alpha_1}\|C_2\|\|S\|^2\|\epsilon(t)\|^2, \qquad (2.9)
\end{aligned}
$$

$$\text{and}\quad \bar{\epsilon}(t)^T C_2 A^{'} f(\epsilon(t)) + \epsilon^T(t)\bar{C}_2\bar{A}^{'}\bar{f}(\epsilon(t)) \leq \alpha_2 \left\| C_2 A^{'} \right\| \|\epsilon(t)\|^2 + \frac{1}{\alpha_2}\left\| C_2 A^{'} \right\| \|M\|^2 \|\epsilon(t)\|^2, \tag{2.10}$$

$$\bar{\epsilon}(t)^T C_2 B^{'} g(\epsilon(t-\mathcal{T}_0(t))) + \epsilon^T(t)\bar{C}\bar{B}_2^{'}\bar{g}(\epsilon(t-\mathcal{T}_0(t))) \leq \alpha_3 \left\| C_2 B^{'} \right\| \|\epsilon(t)\|^2 + \frac{1}{\alpha_3}\left\| C_2 B^{'} \right\| \|L\|^2 \|\epsilon(t-\mathcal{T}_0(t))\|^2. \tag{2.11}$$

Additionally, We can derive the following inequality as

$$\begin{aligned}
&-\bar{\epsilon}^T C_2 h(r\omega(t)) + \bar{\epsilon}^T r C_1 h(\omega(t)) - \epsilon^T(t)\bar{C}_2\bar{h}(r\omega(t)) + \epsilon^T(t)\bar{r}\bar{C}_1\bar{h}(\omega(t)) \\
&\leq \|C_2\| \|\epsilon(t)\|^2 + \frac{1}{\|C_2\|}\|C_2 h(r\omega(t)\|^2 + \|C_1\| \|\epsilon(t)\|^2 + \frac{1}{\|C_1\|}\|rC_1 h(\omega(t)\|^2 \\
&\leq (\|C_1\| + \|C_2\|)\|\epsilon(t)\|^2 + (\|C_2\| + r\bar{r}\|C_1\|)\|P\|^2.
\end{aligned} \tag{2.12}$$

Similarly

$$\begin{aligned}
&\bar{\epsilon}^T C_2 A^{'} f(r\omega(t)) - \bar{\epsilon}^T r C_1 A f(\omega(t)) + \epsilon^T(t)\bar{C}_2\bar{A}^{'}\bar{f}(r\omega(t)) - \epsilon^T(t)\bar{r}\bar{C}_1\bar{A}\bar{f}(\omega(t)) \\
&\leq (\|C_1 A\| + \|C_2 A'\|)\|\epsilon(t)\|^2 + (\|C_2 A'\| + r\bar{r}\|C_1 A\|)\|Q\|^2, \\
&\bar{\epsilon}^T C_2 B^{'} g(r\omega(t-\mathcal{T}_0(t))) - \bar{\epsilon}^T r C_1 B g(\omega(t-\mathcal{T}_0(t))) + \epsilon^T(t)\bar{C}_2\bar{B}^{'}\bar{g}(r\omega(t-\mathcal{T}_0(t))) \\
&-\epsilon^T(t)\bar{r}\bar{C}_1\bar{B}\bar{g}(\omega(t-\mathcal{T}_0(t))) \leq (\|C_1 B\| + \|C_2 B'\|)\|\epsilon(t)\|^2 + (\|C_2 B'\| \\
&+ r\bar{r}\|C_1 B\|)\|N\|^2,
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
\text{and}\quad & -\bar{\epsilon}^T(t) C_2\tilde{\mathcal{R}}(t) + \bar{\epsilon}^T(t) r C_1\hat{\mathcal{R}}(t) - \epsilon^T(t)\bar{C}_2\bar{\tilde{\mathcal{R}}}(t) + \epsilon^T(t)\bar{r}\bar{C}_1\bar{\hat{\mathcal{R}}}(t) \\
&\leq (\|C_1\| + \|C_2\|)\|\epsilon(t)\|^2 + \|C_2\|\left\|\tilde{\mathcal{R}}\right\|^2 + r\bar{r}\|C_1\|\left\|\hat{\mathcal{R}}\right\|^2.
\end{aligned} \tag{2.14}$$

Substituting the inequalities (2.8)-(2.14) in the equation (2.7), and using the result (2.6), we get

$$\begin{aligned}\dot{V}(t) \leq & \epsilon^T(t)\Big[-2\,\mathrm{Re}\{(k)\} - \alpha_1 \|C_2\| - \frac{1}{\alpha_1}\|C_2\|\,\|S\|^2 + \alpha_2\|C_2A'\| + \frac{1}{\alpha_2}\|C_2A'\|\,\|M\|^2 \\ & + \alpha_3\|C_2B'\| + 2(\|C_1\| + \|C_2\|) + (\|C_1A\| + \|C_2A'\|) + (\|C_1B\| + \|C_2B'\|) \\ & + d_1\Big]\bar{\epsilon}(t) - \epsilon^T(t)d_1\bar{\epsilon}(t) + \epsilon(t-\mathcal{T}_0(t))^T\Big[\frac{1}{\alpha_3}\|C_2B'\|\,\|L\|^2 - d_2\Big]\bar{\epsilon}(t-\mathcal{T}_0(t)) \\ & + \epsilon(t-\mathcal{T}_0(t))^T d_2\bar{\epsilon}(t-\mathcal{T}_0(t)) + (\|C_2\| + r\bar{r}\|C_1\|)\|P\|^2 + (\|C_2A'\| \\ & + r\bar{r}\|C_1A\|)\|Q\|^2 + (\|C_2B'\| + r\bar{r}\|C_1B\|)\|N\|^2 + \|C_2\|\left\|\tilde{\mathcal{R}}\right\|^2 \\ & + r\bar{r}\|C_1\|\left\|\hat{\mathcal{R}}\right\|^2. \\ \leq & -d_1V(t) + d_2V(t-\mathcal{T}_0(t)) + X. \end{aligned} \tag{2.15}$$

Since $d_1 - d_2 > 0$, it follows from Lemma 2.3 that

$$V(t) \leq \underset{-\delta\leq s\leq 0}{sup}(V(s))e^{-\eta t} + \frac{X}{\eta},$$

where $\eta > 0$ is the unique solution of $d_1 - d_2e^{\eta\delta} - \eta = 0$, which implies

$$\|\epsilon(t)\|^2 \leq \underset{-\delta\leq s\leq 0}{sup}(V(s))e^{-\eta t} + \frac{X}{\eta}.$$

Therefore error $\epsilon(t)$ converges exponentially to the region

$$\mathcal{D} = \left\{\epsilon(t) : \|\epsilon(t)\|^2 \leq \theta = \frac{X}{\eta}\right\}.$$

Hence, from Definition 2.1, we can easily get the quasi-projective synchronization between CVNNs (2.2) and (2.1). The proof is completed. □

Particularly, for quasi-synchronization if we take $r = 1$, then the synchronization error between (2.1) and (2.2) is $\epsilon_j(t) = \tilde{\omega}_j(t) - \omega_j(t)$. Now we obtain the error system as

$$\begin{aligned}\dot{\epsilon}(t) =& - C_2 h(\epsilon(t)) + C_2 A^{'} f(\epsilon(t)) + C_2 B^{'} g(\epsilon(t - \mathcal{T}_0(t)) + \Delta C h(\omega(t)) + \Delta A f(\omega(t)) \\ &+ \Delta B g(\omega(t - \mathcal{T}_0(t))) - C_2 \tilde{\mathcal{R}}(t) + C_1 \hat{\mathcal{R}}(t) - k\epsilon(t),\end{aligned} \tag{2.16}$$

where $h(\epsilon(t)) = h(\tilde{\omega}(t)) - h(\omega(t)),\ f(\epsilon(t)) = f(\tilde{\omega}(t)) - f(\omega(t)),\ g(\epsilon(t - \mathcal{T}_0(t))) = g(\tilde{\omega}(t - \mathcal{T}_0(t))) - g(\omega(t - \mathcal{T}_0(t))), \Delta C = C_1 - C_2,\ \Delta A = C_2 A^{'} - C_1 A,\ \Delta B = C_2 B^{'} - C_1 B,\ k = diag\{k_1, k_2, \dots, k_n\}$.

As a result, the following corollary can be derived.

Corollary 2.6. *If the Assumptions 2.2.1-2.2.2 hold then the response system (2.2) is quasi synchronized with the master system (2.1) under the controller (2.3), if there exist constants $b_1 > 0, b_2 > 0$ and $\beta_{\hat{i}} > 0$ $(\hat{i} = 1, 2, 3)$ such that*

$$\begin{aligned}&\lambda_{\min}\Big((2\,\mathrm{Re}\{(k)\} + \beta_1 \|C_2\| + \frac{1}{\beta_1}\|C_2\|\|S\|^2 - \beta_2\|C_2A'\| - \frac{1}{\beta_2}\|C_2A'\|\|M\|^2 \\ &-\beta_3\|C_2B'\| - (\|C_1\| + \|C_2\|) - \|\Delta C\| - \|\Delta A\| - \|\Delta B\| - b_1)I_n\Big) > 0, \\ &\frac{1}{\beta_3}\|C_2B'\|\|L\|^2 - b_2 < 0,\end{aligned}$$

and

$$b_1 - b_2 > 0. \tag{2.17}$$

The region of convergence of the synchronization error is

$$\tilde{\mathcal{D}} = \left\{\epsilon(t) \in \mathbb{C}^n : \|\epsilon(t)\|^2 \leq \frac{X}{\eta}\right\}, \tag{2.18}$$

where $X = \|\Delta C\|\|P\|^2 + \|\Delta A\|\|Q\|^2 + \|\Delta B\|\|N\|^2 + \|C_2\|\left\|\tilde{\mathcal{R}}\right\|^2 + \|C_1\|\left\|\hat{\mathcal{R}}\right\|^2$, *$\eta$ is the unique solution of* $b_1 - b_2 e^{\eta\delta} - \eta = 0$.

Proof. Defining the Lyapunov function as

$$V(t) = \epsilon(t)^T \bar{\epsilon}(t).$$

Taking the first order time derivative and from equation (2.16), we get

$$\begin{aligned}
\dot{V}(t) =& -2\,\mathrm{Re}\{(k)\}\epsilon^T(t)\bar{\epsilon}(t) - [\bar{\epsilon}^T(t)C_2 h(\epsilon(t)) + \epsilon^T(t)\bar{C}_2\bar{h}(\epsilon(t))] + [\bar{\epsilon}^T(t)C_2 A' f(\epsilon(t)) \\
&+ \epsilon^T(t)\bar{C}_2\bar{A}'\bar{f}(\epsilon(t))] + [\bar{\epsilon}^T(t)C_2 B' g(\epsilon(t-\mathcal{T}_0(t))) + \epsilon^T(t)\bar{C}_2\bar{B}'\bar{g}(\epsilon(t-\mathcal{T}_0(t)))] \\
&+ \bar{\epsilon}^T(t)\Delta C h(\omega(t)) + \epsilon^T(t)\bar{\Delta C}\bar{h}(\omega(t)) + \bar{\epsilon}^T(t)\Delta A f(\omega(t)) + \epsilon^T(t)\bar{\Delta A}\bar{f}(\omega(t)) \\
&+ \bar{\epsilon}^T(t)\Delta B g(\omega(t-\mathcal{T}_0(t))) + \epsilon^T(t)\bar{\Delta B}\bar{g}(\omega(t-\mathcal{T}_0(t))) - \bar{\epsilon}^T(t)C_2\tilde{\mathcal{R}}(t) \\
&+ \bar{\epsilon}^T(t)C_1\hat{\mathcal{R}}(t) - \epsilon^T(t)\bar{C}_2\bar{\tilde{\mathcal{R}}}(t) + \epsilon^T(t)\bar{C}_1\bar{\hat{\mathcal{R}}}(t)
\end{aligned} \tag{2.19}$$

Based on (2.8), Lemma 2.4, and Assumption 2.2.1, we have

$$\bar{\epsilon}(t)^T C_2 h(\epsilon(t)) + \epsilon^T \bar{C}_2\bar{h}(\epsilon(t)) \leq \beta_1 \|C_2\| \|\epsilon(t)\|^2 + \frac{1}{\beta_1}\|C_2\| \|S\|^2 \|\epsilon(t)\|^2$$

and

$$\bar{\epsilon}(t)^T C_2 A' f(\epsilon(t)) + \epsilon^T(t)\bar{C}_2\bar{A}'\bar{f}(\epsilon(t)) \leq \beta_2 \left\|C_2 A'\right\| \|\epsilon(t)\|^2 + \frac{1}{\beta_2}\left\|C_2 A'\right\| \|M\|^2 \|\epsilon(t)\|^2,$$

$$\begin{aligned}
\bar{\epsilon}(t)^T C_2 B' g(\epsilon(t-\mathcal{T}_0(t))) + \epsilon^T(t)\bar{C}\bar{B}_2'\bar{g}(\epsilon(t-\mathcal{T}_0(t))) \leq& \beta_3 \left\|C_2 B'\right\| \|\epsilon(t)\|^2 \\
&+ \frac{1}{\beta_3}\left\|C_2 B'\right\| \|L\|^2 \|\epsilon(t-\mathcal{T}_0(t))\|^2.
\end{aligned} \tag{2.20}$$

Furthermore, we can deduce the following inequality as

$$\bar{\epsilon}^T(t)\Delta C h(\omega(t)) + \epsilon^T(t)\bar{\Delta C}\bar{h}(\omega(t)) \leq \|\Delta C\| \|\epsilon(t)\|^2 + \|\Delta C\| \|P\|^2. \tag{2.21}$$

Similarly

$$\bar{\epsilon}^T(t)\Delta A f(\omega(t)) + \epsilon^T(t)\bar{\Delta A}\bar{f}(\omega(t)) \leq \|\Delta A\| \|\epsilon(t)\|^2 + \|\Delta A\| \|Q\|^2, \tag{2.22}$$

$$\bar{\epsilon}^T(t)\Delta B g(\omega(t-\mathcal{T}_0(t))) + \epsilon^T(t)\bar{\Delta B}\bar{g}(\omega(t-\mathcal{T}_0(t))) \leq \|\Delta B\| \|\epsilon(t)\|^2 + \|\Delta B\| \|N\|^2, \tag{2.23}$$

and

$$\begin{aligned}
&-\bar{\epsilon}^T(t)C_2\tilde{\mathcal{R}}(t) + \bar{\epsilon}^T(t)C_1\hat{\mathcal{R}}(t) - \epsilon^T(t)\bar{C}_2\bar{\tilde{\mathcal{R}}}(t) + \epsilon^T(t)\bar{C}_1\bar{\hat{\mathcal{R}}}(t) \\
&\leq (\|C_1\| + \|C_2\|)\|\epsilon(t)\|^2 + \|C_2\| \left\|\tilde{\mathcal{R}}\right\|^2 + \|C_1\| \left\|\hat{\mathcal{R}}\right\|^2.
\end{aligned} \tag{2.24}$$

Substituting the inequalities (2.20)-(2.24) in the equation (2.19), and using the result (2.17), we get

$$\begin{aligned}
\dot{V}(t) \leq& \epsilon^T(t)\Big[-2\,\mathrm{Re}\{(k)\} - \beta_1\|C_2\| - \frac{1}{\beta_1}\|C_2\|\|S\|^2 + \beta_2\|C_2A'\| + \frac{1}{\beta_2}\|C_2A'\| \\
&\times \|M\|^2 + \beta_3\|C_2B'\| + (\|C_1\| + \|C_2\|) + \|\Delta C\| + \|\Delta A\| + \|\Delta B\| + b_1\Big]\bar{\epsilon}(t) \\
&- \epsilon^T(t)b_1\bar{\epsilon}(t) + \epsilon(t-\mathcal{T}_0(t))^T\Big[\frac{1}{\beta_3}\|C_2B'\|\|L\|^2 - b_2\Big]\bar{\epsilon}(t-\mathcal{T}_0(t)) + \epsilon(t-\mathcal{T}_0(t))^T \\
&\times b_2\bar{\epsilon}(t-\mathcal{T}_0(t)) + \|\Delta C\|\|P\|^2 + \|\Delta A\|\|Q\|^2 + \|\Delta B\|\|N\|^2 + \|C_2\|\left\|\tilde{\mathcal{R}}\right\|^2 \\
&+ \|C_1\|\left\|\hat{\mathcal{R}}\right\|^2 \\
\leq& -b_1V(t) + b_2V(t-\mathcal{T}_0(t)) + X.
\end{aligned} \tag{2.25}$$

Since $b_1 - b_2 > 0$, Lemma 2.3 gives, $V(t) \leq \sup_{-\delta\leq s\leq 0}(V(s))e^{-\eta t} + \frac{X}{\eta}$, where $\eta > 0$, which implies $\|\epsilon(t)\|^2 \leq \sup_{-\delta\leq s\leq 0}(V(s))e^{-\eta t} + \frac{X}{\eta}$.
Therefore, error $\epsilon(t)$ converges to the region (2.15). Hence, we can obtain the quasi-synchronization between CVNNs (2.2) and (2.1). The proof is completed. □

Additionally, if we take identical systems such that $A' = A$, $B' = B$, $C_1 = C_2$ and $\tilde{\mathcal{R}} = \hat{\mathcal{R}}$, then the problem changes into synchronization between the systems (2.1) and (2.2). we can derive the following corollary.

Corollary 2.7. *Assume that Assumptions 2.2.1-2.2.2 hold, and there exist constants $b_1 > 0, b_2 > 0$ and $\beta_{\hat{i}} > 0$ $(\hat{i} = 1, 2, 3)$ such that*

$$\lambda_{\min}\Big((2\,\mathrm{Re}\{(k)\} + \beta_1 \|C\| + \frac{1}{\beta_1}\|C\|\,\|S\|^2 - \beta_2\|CA\| - \frac{1}{\beta_2}\|CA\|\,\|M\|^2 - \beta_3\|CB\| - b_1)I_n\Big) > 0,$$

$$\frac{1}{\beta_3}\|CB\|\,\|L\|^2 - b_2 < 0,$$

$$and \;\; b_1 - b_2 > 0. \tag{2.26}$$

Then the response system (2.2) is globally exponential synchronized with the master system (2.1) under the controller (2.3).

***Remark* 2.3.1.** A sufficient criterion is established for ensuring QPS between non-identical systems (2.1) and (2.2). The upper bound of synchronization error is estimated, as well as the connection between it and the controller parameter given in equation (2.3). Theorem 1 shows that the estimated upper bound is dependent on k. As a result, by choosing the suitable controller parameter, the bound can be regulated. The relevant evaluation is shown in the reference [1]. In [1], k is real positive constant but in our case $k \in \mathbb{C}$ and there is no restriction on k. Hence, our conclusions are more generic.

***Remark* 2.3.2.** We limit the behaved functions in this study to constant for the simplicity purpose. Also, the amplification function is unrestricted. We will attempt to break this constraint in future research and investigate the broader one.

2.4 Simulation examples

In this section, an example is taken to validate the efficiency and effectiveness of the above theoretical results.

Example 2.1. *Let us consider the two-dimensional time varying delayed complex-valued CGNNs as the master system as*

$$\dot{\omega}(t) = -C_1[h(\omega(t)) - Af(\omega(t)) - Bg(\omega(t - \mathcal{T}_0(t))) + \hat{\mathcal{R}}(t)], \tag{2.27}$$

and the response system corresponding to the master system (2.24) *as*

$$\dot{\tilde{\omega}}(t) = -C_2[h(\tilde{\omega}(t)) - A^{'}f(\tilde{\omega}(t)) - B^{'}g(\tilde{\omega}(t - \mathcal{T}_0(t))) + \tilde{\mathcal{R}}(t)] + W(t), \tag{2.28}$$

where

$$C_1 = \begin{bmatrix} 0.4 + i & 0 \\ 0 & 0.2 + 0.4i \end{bmatrix}, \ A = \begin{bmatrix} 1.52 + 2i & 1.1 - 2i \\ -1.18 - 2i & 2 + 2i \end{bmatrix},$$
$$B = \begin{bmatrix} 0.8 - 2.1i & 0.4 + i \\ -1 - i & 0.4 + 0.5i \end{bmatrix}, C_2 = \begin{bmatrix} 0.5 + i & 0 \\ 0 & 0.4 + 0.7i \end{bmatrix},$$
$$A' = \begin{bmatrix} 1.58 + 2.2i & 1.2 - 2.1i \\ -1.16 - 1.5i & 1.8 + 2.1i \end{bmatrix}, B' = \begin{bmatrix} 0.9 - 2i & 0.5 + i \\ -1.5 - i & 0.6 + 0.7i \end{bmatrix},$$

with the initial conditions $\omega_1(z) = 1.5 - 2i$, $\tilde{\omega}_1(z) = -2 + 0.5i$, *for* $z \in [-0.5, 0]$ *and* $\omega_2(z) = -2 - i$, $\tilde{\omega}_2(z) = 5 + 2.5i$ *for* $z \in [-1, 0]$. *The behaved and activation* *functions are taken as*

$$h_j(\omega_j(t)) = f_j(\omega_j(t)) = \frac{0.25(1 - exp(-x_j(t)))}{1 + exp(-x_j(t))} + \frac{0.25}{1 + exp(-y_j(t))}i,$$
$$g(\omega_j(t - \mathcal{T}_0(t))) = \frac{0.25(1 - exp(-x_j(t - \mathcal{T}_0(t))))}{1 + exp(-x_j(t - \mathcal{T}_0(t)))} + \frac{0.25}{1 + exp(-y_j(t - \mathcal{T}_0(t)))}i,$$

and $\mathcal{T}_0(t) = 0.2sin^2(t)$, $\delta = 0.2$, $\hat{\mathcal{R}}(t) = (0.1cos(t)+0.2sin(t)i, 0)$, $\tilde{\mathcal{R}}(t) = (0, 0.5cos(t)+0.2sin(t)i)$. *The other parameters can be calculated as* $s_j = 0.5$, $l_j = 0.5$, $m_j = 0.5$, $P_j = Q_j = N_j = 0.3535$, $j = 1, 2$.

Case 1. *Here we choose* $r = 0.1 + 0.5i$, *the controller function given in* (2.3) *with* $k_1 = 30 + 6i, k_2 = 30 + 3i$. *The constants are given by* $d_1 = 2.25$, $d_2 = 1.305$ *and* $\alpha_1 = \alpha_2 = \alpha_3 = 1$. *Now it is confirmed that the conditions given in* (2.6) *hold. Apart from that we have* $X = 1.7508653$ *and* $\eta = 6.39$ *[1]. Therefore, from Theorem 2.1, we obtain* $\mathcal{D} = \left\{\epsilon(t) : \|\epsilon(t)\|^2 \leq \frac{1.7508653}{6.39} = 0.274\right\}$ *as the convergence region, which is shown in Figure 2.1. Hence quasi-projective synchronization of master system* (2.1) *and response system* (2.2) *is achieved.*

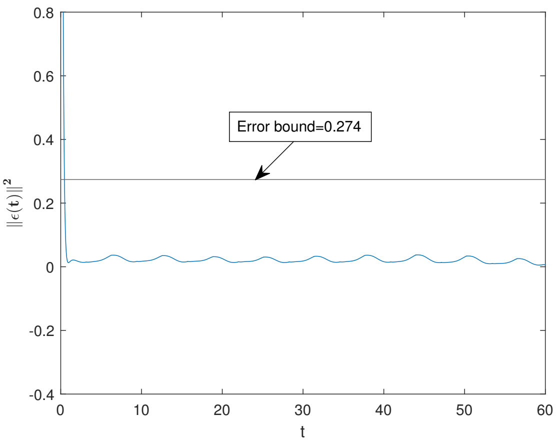

FIGURE 2.1: Synchronization error vs. t for case 1.

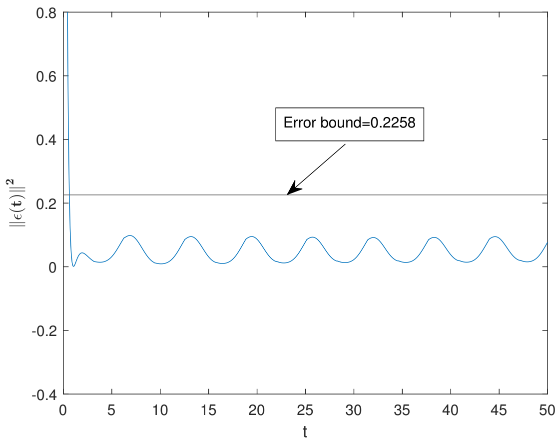

FIGURE 2.2: Synchronization error vs. t for case 2.

Case 2. *Taking* $r = 1$, *and* $k_1 = 31 - 5i$, $k_2 = 31 - 2i$, $b_1 = 5.7232$, $b_2 = 1.502$, $\beta_1 = \beta_2 = \beta_3 = 1$ *and other parameters from Case 1 satisfying Corollary 1,*

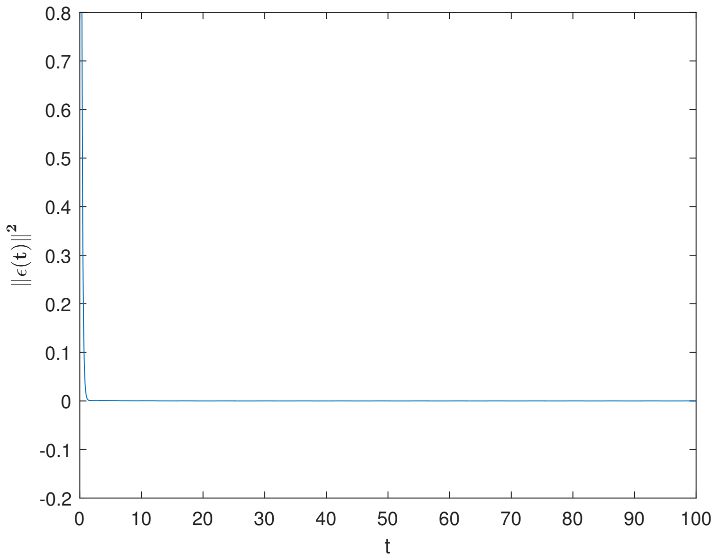

FIGURE 2.3: Synchronization error vs. t for case 3 for two identical CGNNs.

we have obtained $X = 0.67563$ *and* $\eta = 2.991$. *Hence, the quasi synchronization of the master system* (2.1) *and response system* (2.2) *can be accomplished in this case and the error bound will be* $\tilde{\mathcal{D}} = \left\{\epsilon(t) : \|\epsilon(t)\|^2 \leq \frac{0.67563}{2.991} = 0.2258\right\}$, *which is also depicted in Figure 2.2.*

Case 3. *In the case of identical systems, we take* $A' = A = \begin{bmatrix} 1.58 + 2.2i & 1.2 - 2.1i \\ -1.16 - 1.5i & 1.8 + 2.1i \end{bmatrix}$, $B' = B = \begin{bmatrix} 0.9 - 2i & 0.5 + i \\ -1.5 - i & 0.6 + 0.7i \end{bmatrix}$, $C_1 = C_2 = \begin{bmatrix} 0.5 + i & 0 \\ 0 & 0.4 + 0.7i \end{bmatrix}$, *and* $\tilde{\mathcal{R}} = \hat{\mathcal{R}} = (0.1cos(t) + 0.2sin(t)i, 0)$. *The calculated results show that conditions* (2.23) *are satisfied. According to Corollary 2, master system* (2.1) *and response system* (2.2) *are globally exponentially synchronized, and the error curve is depicted in Figure 2.3.*

2.5 Conclusions

The study investigates the quasi-projective synchronization of complex-valued CGNNs with time-varying delays. The main aim is achieving quasi-projective synchronization under a controller, accounting for non-identical parameters and projective coefficients among the system networks' states. Instead of the real decomposition method, the direct method is employed. Various definitions, inequalities, and lemmas are utilized to attain the desired outcomes. Moreover, the synchronization error is evaluated to establish its upper bound and correlation with the controller. Furthermore, several sufficient criteria are derived for specific cases. The efficacy of the method is demonstrated by applying it to a numerical example. The numerical simulation of complex-valued CGNN synchronization underscores the effectiveness of the approach.

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