

Nonsmooth Analysis and Numerical Techniques for Interval and Set Optimization Problems



Thesis submitted in partial fulfillment

for the Award of Degree

Doctor of Philosophy

by

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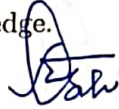
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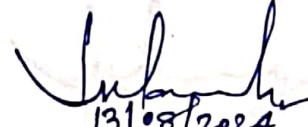
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Dedicated to my parents,

Mr. Nayan Tara Ghosh

and

Mrs. Susmita Ghosh

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List of Symbols

Symbol	Description
\mathbb{R}	The collection of real numbers
\mathbb{R}_+	The collection of non-negative real numbers
\mathbb{R}_-	The collection of non-positive real numbers
$I(\mathbb{R})$	The collection of all compact intervals
$\mathcal{N}_{\mathcal{Y}}(\widehat{\mathbf{Y}})$	The normal cone at $\widehat{\mathbf{Y}}$ on the set $\mathcal{Y} \subseteq I(\mathbb{R})^n$
$\mathcal{T}_{\mathcal{Y}}(\widehat{\mathbf{Y}})$	The tangent cone at $\widehat{\mathbf{Y}}$ on the set $\mathcal{Y} \subseteq I(\mathbb{R})^n$
$\text{Cl}(\mathbf{S})$	Closure of the set $\mathbf{S} \subseteq I(\mathbb{R})^n$
$\mathbf{0}$	$[0, 0]$
$\mathbf{1}$	$[1, 1]$
$-\infty$	$[-\infty, -\infty]$
∞	$[\infty, \infty]$
$I(\mathbb{R})^n$	$I(\mathbb{R}) \times I(\mathbb{R}) \times I(\mathbb{R}) \times \cdots \times I(\mathbb{R})$ (n times)
$I(\mathbb{R}_-, \mathbb{R}_+)$	$\{[\underline{b}, \bar{b}] : \underline{b} \in \mathbb{R}_-, \bar{b} \in \mathbb{R}_+\}$
$I(\mathbb{R}_+)$	$\{[\underline{v}, \bar{v}] : \underline{v}, \bar{v} \in \mathbb{R}_+\}$
$I(\mathbb{R}_-)$	$\{[\underline{v}, \bar{v}] : \underline{v}, \bar{v} \in \mathbb{R}_-\}$
$\mathcal{B}(x^*, \delta)$	The open ball centered at x^* with radius δ
$\mathcal{N}(\bar{x})$	A neighborhood of $\bar{x} \in \mathbb{R}^n$
$\ \cdot\ _{I(\mathbb{R})}$	Norm on $I(\mathbb{R})$
$\widehat{\mathbf{A}}, \widehat{\mathbf{B}}, \widehat{\mathbf{C}}, \dots$	Bold capitals with a hat represents elements of $I(\mathbb{R})^n$ ($n \geq 2$)
$\overline{\mathbb{R}}$	$\mathbb{R} \cup \{-\infty, +\infty\}$
$\overline{I(\mathbb{R})}$	$I(\mathbb{R}) \cup \{-\infty, +\infty\}$
\mathbb{R}_+^m	The non-negative of \mathbb{R}^m
\mathbb{R}_{++}^m	The positive hyper-octant of \mathbb{R}^m
$\mathcal{P}(\mathbb{R}^m)$	The class of all non-trivial subsets of \mathbb{R}^m
$\text{int}(\mathcal{A})$	Interior of $\mathcal{A} \in \mathcal{P}(\mathbb{R}^m)$
$\text{bd}(\mathcal{A})$	Boundary of $\mathcal{A} \in \mathcal{P}(\mathbb{R}^m)$
$[k]$	$\{1, 2, \dots, k\}$

\top	The transpose of a vector
$ \mathcal{A} $	Cardinality of a finite set \mathcal{A}
$\ \cdot\ $	Norm of a vector .

Abbreviations

Abbreviation	Description
IVF	Interval-Valued Function
IOP	Interval Optimization Problem
NIOP	Non-smooth Interval Primal Optimization Problem
NIOD	Non-smooth Interval Dual Optimization Problem
VOP	Vector Optimization Problem
SOP	Set Optimization Problem
MOP	Multiobjective Optimization Problem
KKT	Karush-Kuhn-Tucker
TRM	Trust-Region Method
NTRM	Non-monotone Trust-Region Method
Max-NTRM	Max type Non-monotone Trust-Region Method
Avg-NTRM	Average type Non-monotone Trust-Region Method