Chapter 2

Smooth hypersurfaces of positive Gaussian curvature

In this chapter, we will prove that the Weyl transform of a smooth measure supported on a smooth hypersurface of positive Gaussian curvature is compact, and belongs to the p-Schatten class if $p > n \geq 6$.

2.1 Introduction

Recall that the analog of the Riemann-Lebesgue lemma for the Weyl transform is the fact that if $f \in L^1(\mathbb{R}^{2n})$, then $W(f)$ is a compact operator. Moreover, if $f \in L^p(\mathbb{R}^{2n})$, $1 \leq p \leq 2$, then $W(f)$ is a compact operator (see [67]), and $W(f) \in S^{p'}(\mathcal{H})$, where $\frac{1}{p} + \frac{1}{p'} = 1$ (see e.g., [9, 38]).

In [47], Simon proved that if $r \in (2, \infty)$, then there exists a function $f \in L^r(\mathbb{R}^{2n})$ such that $W(f) \notin \mathcal{B}(\mathcal{H})$ (defined in the sense of tempered distributions). If $r \in (2, \infty]$, and $f \in L^r(\mathbb{R}^{2n})$, a sufficient condition for $W(f)$ to be a compact operator was given by Wong in [67]. He proved that if $f \in L^r(\mathbb{R}^{2n})$, $2 < r \leq \infty$ is such that $\widehat{f} \in L^{r'}(\mathbb{R}^{2n})$, where $\frac{1}{r} + \frac{1}{r'} = 1$, then $W(f)$ is a compact operator on $\mathcal{H}$.

As discussed in Chapter 1, the Fourier transform of a measure need not vanish at infinity. However, with appropriate curvature assumptions on the support of a measure, the Fourier transform of the measure belongs to $\mathcal{C}_0$. In fact it belongs to some L^p space (see Theorem 1.4.3). We will prove an analog of this result for the Weyl transform.

Observe that if λ is absolutely continuous with respect to the Lebesgue measure m on $\mathbb{R}^{2n}$, i.e., $\lambda = fm$ for some $f \in L^1(\mathbb{R}^{2n})$, then $W(\lambda) = W(f)$, and hence $W(\lambda)$ is compact. In general, the Weyl transform of a measure is not compact. For example, the Weyl transform of the Dirac measure δ_0 is the identity operator, which is not compact.

In [54], Thangavelu proved that if μ_r is the normalized surface measure on the sphere $|z| = r$ in $\mathbb{C}^n$, then

$$W(\mu_r) = \sum_{k=0}^{\infty} \frac{k!(n-1)!}{(k+n-1)!} \varphi_k(r) P_k,$$

where φ_k are the Laguerre functions of type $(n-1)$ and P_k is the orthogonal projection of $\mathcal{H}$ onto the k^{th} eigenspace of the harmonic oscillator (see [54, Proposition 4.1]). It follows that $W(\mu_r)$ is a compact operator. By the asymptotic properties of the Laguerre functions (see [52, Theorem 8.22.1]), it follows that

$$W(\mu_r) \in S^p(\mathcal{H}) \tag{2.1}$$

if and only if $p > 4n/(2n-1)$.

Over the years, many mathematicians have worked on the problem of finding conditions on the kernel of an integral operator under which the corresponding integral operator belongs to some Schatten class (see e.g., [1, 6, 7]). The conditions on the integral kernel under which the corresponding integral operator defined on a smooth compact manifold belongs to Schatten class is described in [6].

The main result of this chapter is the following theorem, which is an analog of Theorem 1.4.3 for the Weyl transform, and is proved in Section 2.4.

Theorem 2.1.1. *Suppose S is a compact connected smooth hypersurface in $\mathbb{R}^{2n}$, $n \geq 2$, whose Gaussian curvature is positive everywhere. Let $\mu = \psi\sigma$ be a smooth measure on S. Then $W(\mu)$ is a compact operator. Moreover if $p > n \geq 6$, then*

$$W(\mu) \in S^p(\mathcal{H}).$$

2.2 Compact operators and the Schatten classes

Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{B}(\mathcal{H})$ denote the set of bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called compact if the image of a bounded set under T is relatively compact. The set of compact operators on $\mathcal{H}$, denoted by $\mathcal{K}(\mathcal{H})$, is a closed subspace of $\mathcal{B}(\mathcal{H})$ in the norm topology. Moreover, $\mathcal{K}(\mathcal{H})$ forms a two-sided ideal of $\mathcal{B}(\mathcal{H})$. The following results are well known and easy to find in the literature, see for example [3, 27, 48].

Let $T \in \mathcal{K}(\mathcal{H})$. Let T^* denote the adjoint of T. An operator $T \in \mathcal{B}(\mathcal{H})$ is called self-adjoint if $T^* = T$. The following theorem is a fundamental result in functional analysis, and is known as the spectral theorem for compact self-adjoint operators.

Theorem 2.2.1. *Let T be a compact self-adjoint operator on $\mathcal{H}$. Then $\mathcal{H}$ has an orthonormal basis consisting of eigenvectors of T. Moreover, if $\{\lambda_1, \lambda_2, \dots\}$ are the distinct non-zero eigenvalues of T, and P_n is the projection of $\mathcal{H}$ onto $\ker(T - \lambda_n I)$,* *then* $P_n P_m = P_m P_n = 0$ *if* $n \neq m$*, each* λ_n *is real, and*

$$T = \sum_{n=1}^{\infty} \lambda_n P_n,$$

where the series converges to T *in the metric defined by the norm of* $\mathcal{B}(\mathcal{H})$*. (Of course, this may only be a finite sum).*

Let $T \in \mathcal{K}(\mathcal{H})$. Then $|T| = (T^*T)^{1/2}$ is a compact self-adjoint operator. Therefore, by the spectral theorem, there exists an orthonormal basis $\{\varphi_k \mid k \in \mathbb{N}\}$ of $\mathcal{H}$ consisting of eigenvectors of $|T|$. Let $s_k(T)$ denote the eigenvalue of $|T|$ corresponding to φ_k, $k \in \mathbb{N}$. The non-negative numbers $\{s_k(T)\}_{k=1}^{\infty}$ are called the *singular values* of T.

Theorem 2.2.2. *Let* $T \in \mathcal{K}(\mathcal{H})$*. Then*

$$T = \sum_{k=1}^{\infty} s_k(T) \varphi_k \otimes \psi_k,$$

where $\{\varphi_k\}_{k=1}^{\infty}$ *and* $\{\psi_k\}_{k=1}^{\infty}$ *are orthonormal basis of* $\mathcal{H}$*, and the series converges to* T *in the metric defined by the norm of* $\mathcal{B}(\mathcal{H})$*.*

As a consequence of Theorem 2.2.2, we have the following results.

Theorem 2.2.3. *The space of finite rank operators on* $\mathcal{H}$*, i.e., operators whose range is finite dimensional, is dense in* $\mathcal{K}(\mathcal{H})$ *in the norm topology.*

Theorem 2.2.4. *Let* $T \in \mathcal{B}(\mathcal{H})$ *be a self adjoint operator. If* T^k *is compact for some* $k \in \mathbb{N}$*, then* T *is compact.*

Theorem 2.2.5. *Let* $T \in \mathcal{B}(\mathcal{H})$*. Then* T *is compact if and only if* T^*T *is compact.*

We now define the p-Schatten class on $\mathcal{H}$. Let $T \in \mathcal{K}(\mathcal{H})$. Let $\{s_k(T)\}_{k=1}^{\infty}$ be the sequence of singular values of T. For $1 \leq p < \infty$, the *p-Schatten class* on $\mathcal{H}$, denoted $S^p(\mathcal{H})$, consists of the compact operators T such that

$$\sum_{k=1}^{\infty} |s_k(T)|^p < \infty.$$

For $1 \leq p < \infty$, define the norm on $S^p(\mathcal{H})$ by

$$\|T\|_{S^p} = \left(\sum_{k=1}^{\infty} |s_k(T)|^p \right)^{1/p}.$$

Then $(S^p(\mathcal{H}), \|\cdot\|_{S^p})$ is a Banach space. Let $S^{\infty}(\mathcal{H}) = \mathcal{B}(\mathcal{H})$, and let $\|T\|_{S^{\infty}}$ be the operator norm of T. The family $\{S^p(\mathcal{H}) \,|\, p \in [1, \infty]\}$ is a complex interpolation scale (see [41]).

If $\{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$, and $T \in S^1(\mathcal{H})$, define

$$\operatorname{tr}(T) = \sum_k \langle T\varphi_k, \varphi_k \rangle.$$

Then $\operatorname{tr}(T)$ is independent of the choice of orthonormal basis (see [3]). The number $\operatorname{tr}(T)$ is called the trace of T. Moreover if $T \in S^p(\mathcal{H})$, $1 \leq p < \infty$, then

$$\|T\|_{S^p}^p = \operatorname{tr}((T^*T)^{p/2}).$$

The elements of $S^1(\mathcal{H})$ are called trace class operators, and the elements of $S^2(\mathcal{H})$ are called Hilbert-Schmidt operators. The space $S^2(\mathcal{H})$ is a Hilbert space with respect to the inner-product defined by

$$\langle A, B \rangle_{S^2} = \operatorname{tr}(B^*A), \quad A, B \in S^2(\mathcal{H}).$$

Now we list some important properties of the Schatten classes. For a detailed discussion and proof of these properties, we refer to [3, 48, 37].

Theorem 2.2.6. *The space of finite rank operators is dense in* $S^p(\mathcal{H})$, $1 \le p < \infty$.

Theorem 2.2.7. *If* $1 \le p \le q \le \infty$, *then* $S^p(\mathcal{H}) \subseteq S^q(\mathcal{H})$.

Theorem 2.2.8. *If* $1 \le p \le \infty$, *then* $S^p(\mathcal{H})$ *is a two-sided ideal in* $\mathcal{B}(\mathcal{H})$, *i.e., if* $T \in S^p(\mathcal{H})$ *and* $A \in \mathcal{B}(\mathcal{H})$, *then* $TA, AT \in S^p(\mathcal{H})$. *Moreover,* $\|TA\|_{S^p} \le \|A\|_{\mathrm{op}} \|T\|_{S^p}$ *and* $\|AT\|_{S^p} \le \|A\|_{\mathrm{op}} \|T\|_{S^p}$.

Theorem 2.2.9. *Suppose* $1 \le p \le \infty$. *If* $T \in S^p(\mathcal{H})$ *and* $A, B \in \mathcal{B}(\mathcal{H})$ *are unitary operators, then* $ATB \in S^p(\mathcal{H})$, *and* $\|ATB\|_{S^p} = \|T\|_{S^p}$.

Theorem 2.2.10 (Hölder's inequality)**.** *Let* $p, q, r \in [1, \infty]$ *be such that* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. *If* $T \in S^p(\mathcal{H})$ *and* $S \in S^q(\mathcal{H})$, *then* $ST \in S^r(\mathcal{H})$, *and*

$$\|ST\|_{S^r} \le \|T\|_{S^p} \|S\|_{S^q} .$$

Theorem 2.2.11. *Suppose* $1 \le p < \infty$. *Let* p' *be the conjugate of* p, *i.e.,* $\frac{1}{p} + \frac{1}{p'} = 1$. *Let* $S^p(\mathcal{H})^*$ *denote the topological dual of* $S^p(\mathcal{H})$. *Then* $S^p(\mathcal{H})^* = S^{p'}(\mathcal{H})$ *in the sense that any* $l \in S^p(\mathcal{H})^*$ *has the form*

$$l(A) = \mathrm{tr}(B^* A)$$

for some $B \in S^{p'}(\mathcal{H})$, *and* $\|B\|_{S^{p'}} = \|l\|_{\mathrm{op}}$.

Theorem 2.2.12. *Let* $\mathcal{K}(\mathcal{H})^*$ *denote the topological dual of* $\mathcal{K}(\mathcal{H})$. *Then* $\mathcal{K}(\mathcal{H})^* = S^1(\mathcal{H})$ *in the sense that any* $l \in \mathcal{K}(\mathcal{H})^*$ *has the form*

$$l(A) = \mathrm{tr}(B^* A)$$

for some $B \in S^1(\mathcal{H})$, *and* $\|B\|_{S^1} = \|l\|_{\mathrm{op}}$.

We now discuss the conditions under which an integral operator is compact. Recall that a bounded linear operator T on $\mathcal{H} = L^2(\mathbb{R}^n)$ is called an integral operator if there exists a function $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that

$$(T\varphi)(t) = \int_{\mathbb{R}^n} K(t,u)\varphi(u)\,du, \quad \varphi \in \mathcal{H}.$$

This function K is called the kernel of the operator T. It is well known that if $K \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, then $T \in S^2(\mathcal{H})$ and hence T is compact (see [48, Theorem 3.8.5] for the proof).

We now state the generalized Young's inequality for the particular case of $\mathbb{R}^n$. However, this theorem is true for an integral operator defined on an arbitrary measure space. For a proof, we refer to [11].

Theorem 2.2.13. *Suppose $1 \leq p \leq \infty$ and $C > 0$. Suppose $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ satisfies*

$$\sup_{u\in\mathbb{R}^n} \int_{\mathbb{R}^n} |K(t,u)|\;dt \leq C, \quad \textit{and} \quad \sup_{t\in\mathbb{R}^n} \int_{\mathbb{R}^n} |K(t,u)|\;du \leq C.$$

If $f \in L^p(\mathbb{R}^n)$, the function Tf defined by

$$(Tf)(t) = \int_{\mathbb{R}^n} K(t,u)f(u)\,du, \quad t \in \mathbb{R}^n$$

is well defined almost everywhere, and is in $L^p(\mathbb{R}^n)$. Moreover, $\|Tf\|_p \leq C\,\|f\|_p$.

The following theorem, which is an immediate consequence of Theorem 2.2.13, is useful in proving the compactness of an integral operator.

Theorem 2.2.14. *If $T \in \mathcal{B}(\mathcal{H})$ is an integral operator with kernel K such that*

$$\lim_{\|u\|\to\infty} \int_{\mathbb{R}^n} |K(t,u)|\;dt = 0 \quad \textit{and} \quad \lim_{\|t\|\to\infty} \int_{\mathbb{R}^n} |K(t,u)|\;du = 0,$$

then T is a compact operator.

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ be an integral operator with kernel K such that

$$\lim_{\|u\|\to\infty} \int_{\mathbb{R}^n} |K(t,u)|\ dt = 0, \quad \text{and} \quad \lim_{\|t\|\to\infty} \int_{\mathbb{R}^n} |K(t,u)|\ du = 0.$$

Let $\varepsilon > 0$. Then there exists a natural number N such that if $\|t\| > N$, then

$$\int_{\mathbb{R}^n} |K(t,u)|\ du < \varepsilon,$$

and if $\|u\| > N$, then

$$\int_{\mathbb{R}^n} |K(t,u)|\ dt < \varepsilon.$$

Define $K_N : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ by

$$K_N(t,u) = \begin{cases} K(t,u), & \text{if } \|t\| \leq N \text{ and } \|u\| \leq N, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $K_N \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Therefore T_{K_N}, the integral operator with kernel K_N, is compact. Observe that, for large enough N,

$$\sup_{t\in\mathbb{R}^n} \int_{\mathbb{R}^n} |K(t,u) - K_N(t,u)|\ du < \varepsilon, \quad \text{and}$$
$$\sup_{u\in\mathbb{R}^n} \int_{\mathbb{R}^n} |K(t,u) - K_N(t,u)|\ dt < \varepsilon.$$

Let $f \in L^2(\mathbb{R}^n)$. It follows from Theorem 2.2.13 that for large enough N,

$$\|(T - T_{K_N})f\|_2 < \varepsilon \|f\|_2 .$$

Therefore for large enough N,

$$\|T - T_{K_N}\|_{\mathrm{op}} < \varepsilon.$$

Since $\mathcal{K}(\mathcal{H})$ is a closed subspace of $\mathcal{B}(\mathcal{H})$, it follows that T is compact. $\square$

2.3 Harmonic oscillator

Recall that the Laplacian is the differential operator $\Delta = \sum_{i=0}^{n} \frac{\partial^2}{\partial t_i^2}$.

Definition 2.3.1. The *harmonic oscillator* is the differential operator

$$H = -\Delta + \|t\|^2 .$$

For $k = 0, 1, 2, \ldots,$ and $t \in \mathbb{R}$, define the k^{th} Hermite polynomial $H_k(t)$ by the equation

$$H_k(t) = (-1)^k e^{t^2} \left(\frac{d}{dt}\right)^k e^{-t^2}.$$

The normalized Hermite functions are defined by

$$h_k(t) = (2^k \sqrt{\pi} k!)^{-\frac{1}{2}} H_k(t) e^{-\frac{1}{2}t^2}.$$

Then $\{h_k\}_{k=0}^{\infty}$ forms an orthonormal basis for $L^2(\mathbb{R})$. The higher dimensional Hermite functions are then obtained by taking tensor products.

For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we define the Hermite function Φ_α by

$$\Phi_\alpha(x) = \prod_{j=1}^{n} h_{\alpha_j}(x_j).$$

Then the family $\{\Phi_\alpha \mid \alpha \in \mathbb{N}_0^n\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. For a detailed discussion, see [55, 57]. Moreover, Φ_α is an eigenfunction of the harmonic oscillator corresponding to the eigenvalue $(2\,|\alpha|+n)$. The multiplicity of the eigenvalue $(2\,|\alpha|+n)$ is

$$\frac{1}{(n-1)!}\prod_{i=1}^{n-1}\left(|\alpha|+i\right).$$

Therefore, for $p > n$,

$$H^{-1} \in S^p(\mathcal{H}). \tag{2.2}$$

Since $S^p(\mathcal{H})$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$, in order to show that an operator is in $S^p(\mathcal{H})$ for $p > n$, it suffices to prove that the composition of the harmonic oscillator with the given operator is bounded. In this chapter, we use this technique to conclude the Schatten class membership of the Weyl transform of a smooth measure supported on a compact connected smooth hypersurface of positive Gaussian curvature.

2.4 Proof of the main result

We will prove Theorem 2.1.1 in this section. Let S be a compact connected smooth hypersurface in $\mathbb{R}^{2n}$, $n \geq 2$, and let μ be a smooth measure on S. Assume that S has positive Gaussian curvature everywhere. Observe that this actually implies that S has positive definite second fundamental form, i.e., S is strictly convex (see [23, p122, Theorem 2.1]).

Let $\Pi_i : S \to \mathbb{R}^n$, $i = 1, 2$, be the projection maps on S defined by

$$\Pi_1(x_1, \ldots, x_n, y_1, \ldots, y_n) = (x_1, \ldots, x_n), \quad \text{and}$$
$$\Pi_2(x_1, \ldots, x_n, y_1, \ldots, y_n) = (y_1, \ldots, y_n).$$

We will first prove Theorem 2.1.1 under the following additional assumption.

Assumption 2.4.1. The support of μ is contained in the set of regular points of Π_1.

Let U_1 denote the set of regular points of Π_1 and

$$W_1 = \{x \in \Pi_1(S) \mid x \text{ is a regular value for } \Pi_1\}.$$

Then W_1 is open by the implicit function theorem. Observe that $\Pi_1(U_1) = W_1$. Indeed, if x is the image under Π_1 of a regular point of Π_1, then x cannot be the image of a critical point of Π_1 because S is strictly convex.

For $x \in \mathbb{R}^n$, let S_x denote the set $\Pi_1^{-1}\{x\}$.

Lemma 2.4.2. *Whenever $x \in W_1$, the set $\Pi_2(S_x)$ is a smooth hypersurface in $\mathbb{R}^n$ whose Gaussian curvature is bounded below by a positive constant independent of x.*

We need the following technical lemma to prove Lemma 2.4.2. For a smooth manifold M and $x \in M$, let T_xM denote the tangent space of M at x.

Lemma 2.4.3. *Let M be a compact connected smooth hypersurface in $\mathbb{R}^n$ with positive Gaussian curvature. Let H be a plane in $\mathbb{R}^n$, and suppose H intersects M transversally. Let $N = M \cap H$. Then N is a smooth hypersurface in H, and for all $y \in N$ and $X, Y \in T_yN$, we have*

$$K_{(N \subseteq H)}(X, Y) \geq K_{(M \subseteq \mathbb{R}^n)}(X, Y),$$

where $K_{(N \subseteq H)}$ and $K_{(M \subseteq \mathbb{R}^n)}$ are the second fundamental forms with respect to the inward pointing normal on the smooth hypersurfaces N in H and M in $\mathbb{R}^n$, respectively.

Proof. Let M be a compact connected smooth hypersurface in $\mathbb{R}^n$ with positive Gaussian curvature. By the Jordan-Brouwer separation theorem (see [18, p89]), $M = \partial\Omega$ where Ω is a bounded open set in $\mathbb{R}^n$. Let $\vec{n}_M$ denote the unit normal to M which points into Ω. Let H be a plane in $\mathbb{R}^n$ which intersects M transversally, and let $N = M \cap H$. It is a standard fact from differential topology that N is a smooth hypersurface in H (see [18, p30]). Let $\vec{n}_N$ denote the unit normal vector to N which points into $\Omega \cap H$. Observe that $\vec{n}_N \cdot \vec{n}_M > 0$.

If W is a submanifold of the Riemannian manifold Z, let $\vec{K}_{(W\subseteq Z)}$ denote the vector-valued second fundamental form of W in Z, i.e.,

$$\vec{K}_{(W\subseteq Z)}(X,Y) = \mathrm{Proj}_{TZ\ominus TW}\, \nabla_X Y,$$

where ∇ is the Levi-Civita connection on Z, and $TZ \ominus TW$ denotes the orthogonal complement of the tangent bundle of W in the tangent bundle of Z.

Let $x \in N$, and let $X, Y \in T_xN$. We have orthogonal decompositions

$$\begin{aligned}
\vec{K}_{(N\subseteq\mathbb{R}^n)}(X,Y) &= \vec{K}_{(N\subseteq H)}(X,Y) + \vec{K}_{(H\subseteq\mathbb{R}^n)}(X,Y), \qquad \text{and}\\
\vec{K}_{(N\subseteq\mathbb{R}^n)}(X,Y) &= \vec{K}_{(N\subseteq M)}(X,Y) + \vec{K}_{(M\subseteq\mathbb{R}^n)}(X,Y).
\end{aligned}$$

Since $\vec{K}_{(H\subseteq\mathbb{R}^n)}(X,Y) = 0$, it follows that

$$\begin{aligned}
\vec{K}_{(N\subseteq H)}(X,Y) &= \vec{K}_{(N\subseteq\mathbb{R}^n)}(X,Y)\\
&= \vec{K}_{(N\subseteq M)}(X,Y) + \vec{K}_{(M\subseteq\mathbb{R}^n)}(X,Y).
\end{aligned}$$

Since $\vec{K}_{(N\subseteq M)}(X,Y)$ is tangent to M, it follows that

$$\begin{aligned} K_{(N\subseteq H)}(X,Y)\vec{n}_N \cdot \vec{n}_M &= \vec{K}_{(N\subseteq H)}(X,Y)\cdot \vec{n}_M \\ &= \vec{K}_{(M\subseteq \mathbb{R}^n)}(X,Y)\cdot \vec{n}_M \\ &= K_{(M\subseteq \mathbb{R}^n)}(X,Y). \end{aligned}$$

Therefore

$$K_{(N\subseteq H)}(X,Y) = \frac{K_{(M\subseteq \mathbb{R}^n)}(X,Y)}{\vec{n}_M \cdot \vec{n}_N} \geq K_{(M\subseteq \mathbb{R}^n)}(X,Y).$$

□

Proof of Lemma 2.4.2. Let $x \in W_1$. Let $H = \{(x,y)\,|\,y \in \mathbb{R}^n\}$. Then $S_x = S \cap H$. By Lemma 2.4.3, it follows that S_x is a smooth hypersurface in H. Moreover, if $y \in S_x$ and $X, Y \in T_y S_x$, then $K_{(S_x\subseteq H)}(X,Y) \geq K_{(S\subseteq \mathbb{R}^{2n})}(X,Y)$. Since Π_2 is an isometry between H and $\mathbb{R}^n$, it follows that $\Pi_2(S_x)$ is a smooth hypersurface in $\mathbb{R}^n$ with $K_{(\Pi_2(S_x)\subseteq \mathbb{R}^n)}(X,Y) \geq K_{(S\subseteq \mathbb{R}^{2n})}(X,Y)$. Therefore the smallest principal curvature of $\Pi_2(S_x)$ is greater than or equal to the smallest principal curvature of S. Therefore the Gaussian curvature of S_x is bounded below by the $(n-1)$-th power of the smallest principal curvature of S. □

Let $\mathcal{J}_{\Pi_1}$ denote the normal Jacobian of Π_1, i.e., the absolute value of the determinant of $d\Pi_1$ restricted to the orthogonal complement of its kernel. Then $\mathcal{J}_{\Pi_1}$ is strictly positive on U_1, and hence has a positive minimum on the support of μ.

By the coarea formula (see [2, p159]), we have

$$\begin{aligned} (W(\mu)\varphi)(t) &= \int_S e^{\pi i(x\cdot y + 2y\cdot t)}\varphi(t+x)\,d\mu(x,y) \\ &= \int_S e^{\pi i(x\cdot y + 2y\cdot t)}\varphi(t+x)\psi(x,y)\,d\sigma(x,y) \\ &= \int_{\mathbb{R}^n} \varphi(t+x) \int_{S_x} \frac{e^{\pi i(x+2t)\cdot y}\psi(x,y)}{\mathcal{J}_{\Pi_1}(x,y)}\,d\sigma_x(y)\,dx, \end{aligned}$$

where σ_x denotes the volume measure on S_x. Let $\psi_1 = \psi/\mathcal{J}_{\Pi_1}$. Then ψ_1 is a compactly supported smooth function on U_1. Observe that

$$\int_{S_x} e^{\pi i(x+2t)\cdot y}\psi_1(x,y)\, d\sigma_x(y) = \widehat{\eta_x}(x+2t),$$

where $\eta_x = \Pi_{2*}\left(\psi_1(x,\cdot)\, d\sigma_x\right)$ is the push-forward of the measure $\psi_1(x,\cdot)\, d\sigma_x$ by Π_2, which is a smooth measure supported on $\Pi_2(S_x)$. By Lemma 2.4.2, $\Pi_2(S_x)$ is a smooth hypersurface in $\mathbb{R}^n$ of positive Gaussian curvature. Since η_x is a smooth measure supported on $\Pi_2(S_x)$, it follows from Theorem 1.4.2 that there exists a constant C independent of x such that

$$|\widehat{\eta_x}(t)| \le C\|t\|^{(1-n)/2}.$$

Also, since the measure η_x is nonzero only when $x \in \Pi_1(S)$, it follows that $\widehat{\eta_x}(t)$ is nonzero only when $x \in \Pi_1(S)$.

Thus $W(\mu)$ is an integral operator given by the kernel

$$k(t,u) = \widehat{\eta_{u-t}}(u+t). \tag{2.3}$$

Moreover, the kernel k satisfies

$$|k(t,u)| \le C\|t+u\|^{(1-n)/2}, \tag{2.4}$$

and $k(t,u) = 0$ if $\|t-u\| > R$ for some positive constants C and R.

If $\|t\| > R$ and $\|t-u\| \le R$, it follows from the parallelogram law that

$$\|t\| \le \|u+t\|.$$

Therefore

$$\lim_{\|t\|\to\infty} \int |k(t,u)| \; du = 0.$$

Similarly, we conclude that

$$\lim_{\|u\|\to\infty} \int |k(t,u)| \; dt = 0.$$

Compactness of $W(\mu)$ follows from Theorem 2.2.14.

Recall that the harmonic oscillator is the differential operator $H = -\Delta + \|t\|^2$. It follows from the dominated convergence theorem that

$$(HW(\mu)\varphi)(t) = \int (-\Delta_t + \|t\|^2) k(t,u)\varphi(u)\, du.$$

Let $x \in \Pi_1(U_1)$. Let $S_x = \Pi_1^{-1}\{x\}$, and let σ_x denote the volume measure on S_x. Let $i_x : S_x \to U_1$ denote the inclusion map. Observe that there exists a differential form ω on U_1 such that the measure on S_x induced by $i_x^*\omega$ is σ_x, where $i_x^*\omega$ denotes the pullback of ω by i_x. Then

$$\int_{S_x} e^{\pi i(x+2t)\cdot y} \psi_1(x,y)\, d\sigma_x = \int_{S_x} e^{\pi i(x+2t)\cdot y} \psi_1(x,y)\, \omega.$$

Fix a point $x \in \Pi_1(U_1)$. Let $\tau \in C_0^\infty(\Pi_1(U_1))$ be identically one in a neighborhood of x. There exists a unique vector field X_j on U_1 such that $X_j \perp \ker(d\Pi_1)$ and $d\Pi_1(X_j) = \tau \frac{\partial}{\partial x_j}$. Clearly, the support of X_j is a compact subset of U_1, and so there is a one parameter group φ_s of diffeomorphisms of U_1 generated by X_j. Then, at the point x, we have

$$\begin{aligned}\frac{\partial}{\partial x_j}\int_{S_x} e^{\pi i(x+2t)\cdot y}\psi_1(x,y)\,d\sigma_x &= \frac{\partial}{\partial x_j}\int_{S_x} e^{\pi i(x+2t)\cdot y}\psi_1(x,y)\,\omega \\ &= \int_{S_x} \frac{d}{ds}\bigg|_{s=0} \varphi_s^*(e^{\pi i(x+2t)\cdot y}\psi_1(x,y)\,\omega) \\ &= \int_{S_x} \mathcal{L}_{X_j}\left(e^{\pi i(x+2t)\cdot y}\psi_1(x,y)\,\omega\right),\end{aligned}$$

where $\mathcal{L}_{X_j}(\nu)$ denotes the Lie derivative of the differential form ν with respect to the vector field X_j. We may write

$$X_j = \tau(x)\frac{\partial}{\partial x_j} + a_{1j}(x,y)\frac{\partial}{\partial y_1} + \cdots + a_{nj}(x,y)\frac{\partial}{\partial y_n},$$

where $a_{1j},\ldots,a_{nj} \in C_0^\infty(U_1)$. Since $i_x^*\omega$ is a non-vanishing $(n-1)$-form on the $(n-1)$-dimensional manifold S_x, we may write

$$i_x^*\mathcal{L}_{X_j}\omega = f(x,y)i_x^*\omega$$

for some $f \in C^\infty(U_1)$. Since X_j is compactly supported, in fact $f \in C_0^\infty(U_1)$. Therefore

$$\begin{aligned}&i_x^*\mathcal{L}_{X_j}\left[e^{\pi i(x+2t)\cdot y}\psi_1(x,y)\,\omega\right] \\ &= \left(X_j e^{\pi i(x+2t)\cdot y}\right)\psi_1(x,y)i_x^*\omega + e^{\pi i(x+2t)\cdot y}\left(X_j\psi_1\right)(x,y)i_x^*\omega + e^{\pi i(x+2t)\cdot y}\psi_1(x,y)i_x^*\mathcal{L}_{X_j}\omega \\ &= \pi i\left(\tau(x)y_j + \sum_{k=1}^n (x_k+2t_k)a_{kj}(x,y)\right)e^{\pi i(x+2t)\cdot y}\psi_1(x,y)\,d\sigma_x \\ &\quad + e^{\pi i(x+2t)\cdot y}\left(X_j\psi_1\right)(x,y)\,d\sigma_x + e^{\pi i(x+2t)\cdot y}\psi_1(x,y)f(x,y)\,d\sigma_x \\ &= \pi i\sum_{k=1}^n (x_k+2t_k)a_{kj}(x,y)e^{\pi i(x+2t)\cdot y}\psi_1(x,y)\,d\sigma_x \\ &\quad + e^{\pi i(x+2t)\cdot y}\left[\pi i\tau(x)y_j\psi_1(x,y) + \left(X_j\psi_1\right)(x,y) + \psi_1(x,y)f(x,y)\right]d\sigma_x.\end{aligned}$$

By Lemma 2.4.2 and Theorem 1.4.2, there exists constants C_1 and C_2 such that for large enough t,

$$\left|\int_{S_x} a_{kj}(x,y)e^{\pi i(x+2t)\cdot y}\psi_1(x,y)\,d\sigma_x\right| \le C_1\|x+2t\|^{(1-n)/2},$$

and

$$\begin{aligned}&\left|\int_{S_x} e^{\pi i(x+2t)\cdot y}\left[\pi i\tau(x)y_j\psi_1(x,y)+(X_j\psi_1)(x,y)+\psi_1(x,y)f(x,y)\right]\,d\sigma_x\right|\\ &\le C_2\|x+2t\|^{(1-n)/2}.\end{aligned}$$

Therefore

$$\left|\frac{\partial}{\partial x_j}\int_{S_x} e^{\pi i(x+2t)\cdot y}\psi_1(x,y)\,d\sigma_x\right| \le C_3\|x+2t\|^{(3-n)/2},$$

for large enough t. Similarly,

$$\left|\frac{\partial^2}{\partial {x_j}^2}\int_{S_x} e^{\pi i(x+2t)\cdot y}\psi_1(x,y)\,d\sigma_x\right| \le C_4\|x+2t\|^{(5-n)/2},$$

for large enough t.

Therefore

$$\left|\Delta_x\int_{S_x} e^{\pi i(x+2t)\cdot y}\psi_1(x,y)\,d\sigma_x\right| \le C_5\|x+2t\|^{(5-n)/2},$$

for large enough t. It follows that

$$|\Delta_t k(t,u)| \le C_5\|t+u\|^{(5-n)/2},$$

for large enough t. Therefore, by equation (2.4),

$$\begin{aligned}\int \left|(-\Delta_t+\|t\|^2)k(t,u)\right|\,du &\le C_5\int \|t+u\|^{(5-n)/2}\,du + C\,\|t\|^2\int \|t+u\|^{(1-n)/2}\,du\\ &\le C_6\|t\|^{(5-n)/2}\end{aligned}$$

for large enough t. Hence, when $n \geq 6$, we have

$$\lim_{\|t\|\to\infty} \int \left|(-\Delta_t + \|t\|^2)k(t,u)\right| \, du = 0.$$

Similarly, when $n \geq 6$, we have

$$\lim_{\|u\|\to\infty} \int \left|(-\Delta_t + \|t\|^2)k(t,u)\right| \, dt = 0.$$

By Theorem 2.2.14, it follows that $HW(\mu)$ is a compact operator when $n \geq 6$. Since $S^p(\mathcal{H})$ is an ideal of $\mathcal{B}(\mathcal{H})$ and $H^{-1} \in S^p(\mathcal{H})$ for $p > n$, it follows that

$$W(\mu) \in S^p(\mathcal{H})$$

for $p > n \geq 6$.

This completes the proof of Theorem 2.1.1 under Assumption 2.4.1. Now we consider the general case.

Lemma 2.4.4. *Let $w \in S$. Then either Π_1 or Π_2 is a submersion at w.*

Proof. Let $w \in S$ be such that both Π_1 and Π_2 are not submersions at w. Let $d_w\Pi_1$ and $d_w\Pi_2$ denote the differentials of Π_1 and Π_2 at w, respectively. Then $d_w\Pi_1$ and $d_w\Pi_2$ are not surjective, i.e.,

$$\text{Rank}(d_w\Pi_1) \leq n-1 \quad \text{and} \quad \text{Rank}(d_w\Pi_2) \leq n-1.$$

Therefore, by the rank-nullity theorem,

$$\text{Nullity}(d_w\Pi_1) \geq n \quad \text{and} \quad \text{Nullity}(d_W\Pi_2) \geq n.$$

It follows that the intersection of the null-spaces $d_w\Pi_1$ and $d_w\Pi_2$ is nonzero. This contradicts the fact that $d_w\Pi_1 \oplus d_w\Pi_2$ is injective. Therefore, either Π_1 or Π_2 is a submersion at w. □

Let U_2 be the set of regular points of Π_2. By Lemma 2.4.4, $S = U_1 \cup U_2$. Let $\{\varphi_1, \varphi_2\}$ be a partition of unity subordinate to $\{U_1, U_2\}$.

Then the measure $\varphi_1\mu$ satisfies Assumption 2.4.1. Hence $W(\varphi_1\mu)$ is a compact operator, and $W(\varphi_1\mu) \in S^p(\mathcal{H})$ for $p > n \geq 6$.

Define $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$J(x, y) = (-y, x).$$

Then JS is also a compact connected smooth hypersurface in $\mathbb{R}^{2n}$, $n \geq 2$, whose Gaussian curvature is positive everywhere, and $J_*(\varphi_2\mu)$, the push-forward of the measure $\varphi_2\mu$ by J, is a smooth measure on JS, which in addition satisfies Assumption 2.4.1. Therefore $W(J_*(\varphi_2\mu))$ is a compact operator, and $W(J_*(\varphi_2\mu)) \in S^p(\mathcal{H})$ for $p > n \geq 6$.

It follows from equation (1.2) that

$$\rho(x, y, 1) = \mathcal{F}^{-1}\rho(-y, x, 1)\mathcal{F} = \mathcal{F}^{-1}\rho(J(x, y), 1)\mathcal{F}.$$

Therefore

$$W(\varphi_2\mu) = \mathcal{F}^{-1}W(J_*(\varphi_2\mu))\mathcal{F}.$$

Since the set of compact operators forms a two-sided ideal in $\mathcal{B}(\mathcal{H})$, it follows that $W(\varphi_2\mu)$ is a compact operator. Moreover, it follows from Theorem 2.2.8 that if $p > n \geq 6$, then $W(\varphi_2\mu) \in S^p(\mathcal{H})$. Hence $W(\mu) = W(\varphi_1\mu) + W(\varphi_2\mu)$ is compact, and $W(\mu) \in S^p(\mathcal{H})$ if $p > n \geq 6$. This completes the proof of Theorem 2.1.1.

2.5 Conclusion

We conclude that if μ is a smooth measure supported on a compact connected smooth hypersurface of positive Gaussian curvature in $\mathbb{R}^{2n}$, $n \geq 2$, then the Weyl transform of μ is compact and belongs to the p-Schatten class if $p > n \geq 6$.

The result about the decay of the Fourier transform of a smooth measure supported on a hypersurface was with the assumption that the Gaussian curvature of the support of the measure is non-zero everywhere. However, we needed the positivity of the Gaussian curvature in the proof of Lemma 2.4.2, which is an essential tool in the proof of the main theorem. Observe that Lemma 2.4.2 fails for a hypersurface of negative Gaussian curvature, for example, it fails for a hyperboloid. However, Theorem 2.1.1 is true for a hypersurface of negative Gaussian curvature, and this will be proved in Chapter 4.

Another limitation of Theorem 2.1.1 is that we take $n \geq 2$, i.e., the case of a hypersurface (curve) in $\mathbb{R}^2$ is not considered. In Chapter 3, we will prove that the Weyl transform of a smooth measure supported on a compact connected smooth hypersurface of non-zero curvature in $\mathbb{R}^2$ is compact.

Since the Weyl transform of the normalized surface measure on a sphere in $\mathbb{R}^{2n}$ belongs to $S^p(\mathcal{H})$ if and only if $p > 4n/(2n-1)$, with no restriction on n, we suspect that the estimates obtained in this chapter for the Schatten class membership of the Weyl transform of a smooth measure supported on a compact connected smooth hypersurface of positive Gaussian curvature are not sharp.

Inspired by this example, we made the following conjecture in [32].

Conjecture 2.5.1. Let S be a compact connected smooth hypersurface in $\mathbb{R}^{2n}$ such that S has a positive Gaussian curvature everywhere, and let μ be a smooth measure on S. Then $W(\mu) \in S^p(\mathcal{H})$ if $p > 4n/(2n-1)$, with no restriction on n.

This conjecture was recently settled by Luef and Samuelsen [29]. This will be discussed in Chapter 4.