Chapter 4

Parameter Estimation for a Class of Uncertain Systems: An Adaptive Super-Twisting Approach

4.1 Introduction

The problem of parameter estimation is one of the most significant challenges in control theory. In many real-world applications, some parameters that govern a system or process are unknown, and it is crucial to estimate these values to understand and predict the system's behavior. Parameter estimation is a critical problem in various fields, such as engineering, physics, economics, and biology, and it is employed for making predictions, designing experiments, and optimizing systems [6]. Estimating parameters for nonlinear uncertain dynamical systems is an important task in control theory, where several control design techniques assume the accurate system model is known. However, real systems cannot be precisely modeled, leading to the development of system identification and parameter estimation techniques. This chapter emphasize on the parameter estimation based on an adaptive super-twisting approach in the presence of uncertainty.

The motivation for the work here is drawn from [74], where an adaptive super-twisting sliding mode control law is proposed with considering the bounds of uncertainties and perturbations are not known. A part of the work is also motivated by the design of nonlinear controller based on sliding mode technique for unknown parameter estimation, as noted by [71].

The main contributions of this chapter are as follows:

1. The proposed approach is used to estimate the unknown constant parameters in finite time under the assumption that the parameters are bounded.

2. In the proposed approach, robustness of the estimation of the parameters is achieved for a class of uncertain system.

3. The control design parameters are updated using adaptive laws which prevent over-estimation. Moreover, verification of the persistence of excitation condition is not required here.

This chapter is organized as follows. In Section 4.2, the problem formulation is described in details. The main contribution of the chapter is presented in Section 4.3. This includes the adaptive super-twisting algorithm based parameter estimation technique. The Lyapunov based stability proof is also presented. Section 4.4 presents an illustrative example. Section 4.5 validates the proposed technique using the examples of pendulum system and single link manipulator system. Simulation results show the effectiveness of the proposed scheme. Section 4.6 summarizes the chapter.

4.2 Problem Formulation

Consider the nonlinear uncertain dynamical system

$$\dot{z} = f(t, z; \varphi) + h(z)u; \qquad z(t_0) = z_0 \tag{4.1}$$

where $z = [z_1, z_2, \cdots, z_n] \in \mathbb{R}^n$ represents the known system state, $u \in \mathbb{R}$ is the control input, $\varphi \in D \subset \mathbb{R}^n$ denotes the unknown constant parameters of the system (D being a compact set). The mapping $f : [0\ \infty) \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a nonlinear function. $h : \mathbb{R}^n \to \mathbb{R}^n$ is a vector function and $t_0 \geq 0$ is the initial time.

Assume that the system (4.1) can be represented in the following generalized controller canonical form:

$$\begin{aligned} &\dot{z}_1 = z_2,\ \dot{z}_2 = z_3, \cdots, \\ &\dot{z}_n = \varphi^\top \phi(z) + g(z)u + \delta(z, t) \end{aligned} \tag{4.2}$$

where $\varphi = [\varphi_1, \varphi_2, \cdots, \varphi_n]^\top \in \mathbb{R}^n$ denotes the unknown constant parameters. $g(z)$ is the known scalar function. $\delta(z,t) \in \mathbb{R}$ represents the unmodelled dynamics, which are treated as model uncertainty and external disturbances. It is assumed that

$$|\delta(z,t)| \leq \delta_0 \tag{4.3}$$

where $\delta_0 > 0$ is a constant bound. $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is the known function, such that,

$$\phi(z) = [\phi_1(z), \phi_2(z), \cdots, \phi_n(z)]^T$$

$\phi_i(z)$ is a scalar function in z, and $i \in \{1, 2, \cdots, n\}$.
Before proceeding to the design of the estimator, the following assumptions are mentioned:

Assumption 3 *The states of the system (4.2) are available for measurement.*

Assumption 4 *A sliding surface $s \in \mathbb{R}$ is designed in such a manner that it yields a desirable compensated dynamics in the sliding mode $s = 0$.*

Suppose that the relative degree of the system (4.1) with sliding variable s, with respect to control signal u is one and the internal dynamics is stable. Thus, the dynamics of the sliding manifold is

$$\dot{s} = \frac{\partial s}{\partial t} + \frac{\partial s}{\partial z} f(t, z; \varphi) + \frac{\partial s}{\partial z} h(z) u = \Phi(t, z; \varphi) + \psi u = \Phi(t, z; \varphi) - \bar{u} \tag{4.4}$$

where $\Phi(t, z; \varphi) = \frac{\partial s}{\partial t} + \frac{\partial s}{\partial z} f(t, z; \varphi)$, $\psi = \frac{\partial s}{\partial z} h(z)$ and $\bar{u} = -\psi u$.

4.3 Parameter Estimation using Adaptive Super-Twisting Algorithm

In this part, an adaptive super-twisting algorithm based approach to design an identifier input for estimating the unknown constant parameter of the system is discussed.

In view of system (4.2), the parameter estimation system is given as

$$\begin{aligned} \dot{\hat{z}}_1 &= \hat{z}_2, \ \dot{\hat{z}}_2 = \hat{z}_3 \cdots, \\ \dot{\hat{z}}_n &= \hat{\varphi}^T \phi(z) + g(z) u + v \end{aligned} \tag{4.5}$$

where $\hat{z} = [\hat{z}_1, \hat{z}_2, \cdots, \hat{z}_n]^T \in \mathbb{R}^n$ denotes estimated state, $v \in \mathbb{R}$ represents the identifier input. $\hat{\varphi} = [\hat{\varphi}_1, \hat{\varphi}_2, \cdots, \hat{\varphi}_n]^T \in \mathbb{R}^n$ is the estimated parameter. $\phi(z)$ is the known function.

$g(z)$ is the known scalar function.

The error in state estimation is

$$e = z - \hat{z}, \quad e = [e_1, e_2, \cdots, e_n]^T \in \mathbb{R}^n. \tag{4.6}$$

The error in parameter estimation is represented as

$$\theta = \varphi - \hat{\varphi}, \text{ with } \theta = [\theta_1, \theta_2, \cdots, \theta_n]^T \in \mathbb{R}^n. \tag{4.7}$$

In accordance with (4.6) and (4.7), the dynamics of state estimation error is represented as

$$\begin{aligned} &\dot{e}_1 = e_2, \ \dot{e}_2 = e_3, \cdots, \\ &\dot{e}_n = \theta^T \phi(z) + \delta(z,t) - v. \end{aligned} \tag{4.8}$$

In view of (4.4), for the system (4.8), the sliding surface s is considered in such a way that its dynamics can be represented as

$$\dot{s} = \theta^T \phi(z) + \delta(z,t) + \Theta(z, \hat{z}) - v \tag{4.9}$$

where $\Theta(z, \hat{z})$ is the known scalar function.

The adaptive super-twisting algorithm based identifier input v is designed as

$$v = \ \Theta(z, \hat{z}) + \alpha |s|^{1/2} \text{sign}(s) + \int_0^t \frac{\beta}{2} \text{sign}(s) d\tau \tag{4.10}$$

where the adaptive gains α and β are updated by

$$\begin{aligned} \dot{\alpha} &= \begin{cases} w_1 \sqrt{\frac{\gamma_1}{2}} \text{sign}(|s| - \mu), & \text{if } \alpha > \alpha_p \\ \zeta, & \text{if } \alpha \leq \alpha_p \end{cases} \\ \beta &= 2\epsilon\alpha \end{aligned} \tag{4.11}$$

where w_1, γ_1, μ, ζ, ϵ and α_p are positive constants and $\alpha(0) > \alpha_p$. Substituting (4.10) into (4.9), yields

$$\dot{s} = \theta^T \phi(z) + \delta(z,t) - \alpha |s|^{\frac{1}{2}} \text{sign}(s) \underbrace{- \int_0^t \frac{\beta}{2} \text{sign}(s) d\tau}_{v_1}. \tag{4.12}$$

Further, one can represent (4.12) as

$$\begin{aligned} \dot{s} &= \theta^T \phi(z) + \delta - \alpha |s|^{1/2} \text{sign}(s) + v_1 \\ \dot{v}_1 &= -\frac{\beta}{2} \text{sign}(s). \end{aligned} \tag{4.13}$$

Assumption 5 *The function* $\theta^T\phi(z) \in \mathbb{R}$ *and* $\delta(z,t) \in \mathbb{R}$ *satisfy the inequality:*

$$|\theta^T\phi(z) + \delta(z,t)| \leq \Delta|s|^{1/2}$$

where θ *is the error in parameter estimation,* $\phi(z)$ *is the known function and* Δ *is a positive constant.*

Theorem 2 *Consider the system* (4.2)*, for which the parameter estimation system is designed as* (4.5) *and the associated error dynamics is defined as in* (4.8)*. Suppose there exists an identifier input* (4.10) *(with the chosen sliding surface s satisfying Assumption 4) and the adaptive laws* (4.11) *such that the dynamics of the estimated parameters satisfies:*

$$\dot{\hat{\varphi}} = \left((\lambda + 4\epsilon^2)sign(s) + \epsilon|s|^{-\frac{1}{2}} \int_0^t \beta sign(s) d\tau\right)\phi(z), s \neq 0 \tag{4.14}$$

with the adaptive gain

$$\alpha > \frac{4\epsilon[4\epsilon^2(\delta_0^2 - \delta_0 + 1) + \epsilon(2\delta_0^2\lambda + 2\lambda + 1 - 2\delta_0) - \delta_0\lambda(1-\delta_0)] + \lambda^2}{4\lambda\epsilon[2\delta_0 - 4\delta_0\epsilon - 1]}$$

where λ *and* ϵ *are positive constants, and* δ_0 *is the upper bound of the uncertainty* δ*. Then, the estimated parameter* $\hat{\varphi}$ *converges to the actual parameter* φ *in finite time.*

Proof: For the system (4.13), let us consider new state variables

$$x = [x_1 \; x_2]^T = [|s|^{1/2}\text{sign}(s) \; v_1]^T. \tag{4.15}$$

From (4.15), one gets

$$x_1 = |s|^{1/2}\text{sign}(s) \Rightarrow |x_1| = |s|^{1/2}. \tag{4.16}$$

Alternatively, one can write (4.13) as:

$$\dot{x} = \frac{1}{2|x_1|}(Ax + B) \tag{4.17}$$

where

$$A = \begin{bmatrix} -\alpha & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} \theta^T\phi(z) + \delta(z,t) \\ -2|x_1|\frac{\beta}{2}\text{sign}(s) \end{bmatrix}. \tag{4.18}$$

Let us consider the Lyapunov function

$$V(x,\theta,\alpha,\beta) = V_1(x,\theta) + \frac{1}{2\gamma_1}(\alpha - \alpha_1)^2 + \frac{1}{2\gamma_2}(\beta - \beta_1)^2 \tag{4.19}$$

where γ_1, γ_2, $\alpha_1 > 0$ and $\beta_1 > 0$ are positive constants and

$$V_1(x,\theta) = x^T P x + \frac{1}{2}\theta^T\theta. \tag{4.20}$$

The positive-definite matrix P is defined as

$$\begin{bmatrix} \lambda + 4\epsilon^2 & -2\epsilon \\ -2\epsilon & 1 \end{bmatrix}, \ \lambda > 0, \ \epsilon > 0. \tag{4.21}$$

Taking the time derivative of (4.19), one gets

$$\dot{V} = \dot{V}_1(x,\theta) + \frac{1}{\gamma_1}(\alpha - \alpha_1)\dot{\alpha} + \frac{1}{\gamma_2}(\beta - \beta_1)\dot{\beta}. \tag{4.22}$$

Taking the time derivative of (4.20), one obtains

$$\begin{aligned} \dot{V}_1(x,\theta) &= \frac{1}{|x_1|}\Big(x^T A^T P x + B^T P x\Big) + \theta^T\dot{\theta} \\ &= \frac{1}{|x_1|}\Big(- \alpha(\lambda + 4\epsilon^2)x_1^2 + (2\alpha\epsilon + \lambda + 4\epsilon^2)x_1x_2 - 2\epsilon x_2^2 \\ &\qquad + (\lambda + 4\epsilon^2)x_1\theta^T\phi(z) - 2\epsilon x_2\theta^T\phi(z) + (\lambda + 4\epsilon^2)x_1\delta \\ &\qquad - 2\epsilon x_2\delta + 2\epsilon\beta x_1|x_1|\text{sign}(s) - \beta|x_1|x_2\text{sign}(s)\Big) + \theta^T\dot{\theta} \end{aligned} \tag{4.23}$$

since φ is an actual parameter which is constant. Hence, $\dot{\varphi}$ is zero. Using (4.3), one can write (4.23) as

$$\begin{aligned} \dot{V}_1 \leq \frac{1}{|x_1|}\Big(&- \alpha(\lambda + 4\epsilon^2)x_1^2 + (2\alpha\epsilon + \lambda + 4\epsilon^2)x_1x_2 - 2\epsilon x_2^2 \\ &+ (\lambda + 4\epsilon^2)x_1\delta_0 + 2\epsilon x_2\delta_0 + (2\epsilon x_1 - x_2)\beta|x_1|\text{sign}(s)\Big) \\ &+ \theta^T\left\{((\lambda + 4\epsilon^2)\text{sign}(s) - 2\epsilon\frac{x_2}{|x_1|})\phi(z) - \dot{\hat{\varphi}}\right\}. \end{aligned} \tag{4.24}$$

From the design criterion (4.14), one obtains

$$\begin{aligned} \dot{V}_1(x,\theta) \leq \frac{1}{|x_1|}\Big(&- \alpha(\lambda + 4\epsilon^2)x_1^2 + (2\alpha\epsilon + \lambda + 4\epsilon^2)x_1x_2 - 2\epsilon x_2^2 \\ &+ (\lambda + 4\epsilon^2)x_1\delta_0 + 2\epsilon x_2\delta_0 + 2\epsilon\beta x_1|x_1|\text{sign}(s) - \beta|x_1|x_2\text{sign}(s)\Big). \end{aligned} \tag{4.25}$$

Now using (4.16), one obtains

$$\begin{aligned}
\dot{V}_1(x,\theta) \leq \frac{1}{|x_1|}\Big(&-\alpha(\lambda+4\epsilon^2)x_1^2 + (2\alpha\epsilon+\lambda+4\epsilon^2)x_1x_2 \\
&- 2\epsilon x_2^2 + (\lambda+4\epsilon^2)x_1\delta_0 + 2\epsilon x_2\delta_0 + 2\epsilon\beta x_1^2 - \beta x_1x_2\Big) \\
\leq \frac{1}{|x_1|}\Big(&-\alpha(\lambda+4\epsilon^2)x_1^2 + (2\alpha\epsilon+\lambda+4\epsilon^2)x_1x_2 - 2\epsilon x_2^2 + \\
&(\lambda+4\epsilon^2)(x_1^2+x_1)\delta_0 + 2\epsilon(x_2^2+x_2)\delta_0 + 2\epsilon\beta x_1^2 - \beta x_1x_2\Big) \\
= \frac{1}{|x_1|}\Big(&\big((\lambda+4\epsilon^2)(-\alpha+\delta_0)+2\epsilon\beta\big)\, x_1^2 - 2\epsilon(1-\delta_0)x_2^2 \\
&+ (2\alpha\epsilon+\lambda+4\epsilon^2-\beta)x_1x_2\Big) + W
\end{aligned} \tag{4.26}$$

where

$$\begin{aligned}
W =& \frac{1}{|x_1|}\Big((\lambda+4\epsilon^2)\delta_0x_1 + 2\epsilon\delta_0x_2\Big) \leq \frac{1}{|x_1|}\big((\lambda+4\epsilon^2)\delta_0 + 2\epsilon\delta_0\big)\,||x|| \\
=& \frac{\kappa}{|x_1|}||x|| \text{ with } \kappa = (\lambda+4\epsilon^2)\delta_0 + 2\epsilon\delta_0.
\end{aligned}$$

One can rewrite (4.26) as

$$\dot{V}_1(x,\theta) \leq -\frac{1}{|x_1|}\begin{bmatrix} x_1 & x_2 \end{bmatrix} Q \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + W \leq -\frac{x^TQx}{|x_1|} + \frac{\kappa||x||}{|x_1|} \tag{4.27}$$

where the matrix Q is given as

$$\begin{bmatrix} -\left((\lambda+4\epsilon^2)(-\alpha+\delta_0)+2\epsilon\beta\right) & \star \\ -\frac{1}{2}(2\alpha\epsilon+\lambda+4\epsilon^2-\beta) & 2\epsilon(1-\delta_0) \end{bmatrix}. \tag{4.28}$$

As $\beta = 2\epsilon\alpha$, Q is positive definite with a minimal eigenvalue $\lambda_{\min}(Q) \geq 2\epsilon$ if

$$\alpha > \frac{4\epsilon[4\epsilon^2(\delta_0^2-\delta_0+1) + \epsilon(2\delta_0^2\lambda+2\lambda+1-2\delta_0) - \delta_0\lambda(1-\delta_0)] + \lambda^2}{4\lambda\epsilon[2\delta_0-4\delta_0\epsilon-1]}. \tag{4.29}$$

By using Rayleigh inequality $\lambda_{\min}(Q)||x||^2 \leq x^TQx \leq \lambda_{\max}(Q)||x||^2$, (4.27) becomes

$$\begin{aligned}
\dot{V}_1(x,\theta) \leq& \frac{-\lambda_{\min}(Q)||x||^2}{|x_1|} + \frac{\kappa||x||}{|x_1|} = \frac{-\lambda_{\min}(Q)(1-\tau)||x||^2}{|x_1|} \\
&+ \frac{\kappa}{|x_1|}||x|| - \tau\frac{\lambda_{\min}(Q)}{|x_1|}||x||^2,\ \ 0<\tau<1.
\end{aligned} \tag{4.30}$$

From (4.30), one obtains

$$\dot{V}_1(x,\theta) \leq -\frac{\lambda_{\min}(Q)(1-\tau)}{|x_1|}||x||^2 \tag{4.31}$$

if $||x|| > \kappa/(\tau\lambda_{\min}(Q))$.

By using the relation $|x_1| \leq ||x||$, the Eq. (4.31) becomes

$$\dot{V}_1(x,\theta) \leq -\lambda_{\min}(Q)(1-\tau)||x||. \tag{4.32}$$

By using the inequality $\lambda_{\min}(P)||x||^2 \leq x^T P x \leq \lambda_{\max}(P)||x||^2$, the Eq. (4.20) holds the inequality

$$\lambda_{\min}(P)||x||^2 \leq x^T P x \leq V_1(x,\theta). \tag{4.33}$$

If there exits $\sigma > 0$ such that

$$V_1(x,\theta) \leq \sigma\lambda_{\max}(P)||x||^2 \tag{4.34}$$

then, (4.32) can be written as:

$$\dot{V}_1(x,\theta) \leq -\lambda_{\min}(Q)(1-\tau)\frac{V_1^{1/2}(x,\theta)}{(\sigma\lambda_{\max}(P))^{1/2}} \tag{4.35}$$

(4.35) can also be represented as

$$\dot{V}_1(x,\theta) \leq -cV_1^{1/2},\ c = \frac{\lambda_{\min}(Q)(1-\tau)}{(\sigma\lambda_{\max}(P))^{1/2}}. \tag{4.36}$$

In accordance with (4.36), can state (4.22) as

$$\begin{aligned}\dot{V}(x,\theta,\alpha,\beta) \leq& -cV_1^{1/2} + (\alpha-\alpha_1)\frac{\dot{\alpha}}{\gamma_1} + (\beta-\beta_1)\frac{\dot{\beta}}{\gamma_2}\\ =& -cV_1^{1/2} - \frac{w_1|\alpha-\alpha_1|}{\sqrt{2\gamma_1}} + \frac{w_1|\alpha-\alpha_1|}{\sqrt{2\gamma_1}} - \frac{w_2|\beta-\beta_1|}{\sqrt{2\gamma_2}}\\ &+ \frac{w_2|\beta-\beta_1|}{\sqrt{2\gamma_2}} + (\alpha-\alpha_1)\frac{\dot{\alpha}}{\gamma_1} + (\beta-\beta_1)\frac{\dot{\beta}}{\gamma_2}.\end{aligned} \tag{4.37}$$

Using a well-known inequality $(a^2+b^2+c^2)^{1/2} \leq |a|+|b|+|c|$ and (4.19), one obtains

$$-cV_1^{1/2} - \frac{w_1}{\sqrt{2\gamma_1}}|\alpha-\alpha_1| - \frac{w_2}{\sqrt{2\gamma_2}}|\beta-\beta_1| \leq -c_0V^{1/2} \tag{4.38}$$

where $c_0 = \min(c, w_1, w_2)$.

In view of (4.38), one can rewrite (4.37) as

$$\begin{aligned}\dot{V}(x,\theta,\alpha,\beta) \leq& -c_0V^{1/2}(x,\theta,\alpha,\beta) + \frac{\dot{\alpha}}{\gamma_1}(\alpha-\alpha_1) + \frac{\dot{\beta}}{\gamma_2}(\beta-\beta_1)\\ &+ \frac{w_1}{\sqrt{2\gamma_1}}|\alpha-\alpha_1| + \frac{w_2}{\sqrt{2\gamma_2}}|\beta-\beta_1|.\end{aligned} \tag{4.39}$$

Suppose that the gains $\alpha(t)$ and $\beta(t)$ become bounded by the adaptive law (4.11). Then, there exist positive constants α_1 and β_1, such that, $\alpha(t)-\alpha_1<0$ and $\beta(t)-\beta_1<0$, for all $t\geq 0$.

Now, (4.39) can be represented as:

$$\begin{aligned}\dot{V}(x,\theta,\alpha,\beta)\leq&-c_0V^{1/2}(x,\theta,\alpha,\beta)\\&-|\alpha-\alpha_1|\left(\frac{\dot{\alpha}}{\gamma_1}-\frac{w_1}{\sqrt{2\gamma_1}}\right)-|\beta-\beta_1|\left(\frac{\dot{\beta}}{\gamma_2}-\frac{w_2}{\sqrt{2\gamma_2}}\right)\\ \dot{V}(x,\theta,\alpha,\beta)\leq&-c_0V^{1/2}(x,\theta,\alpha,\beta)+\Xi\end{aligned}\tag{4.40}$$

where $\Xi=-|\alpha-\alpha_1|\left(\frac{\dot{\alpha}}{\gamma_1}-\frac{w_1}{\sqrt{2\gamma_1}}\right)-|\beta-\beta_1|\left(\frac{\dot{\beta}}{\gamma_2}-\frac{w_2}{\sqrt{2\gamma_2}}\right)$.

Now, consider the following cases:

Case-1: Assume that the region $|s|>\mu$ and $\alpha(t)>\alpha_p$ $\forall t\geq 0$.

By choosing $\epsilon=\frac{w_2}{2w_1}\sqrt{\gamma_2/\gamma_1}$, in view of (4.11), $\dot{\alpha}=w_1\sqrt{\gamma_1/2}$ and $\dot{\beta}=w_2\sqrt{\gamma_2/2}$. Then, Ξ becomes zero.

Thus, one can represent (4.40) as:

$$\dot{V}(x,\theta,\alpha,\beta)\leq -c_0V^{1/2}(x,\theta,\alpha,\beta).\tag{4.41}$$

It can be observed that the adaptive gain $\alpha(t)$ must hold (4.29) for finite time convergence. Therefore, adaptive gain $\alpha(t)$ shall increase in view of $\dot{\alpha}=w_1\sqrt{\gamma_1/2}$ until (4.29) is satisfied that guarantees the matrix Q is positive definite according to (4.41). Then, finite-time convergence to the region $|s|\leq\mu$ is ensured.

Case-2: Let sliding surface reaches the domain $|s|<\mu$. Then, in view of (4.11), the dynamics of adaptive gain $\alpha(t)$ becomes

$$\dot{\alpha}=\begin{cases}-w_1\sqrt{\frac{\gamma_1}{2}}, & \text{if } \alpha>\alpha_p\\ \zeta, & \text{if } \alpha\leq\alpha_p\end{cases}\tag{4.42}$$

and

$$\Xi=\begin{cases}2\,|\alpha-\alpha_1|\,\frac{w_1}{\sqrt{2\gamma_1}}+2\,|\beta-\beta_1|\,\frac{w_2}{\sqrt{2\gamma_2}}, & \text{if } \alpha>\alpha_p\\ -\,|\alpha_p-\alpha_1+\zeta\cdot t|\left(\frac{\zeta}{\gamma_1}-\frac{w_1}{\sqrt{2\gamma_1}}\right)-|\beta-\beta_1|\left(\frac{2\zeta\epsilon}{\gamma_2}-\frac{w_2}{\sqrt{2\gamma_2}}\right), & \text{if } \alpha\leq\alpha_p.\end{cases}\tag{4.43}$$

It can be noticed that the value of Ξ in (4.43) is valid only for finite time for $\alpha\leq\alpha_p$. For $\alpha\leq\alpha_p$, its value immediately starts increasing such that $\alpha(t)=\alpha_p+\zeta\cdot t;\ \alpha(0)=\alpha_p$. Then, the value of Ξ in (4.43) is valid when $\alpha>\alpha_p$.

In accordance with (4.43), the sign of time derivative of the Lyapunov function (4.40) becomes indefinite. $|s|$ may become more significant than μ with decreases in gains α and β. Then, case-1 would be satisfied such that s reaches the region $|s| \leq \mu$ again in finite time and then may leave the region for a finite time. Therefore, it is ensured that the sliding surface s always remains in the larger region $|s| \leq \psi_1, \;\; \psi_1 > \mu$.

For $|s| \leq \mu$, $|\dot{s}|$ is governed by (4.11) and (4.13) such that

$$|\dot{s}| \leq [(\alpha(t_1) + \Delta)\mu^{1/2}] + \epsilon\alpha(t_1)(t_2 - t_1) = \bar{\psi}_2$$

where t_1 represents time instant when sliding surface s enters the domain $|s| \leq \mu$ and t_2 denotes time it leaves this domain.

For $\mu < |s| \leq \psi_1$

$$|\dot{s}| \leq \Delta\psi_1^{1/2} + \left(\epsilon + \psi_1^{1/2}\right)\left(\alpha(t_2) + w_1\sqrt{\psi_1\gamma_1/2}\right)(t_3 - t_2) = \psi_2^{\star}$$

where t_2 defines time instant when s leaves the domain $|s| \leq \mu$, whereas $t_3 (> t_2)$ is time when it re-enters the same domain thereafter. Then, the domain of sliding mode becomes

$$S = \{s, \;\dot{s} : |s| \leq \psi_1, \;|\dot{s}| \leq \psi_2, \;\psi_1 > \mu\}$$

where $\psi_2 = \max(\bar{\psi}_2, \;\psi_2^{\star})$. ■

Remark 5 *The model to be estimated can be expressed in the form of generalized controller canonical form (GCCF) as given in the Eqn. 4.2. Furthermore, the uncertain parameter is estimated by utilizing adaptive super-twisting algorithm in finite time.*

Now, the adaptive gains $\alpha(t)$ and $\beta(t)$ are given to show the boundedness.

Boundedness of Adaptive Gains: Consider the region: $\mu < |s| \leq \psi_1$, then from (4.11), one can observe that the adaptive gain $\alpha(t)$ can be written as:

$$\alpha(t) = \alpha(0) + w_1\sqrt{\frac{\gamma_1}{2}} \cdot t, \;\; 0 \leq t \leq t_f.$$

Therefore, the adaptive gain $\alpha(t)$ is bounded. This implies that the adaptive gain $\beta(t) = 2\epsilon\alpha(t)$ is also bounded. Moreover, in the region: $|s| \leq \mu$ the adaptive gains $\alpha(t)$ and $\beta(t)$ are decreasing. Thus, in the real second-order sliding mode, the adaptive gains $\alpha(t)$ and $\beta(t)$ are bounded.

Real second-order sliding mode: The inequality (4.29) is satisfied in finite time t_0 since its right-hand side remains bounded and adaptive gain $\alpha(t)$ increases linearly with time in view of (4.11). Suppose that $\mu = 0$. This means that s, $\dot{s}$ approach zero in finite time t_r represented as $t_r \leq 2V^{1/2}(t_0)/\psi_0$ which can also be obtained by directly integrating the inequality (4.40) with $\Xi = 0$. For $\mu > 0$, the domain of sliding mode becomes $S = \{s,\ \dot{s} : |s| \leq \psi_1,\ |\dot{s}| \leq \psi_2,\ \psi_1 > \mu\}$ in finite time $t_f \leq t_r$.

Ideal second-order sliding mode: It should be noted that in adaptive gain filter (4.11), if the detector is eliminated by taking $\mu = 0$, then adaptive laws (4.11) can be changed to $\dot{\alpha} = w_1\sqrt{\gamma_1/2}$ when $s \neq 0$ and zero when $s = 0$ ($\alpha(0) > 0$) and β is equal to $2\epsilon\alpha$. In this way the adaptive STA control law will force the states of the system (4.5) to the ideal second order sliding mode, i.e., $s = \dot{s} = 0$ in finite time. However, the gains $\alpha(t)$ and $\beta(t)$ can be overestimated.

From the discussions mentioned above, the proposed adaptive super-twisting algorithm based parameter estimation with the identifier input (4.10) under uncertainties forces the sliding surface and its derivative to remain inside certain domains in finite time. Therefore, one can observe that the estimated parameter $\hat{\varphi}$ converges to the actual parameter φ in finite time.

4.4 Illustrative Example

Consider the second order nonlinear dynamical system of Duffing oscillator of the form [75]:

$$m\ddot{x}(t) + b\dot{x}(t) + k_1 x(t) + k_2 x^3(t) = f(t) \tag{4.44}$$

where $x(t)$ is the displacement, m represents the system mass which is known, b is the damping coefficient, k_1 and k_2 are the stiffness coefficients and $f(t)$ denotes the force applied to the system.

Alternatively, one can write (4.44) as:

$$\ddot{x}(t) = -\frac{b}{m}\dot{x}(t) + \frac{k_1}{m}x(t) - \frac{k_2}{m}x^3(t) + \frac{1}{m}u \tag{4.45}$$

where $u = f(t)$.

Let us assume the state variables $z_1 = x(t)$ and $z_2 = \dot{x}(t)$ and the state vector $z = [z_1\, z_2]^\top$. Then,

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -\frac{k_1}{m}z_1 - \frac{k_2}{m}z_1^3(t) + \frac{1}{m}u - \frac{b}{m}z_2 \end{aligned} \tag{4.46}$$

Let us assume the bounds as $k_1 \leq k_{1m}$, $k_2 \leq k_{2m}$ and $b \leq b_m$. We can further rewrite (4.46) as

$$\begin{aligned} \dot{z}_1 =& z_2 \\ \dot{z}_2 =& \varphi^T \phi(z) + g(z)u + \delta \end{aligned} \tag{4.47}$$

where $\varphi^T = \begin{bmatrix} \frac{k_1}{m} & \frac{k_2}{m} \end{bmatrix}$ is uncertain constant parameter, $\phi(z) = \begin{bmatrix} -z_1 & -z_1^3 \end{bmatrix}^T$ is known nonlinear function, $g(z) = \frac{1}{m}$ is known function and $\delta = -\frac{b}{m} z_2$ is unmodelled part of the system. Let us consider an estimator system as:

$$\begin{aligned} \dot{\hat{z}}_1 =& \hat{z}_2 \\ \dot{\hat{z}}_2 =& \hat{\varphi}^T \phi(z) + g(z)u + v \end{aligned} \tag{4.48}$$

where v is the identifier input. Consider the error in state estimation as $e = z - \hat{z}$ and the error in parameter estimation as $\theta = \varphi - \hat{\varphi}$. Then, the error dynamics becomes

$$\dot{e}_1 = e_2, \ \dot{e}_2 = \theta^\top \phi(z) + \delta - v \tag{4.49}$$

Observe that we obtain the same form as (4.8). Now let us assume a sliding surface as $s = e_2 + \eta e_1$ then we have $\dot{s} = \dot{e}_2 + \eta \dot{e}_1$, which yields $\dot{s} = \theta^\top \phi(z) + \delta - v + \eta e_2$. Selecting the identifier input as $v = \eta e_2 + \alpha |s|^{\frac{1}{2}} \text{sign}(s) + \int_0^t \frac{\beta}{2} \text{sign}(s(\tau)) d\tau$. Then, $\dot{s} = \theta^\top \phi(z) + \delta - \alpha |s|^{\frac{1}{2}} \text{sign}(s) - \int_0^t \frac{\beta}{2} \text{sign}(s(\tau)) d\tau$. By assuming $v_1 = -\int_0^t \frac{\beta}{2} \text{sign}(s(\tau)) d\tau$, the same form is obtained as in (4.13).

With this example, the following observations are marked:

Remark 6 *As a part of the design, one is free to choose any sliding surface s that yields a form of $\dot{s}$ as given in (4.9).*

Remark 7 *Observe that the parameter estimation problem of any arbitrary-order n^{th}-order system is reduced to a 2^{nd}-order system (4.13), with the estimate of the parameter being governed by the dynamics as given in (4.14).*

Remark 8 *A general form of linear sliding surface can be chosen as*

$$s = \eta_1 e_1 + \eta_2 e_2 + \cdots + \eta_{n-1} e_{n-1} + e_n \tag{4.50}$$

yielding

$$\dot{s} = \eta_1 e_2 + \eta_2 e_3 + \cdots + \eta_{n-1} e_n + \dot{e}_n. \tag{4.51}$$

Observe that (4.51) is equivalent to (4.9). Also, note that the choice of the gains $\eta_1, \eta_2, \cdots, \eta_{n-1}$ is dictated by the fact that the polynomial

$$p^{n-1} + \eta_{n-1}p^{n-2} + \cdots + \eta_1 \tag{4.52}$$

is Hurwitz. This is the general choice for linear sliding surface. However, the choice is not limited to only linear form. Please see Section (4.5), where a nonlinear sliding surface has been considered in Example 1.

4.5 Simulation Results

The effectiveness of the proposed scheme is verified in this section. For this, two practical examples have been taken. The first one is a pendulum system. The second system is a single link manipulator with flexible joints.

Example 1: Consider the pendulum system [14]:

$$\dot{z}_1 = z_2,\ \dot{z}_2 = -\varphi_1 \mathrm{sin} z_1 - \varphi_2 z_2 + \varsigma u + \delta \tag{4.53}$$

where φ_1 and φ_2 are the unknown constant parameters that satisfy $0 < \varphi_1 \leq a$ and $0 < \varphi_2 \leq b$ with known bounds a and b. Assume that $a = 2$ and $b = 0.5$. δ is the model uncertainty and external disturbance and it is taken as $0.8\mathrm{cos}z_1$+0.1sint.

For the system (4.53), the parameter estimation system is defined as:

$$\dot{\hat{z}}_1 = \hat{z}_2,\ \dot{\hat{z}}_2 = -\hat{\varphi}_1 \mathrm{sin} z_1 - \hat{\varphi}_2 z_2 + \varsigma u + v. \tag{4.54}$$

In view of (4.53) and (4.54), the error dynamics is given by

$$\dot{e}_1 = e_2,\ \dot{e}_2 = -\theta_1 \mathrm{sin} z_1 - \theta_2 z_2 + 0.8\mathrm{cos} z_1 + 0.1\mathrm{sin} t - v \tag{4.55}$$

where $\theta_1 = \varphi_1 - \hat{\varphi}_1$ and $\theta_2 = \varphi_2 - \hat{\varphi}_2$ are the errors in the parameter estimation.

For the system (4.55), the sliding surface is

$$s = e_2 + \eta |e_1|^{\varepsilon} \mathrm{sign} e_1 \tag{4.56}$$

where $\eta \in \mathbb{R}_{>0}$ and $0.5 < \varepsilon < 1$.

Taking the time derivative of (4.56), one gets

$$\begin{aligned} \dot{s} =& \dot{e}_2 + \eta\varepsilon |e_1|^{\varepsilon - 1} e_2 \\ =& -\theta_1 \mathrm{sin} z_1 - \theta_2 z_2 + 0.8\mathrm{cos} z_1 + 0.1\mathrm{sin} t - v + \vartheta(e_1) e_2 \end{aligned} \tag{4.57}$$

where $\vartheta(e_1) := \eta\varepsilon|e_1|^{\varepsilon-1}$.

Now, the adaptive super-twisting algorithm based identifier input is designed as

$$\begin{aligned} v =& \vartheta(e_1)e_2 + \alpha|s|^{0.5}\text{sign}(s) - v_1 \\ \dot{v}_1 =& -\frac{\beta}{2}\text{sign}(s). \end{aligned} \tag{4.58}$$

To avoid the singularity

$$\vartheta(e_1) = \begin{cases} 0, & \text{if } e_1 = 0 \text{ and } s \neq 0 \\ \eta\varepsilon|e_1|^{\varepsilon-1}, & \text{otherwise.} \end{cases} \tag{4.59}$$

For the sake of simplicity, u is 0.1. The design parameters of adaptive super-twisting algorithm based parameter estimation are selected as $\eta = 12$, $\varepsilon = 0.91$, $\epsilon = 0.03$, $\lambda = 0.07$ $w_1 = 60$, $\gamma_1 = 0.0009$, $\alpha_p = 0.05, \zeta = 0.08$, $\mu = 0.004$ and $\alpha(0) = 0.45$. The actual value of the parameters are chosen as $\varphi_1 = 0.8$ and $\varphi_2 = 0.2$. It is assumed that the estimated parameters take the initial values as $\hat{\varphi}(0) = \begin{bmatrix} -0.5 & -1 \end{bmatrix}$. For the equation (12) in the reference [144], the values of parameters are selected as $L^\star = 0.08$ and $k = 0.5$. The parameter estimation errors $\theta_1(t)$ and $\theta_2(t)$ are shown in Figure 4.1. It can be clearly noticed that the parameter estimation errors $\theta_1(t)$ and $\theta_2(t)$ converge to a small neighborhood of zero in finite time. Thus, the estimated parameter $\hat{\varphi}$ converges to the actual parameter φ. The trajectory of the sliding surface $s(t)$ is depicted in Figure 4.2. The adaptive gain $\alpha(t)$ is shown in Figure 4.3, indicating an adaptation of the gain magnitude corresponding to the model uncertainty and external disturbance. The identifier input $v(t)$ is illustrated in Figure 4.4.

From the discussions mentioned above, it is noticed that the estimated parameters $\hat{\varphi}_1$ and $\hat{\varphi}_2$ converge to the actual parameters φ_1 and φ_2, respectively in the presence of uncertainty in finite time.

Example 2: Consider the single link manipulator system given by [79]:

$$\begin{aligned} &\dot{z}_1 = z_2,\ \dot{z}_2 = z_3,\ \dot{z}_3 = z_4, \\ &\dot{z}_4 = \varphi_1(z_2^2\text{sin}z_1 - \text{cos}z_1) - \varphi_2\text{sin}z_1 + \varphi_3 z_3 + \varsigma u + \delta \end{aligned} \tag{4.60}$$

where φ_1, φ_2 and φ_3 are the unknown constant parameters that satisfy $0 < \varphi_1 \leq a$, $0 < \varphi_2 \leq b$ and $0 < \varphi_3 \leq c$ with known bound a, b and c. Assume that $a = 2$, $b = 3$ and $c = 1$. δ is the model uncertainty and external disturbance and it is taken as $0.8\text{cos}z_1 + 0.1\text{sint}$.

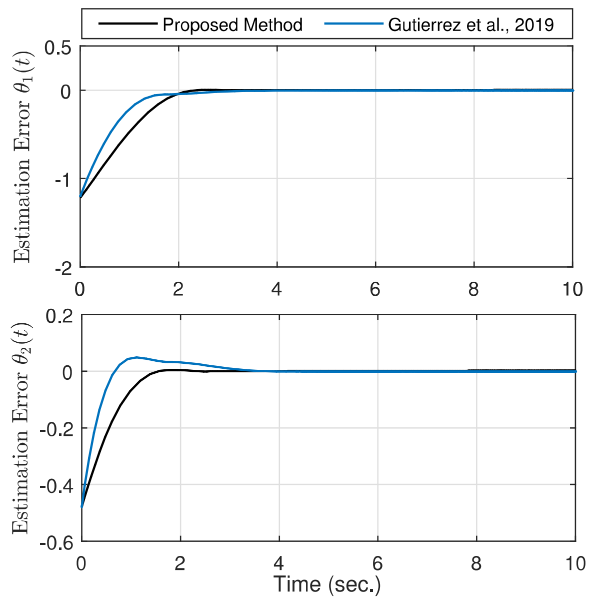

Figure 4.1: Parameter estimation errors $\theta_1(t)$ and $\theta_2(t)$.

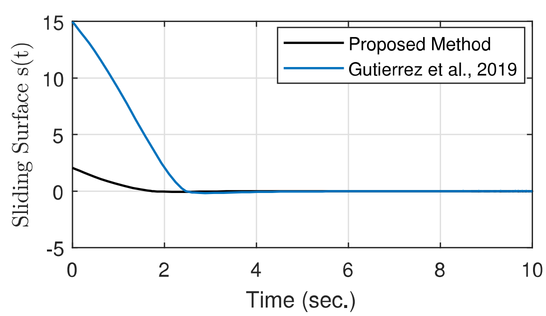

Figure 4.2: Sliding surface $s(t)$.

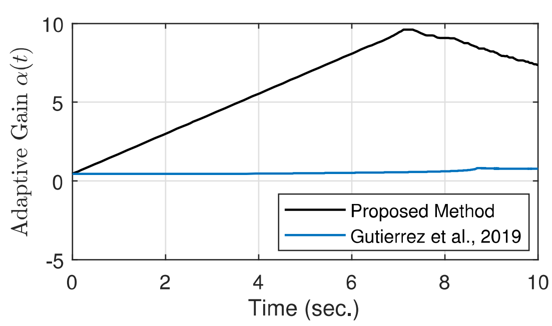

Figure 4.3: Adaptive gain $\alpha(t)$.

For system (4.60), parameter estimation system is defined as:

$$\begin{aligned} &\dot{\hat{z}}_1 = \hat{z}_2,\ \ \dot{\hat{z}}_2 = \hat{z}_3,\ \ \dot{\hat{z}}_3 = \hat{z}_4, \\ &\dot{\hat{z}}_4 = \hat{\varphi}_1(z_2^2 \text{sin} z_1 - \text{cos} z_1) - \hat{\varphi}_2 \text{sin} z_1 + \hat{\varphi}_3 z_3 + \varsigma u + v. \end{aligned} \tag{4.61}$$

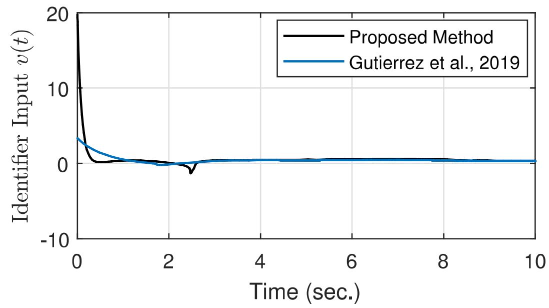

Figure 4.4: Identifier input $v(t)$.

In view of (4.60) and (4.61), the error dynamics is given by

$$\begin{aligned} &\dot{e}_1 = e_2,\ \dot{e}_2 = e_3,\ \dot{e}_3 = e_4, \\ &\dot{e}_4 = \theta_1(z_2^2\text{sin}z_1 - \text{cos}z_1) - \theta_2\text{sin}z_1 + \theta_3 z_3 + 0.8\text{cos}z_1 + 0.1\text{sin}t - v. \end{aligned} \tag{4.62}$$

For the system (4.62), assume the sliding surface as:

$$s = e_4 + \eta_3 e_3 + \eta_2 e_2 + \eta_1 e_1 \tag{4.63}$$

where η_1, η_2 and η_3 are constant gains such that the polynomial in p

$$p^3 + \eta_3 p^2 + \eta_2 p + \eta_1$$

is Hurwitz. Taking the time derivative of (4.63), one gets

$$\begin{aligned} \dot{s} =&\dot{e}_4 + \eta_3\dot{e}_3 + \eta_2\dot{e}_2 + \eta_1\dot{e}_1 \\ =&\theta_1(z_2^2\text{sin}z_1 - \text{cos}z_1) - \theta_2\text{sin}z_1 + \theta_3 z_3 - v + k_3 e_4 + k_2 e_3 + k_1 e_2 + 0.8\text{cos}z_1 + 0.1\text{sin}t. \end{aligned} \tag{4.64}$$

Now, the adaptive super-twisting algorithm based identifier input is designed as

$$\begin{aligned} v =&\eta_3 e_4 + \eta_2 e_3 + \eta_1 e_2 + \alpha|s|^{0.5}\text{sign}(s) - v_1 \\ \dot{v}_1 =& -\frac{\beta}{2}\text{sign}(s). \end{aligned} \tag{4.65}$$

For the sake of simplicity, u is 0.1. The design parameters of adaptive super-twisting algorithm based parameter estimation are taken as $\eta_1 = 1.9$, $\eta_2 = 135$, $\eta_3 = 186$, $\epsilon = 0.0002$, $\lambda = 0.0004$, $w_1 = 186$, $\gamma_1 = 0.1$, $\alpha_p = 0.005, \zeta = 0.001$, $\mu = 0.005$ and $\alpha(0) = 0.1$. The actual value of the parameters are chosen as $\varphi_1 = 0.4$, $\varphi_2 = 0.6$ and $\varphi_3 = 0.3$. The initial conditions of the estimated parameters are set to $\hat{\varphi}(0) = \begin{bmatrix} 0.172 & 0.647 & 0.345 \end{bmatrix}$. For the equation (12) in the reference [144], the values of parameters are selected as

$L^{\star} = 0.08$ and $k = 0.1$. The parameter estimation errors $\theta_1(t)$, $\theta_2(t)$ and $\theta_3(t)$ are shown in Figure 4.5. It can be clearly noticed that the parameter estimation errors $\theta_1(t)$, $\theta_2(t)$ and $\theta_3(t)$ converge to a small neighborhood of zero. Thus, the estimated parameter $\hat{\varphi}$ converges to the actual parameter φ. The trajectory of the sliding surface $s(t)$ is depicted in Figure 4.6. The adaptive gain $\alpha(t)$ is shown in Figure 4.7. The identifier input $v(t)$ is illustrated in Figure 4.8.

The non-monotonicity of the adaptive gain in Figure 4.7 refers to a situation where the adaptive gain, used in an algorithm or system for parameter estimation, does not exhibit a consistent trend of increasing or decreasing as the estimation process progresses. The non-monotonicity of the adaptive gain can arise from various factors such as complex model dynamics and noisy and uncertainty.

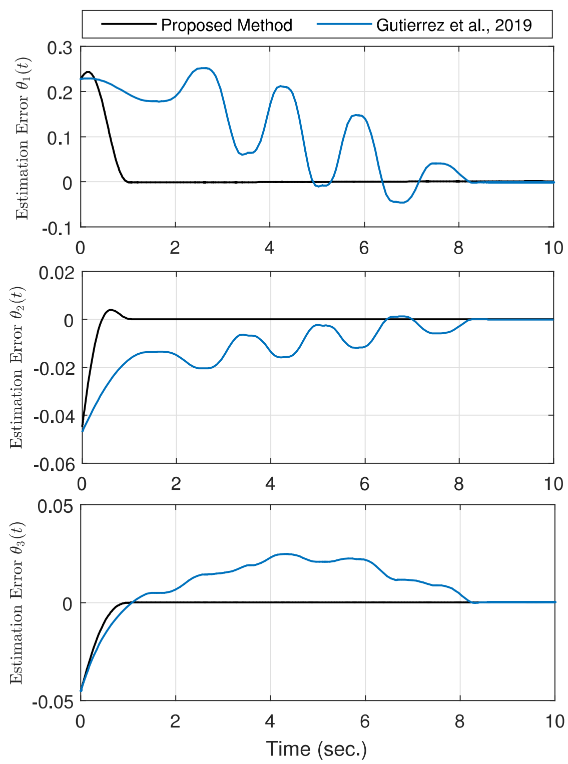

Figure 4.5: Parameter estimation errors $\theta_1(t)$, $\theta_2(t)$ and $\theta_3(t)$.

From the above discussions, one can observe that the estimated parameters converge to the actual parameters in the presence of uncertainty and external disturbance. Furthermore, the proposed method is compared with the existing results [144]. It is noted that the obtained results exhibit faster convergence compared to the simplified version of adaptive super-twisting control in the existing literature [144].

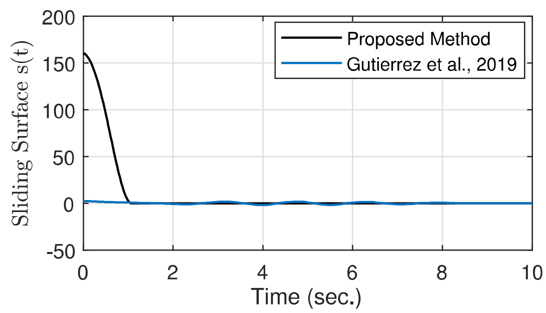

Figure 4.6: Sliding surface $s(t)$.

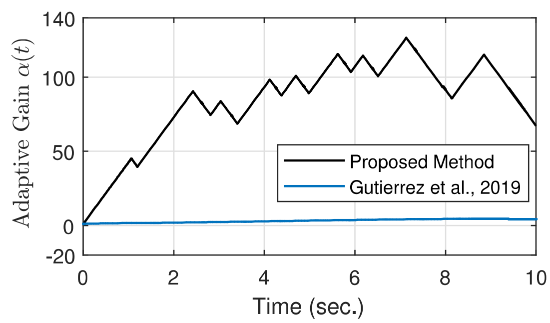

Figure 4.7: Adaptive gain $\alpha(t)$.

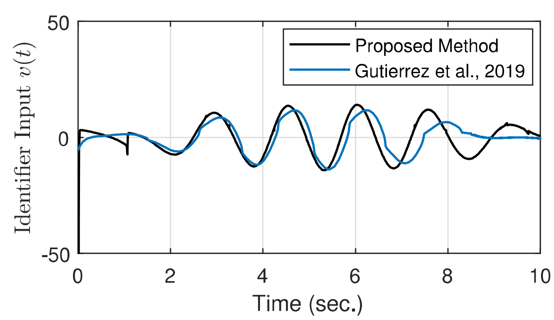

Figure 4.8: Identifier input $v(t)$.

4.6 Summary

In this chapter, the adaptive super-twisting algorithm based method estimates the unknown parameters of the system. The estimated parameter $\hat{\varphi}$ converges to the actual parameter φ in the presence of uncertainty. The Lyapunov stability is also presented. The technique is illustrated by simulation of numerical examples which show the effectiveness of the proposed scheme.