ON QUALITATIVE PROPERTIES OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

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DOCTOR OF PHILOSOPHY

by

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Chapter 7

Conclusions and Future Plan

7.1 Conclusion

This thesis provides a qualitative analysis of nonlinear differential and integro-differential equations of fractional order. The work emphasizes establishing the existence, uniqueness, and controllability results for a class of fractional-order nonautonomous nonlinear functional differential equations involving the Caputo fractional derivative. Additionally, it investigates the existence and uniqueness of solutions for fractional-order nonlinear functional differential equations utilizing the Riesz-Caputo fractional derivative. Chapter 1 provides an overview of key mathematical concepts in FC and outlines the foundational results utilized throughout this thesis. Chapter 2 presents the historical development of FC and offers a concise literature review on the theory of existence, uniqueness, and controllability for various types of differential and fractional differential equations.

Chapter 3 focuses on the mathematical analysis and results of extending a control system from autonomous to nonautonomous evolution within the framework of

fractional control. Using the resolvent operators' properties and the Banach fixed-point theorem, we establish the existence and uniqueness of a mild solution for the system (3.1). Additionally, we examine the controllability of the fractional control system (3.2) using resolvent operator theory and the generalized Banach contraction principle, under suitable assumptions on the system operators $\mathtt{f}$, $\mathcal{G}$, and $\mathtt{g}$. Several examples are provided at the end of the chapter to demonstrate the applicability of the results.

In Chapter 4, we extend the work done in Chapter 3 and investigate the existence of a mild solution and the approximate controllability of the fractional control system (4.10). Under appropriate assumptions on the operators $\mathtt{f}$, $\mathcal{G}$, and $\mathtt{g}$, we establish the existence of a mild solution and explore the approximate controllability of the control system (4.10). The results are obtained using the semigroup technique, Krasnoselskii's fixed-point theorem, and the properties of fractional calculus.

In Chapter 5, we study the nonlinear FDEs with delay-forced terms along with Riesz-Caputo's fractional derivative. The existence of solutions to nonlinear FDEs with delay is first established using the FC technique and the Banach fixed-point theorem. Using the theory of FC, the measure of noncompactness, and the strict set contraction mapping principle, we also investigate the existence of a solution for the problem (5.1) under a different set of assumptions on $\mathtt{f}$. A few instances are presented at the conclusion to demonstrate the competence of the suggested outcomes.

The study of FDEs is continued in previous chapters. In Chapter 6, the problem of nonlinear FDEs with delay-forced terms is enlarged to the nonlinear fractional integro-differential equations with delay-forced terms along with Riesz-Caputo's fractional derivative. Employing FC techniques and multiple fixed-point theorems, we establish a few results regarding both the existence and uniqueness of the system

(6.1). Further, by introducing a partial order in a Banach space of all continuous functions, we look into the existence of extremal solutions of equation (6.1). In the end, a few examples are showcased to evince the proficiency of the offered results.

7.2 Future Plan

In Chapters 3 and 4, we successfully established the existence and uniqueness of mild solutions and demonstrated controllability within the realm of fractional control systems along with Caputo fractional derivative. Further, continuing our study in this direction, Chapter 5 focused on investigating the existence of solutions for FDEs involving the Riesz-Caputo fractional derivative. Then, this analysis was extended to fractional integro-differential equations in Chapter 6, where we explored the existence and uniqueness of solutions under various set of assumptions on the forced term `f`. Moreover, by introducing a partial order in a Banach space of continuous functions, we look into the existence of extremal solutions. Building on this foundation, our future aim is to delve into the existence, uniqueness, and controllability of nonautonomous fractional systems and nonautonomous fractional control systems along with Riesz-Caputo's fractional derivative and examine their stability and dependency on initial data, thereby providing a holistic exploration of the considered problems in this thesis.

Expanding beyond these developments, it is essential to note that there is limited prior work in the domain of nonautonomous non-instantaneous impulsive FDEs and nonautonomous neutral FDEs with both the derivative as we work in our thesis. We are motivated to explore and establish the existence and uniqueness of mild solutions and the concept of controllability for both non-instantaneous impulsive FDEs and non-instantaneous impulsive fractional integro-differential equations with

Caputo and Riesz-Caputo derivatives. Along with this, our research will explore the existence and uniqueness of mild solutions and the controllability of neutral FDEs and neutral fractional integro-differential equations along Caputo and Riesz-Caputo derivatives.

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