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Standard HVM

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We discuss a standard hierarchical VEVs model which predicts the leptonic mixing angles in terms of the Cabibbo angle, and masses of strange and charm quarks as $\sin\theta_{12}^{\ell} \geq 1 - 2\sin\theta_{12}$, $\sin\theta_{23}^{\ell} \geq 1 - \sin\theta_{12}$, and $\sin\theta_{13}^{\ell} \geq \sin\theta_{12} - \frac{m_s}{m_c}$ for the normal mass ordering of neutrinos. This results in very precise predictions of the leptonic mixing angles given by $\sin\theta_{12}^{\ell} = 0.55 \pm 0.00134$, $\sin\theta_{23}^{\ell} = 0.775 \pm 0.00067$, and $\sin\theta_{13}^{\ell} = 0.1413 - 0.1509$. Furthermore, we predict neutrinos to be the Dirac kind disfavoring the inverted mass hierarchy. The standard hierarchical VEVs model predicts a possible new class of the dark matter candidate, named as neutrinic dark matter.

Keywords: Flavour problem; leptonic mixing; dark-matter; .

1. Introduction

The origin of the flavor of the Standard Model (SM) is a fascinating problem that may lead to physics beyond the SM. This problem demands an explanation for the fermionic mass spectrum and mixing of the SM. There exist different ways to solve this problem. For instance, a technicolor framework can solve this problem through the multi-fermion chiral condensates of a Dark-TechniColor (DTC) symmetry.[1,2] An Abelian flavor symmetry can be used as well.[3–13] We can also have a solution based on loop-suppressed couplings to the Higgs,[14] wave-function localization scenario,[15] the compositeness,[16] extra-dimensional framework,[17] and discrete symmetries,[18–22] and flavor deconstruction.[23–41]

Hierarchical VEVs Model (HVM) redefines the flavor problem of the SM in terms of the VEVs hierarchy, and provide an explanation to the quark mixing angles in terms of the ratios of the hierarchical VEVs.[1] In the Ultra-Violet (UV) completion of HVM, the hierarchical VEVs are chiral multi-fermion condensates of a DTC dynamics depending only on a single parameter Λ_{DTC}, i.e. the scale of the underlying non-Ableian DTC gauge theory.[2] For a recent review of the HVM, see Ref. 42.

In the simplest version of the HVM, the SM is subjected to the imposition of the three Abelian discrete symmetries $\mathcal{Z}_2$, $\mathcal{Z}_2'$ an $\mathcal{Z}_2''$, and charged fermionic mass spectrum is explained in terms of the VEVs of the six gauge singlet scalar fields χ_i, assumed to be the bound states of a new strong dynamics, through the dimension-5 operators.[1] In the HVM, the neutrino masses are recovered by adding an additional gauge singlet scalar field χ_7 through the type-I seesaw mechanism.[43] However, no predictions for the neutrino mixing angles were made.

Predicting neutrino mixing angles is one of the most difficult challenges beyond the SM. For a review of neutrino models, see Ref. 44. The reason is that the observed pattern of the neutrino mixing angles is entirely different from that of the quark mixing angles. For instance, a global fit to neutrino oscillation data provides the following values of the neutrino oscillation parameters for the normal mass ordering:[45]

$$\begin{aligned}
&\Delta m_{21}^2 = (7.50\,^{+0.64}_{-0.56}) \times 10^{-5}\mathrm{eV}^2, \quad |\Delta m_{31}^2| = (2.55 \pm 0.08) \times 10^{-3}\mathrm{eV}^2, \\
&\sin\theta_{12}^\ell = (0.564\,^{+0.044}_{-0.043}), \quad \sin\theta_{23}^\ell = (0.758\,^{+0.023}_{-0.099}), \quad \sin\theta_{13}^\ell = (0.1483\,^{+0.0067}_{-0.0069}),
\end{aligned} \tag{1}$$

where the errors are in 3 σ range.

We observe that only $\sin\theta_{13}^\ell$ is closer to the Cabibbo angle $\sin\theta_{12}^q$, and $\sin\theta_{12}^q - \sin\theta_{13}^\ell \approx 0.06923 - 0.08427$. Furthermore, although the mixing angle sin θ_{23}^ℓ is of the order one, it is not exactly one. Moreover, the $\sin\theta_{12}^\ell$ is large, however, it cannot be considered order one.

On the other side, the quark mixing angles are[46]

$$\begin{aligned}
\sin\theta_{12} = (0.225 \pm 0.00067), \quad \sin\theta_{23} = (0.04182\,^{+0.00085}_{-0.00074}), \quad \sin\theta_{13} \\
= (0.00369 \pm 0.00011).
\end{aligned} \tag{2}$$

In the case of quark-mixing, $\sin\theta_{23} \approx \sin\theta_{12}^2$ and remarkable observation is $\sin\theta_{13} < \sin\theta_{12}^3$.

The really important and challenging question is if these observations can be predicted from a fundamental theory. In this work, we shall discuss what we refer to as the Standard HVM (SHVM), and show that the above observations can be explained in the SHVM.

We shall show that the SHVM leads to a multiple Axion-Like Particles (ALPs) scenario. ALPs can play the role of cold dark matter,[47–51] and can address the strong CP problem.[52,53] Due to the multiple ALPs scenarios, the SHVM may be a consequence of a string type dynamics. This is because of the existence of an axiverse, that is, emergence of a large number of ALPs in string theory.[54–58]

A field theory axiverse (π axiverse) is an alternative to string theory axiverse, which also produces a large number of ALPs.[59] The field theory axiverse is based on a confining gauge theory referred as dark QCD.[60–62] There are $N_f^2 - 1$ axionlike states in dark QCD.[59]

The UV completion of the SHVM can come from three strong QCD-like gauge dynamics, namely, $\mathrm{SU}(N)_{\mathrm{TC}}$, $\mathrm{SU}(N)_{\mathrm{DTC}}$, and $\mathrm{SU}(N)_F$, where TC stands for technicolor, DTC is for dark-technicolor, and F represents the strong dynamics of vector-like fermions of a dark-QCD symmetry.[2] Therefore, emergence of ALPs is natural in the SHVM from its UV dynamics. Moreover, the dark-technicolor paradigm is a good example of the field theory axiverse.

The most important fact in support of the SHVM and its UV completion, the dark-technicolor paradigm, comes from a recent development where it is shown that a conformal strong dynamics can provide the eletroweak symmetry breaking.[63] This dynamics in the case of the dark-technicolor paradigm is the $\mathrm{SU}(N)_{\mathrm{TC}}$ symmetry, which generates the SM Higgs field. Hence, the SHVM with dark-technicolor paradigm as a UV completion is an interesting scenario to address the flavor origin of the SM.

2. The Standard HVM

The standard HVM is achieved by enlarging the discrete symmetries used in HVM. We impose a generic $\mathcal{Z}_N \times \mathcal{Z}_M \times \mathcal{Z}_P$ flavor symmetry on the SM. This symmetry naturally emerges through the breaking of three axial $U(1)$ groups in the UV completion of HVM.[2,42] For the sake of completeness, we present an explicit UV construction of the SHVM in Appendix A, where the gauge singlet scalar fields χ_i and the $\mathcal{Z}_N \times \mathcal{Z}_M \times \mathcal{Z}_P$ flavor symmetry naturally arise from a DTC dynamics.

The transformations of the composite scalar fields χ_i under the $\mathcal{G}_{\mathrm{SM}} \equiv \mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times U(1)_Y$ symmetry of the SM are given as

$$\chi_i : (1, 1, 0), \tag{3}$$

where $i = 1 - 6$.

The masses of the charged fermions are produced by the dimension-5 operators written as

$$\mathcal{L} = \frac{1}{\Lambda}[y_{ij}^u \bar{\psi}_{L_i}^q \tilde{\varphi} \psi_{R_i}^u \chi_i + y_{ij}^d \bar{\psi}_{L_i}^q \varphi \psi_{R_i}^d \chi_i + y_{ij}^\ell \bar{\psi}_{L_i}^\ell \varphi \psi_{R_i}^\ell \chi_i] + \mathrm{H.c.}, \tag{4}$$

i and j are generation indices, ψ_L^q, ψ_L^ℓ are the quark and leptonic doublets, ψ_R^u, ψ_R^d, ψ_R^ℓ denote the right-handed up, down-type quarks and leptons, φ and $\tilde{\varphi} = -i\sigma_2\varphi^*$ represent the SM Higgs field, and its conjugate and σ_2 is the second Pauli matrix.

We produce the required pattern of the charged fermions masses for the SHVM by assigning the following generic charges under the $\mathcal{Z}_N \times \mathcal{Z}_M \times \mathcal{Z}_P$ flavor symmetry.

$$\begin{aligned}
&\psi_{L_1}^q : (+, 1, \omega_{14}^{\mathrm{P}-1}), \quad \psi_{L_2}^q : (+, 1, \omega_{14}), \quad \psi_{L_3}^q : (+, 1, \omega_{14}^2),\\
&u_R : (-, 1, \omega_{14}^{\mathrm{P}-3}), \quad c_R : (+, 1, \omega_{14}^6), \quad t_R : (+, 1, \omega_{14}^4),\\
&d_R : (-, 1, \omega_{14}^{12}), \quad s_R : (+, 1, \omega_{14}^{12}), \quad b_R : (+, 1, \omega_{14}^{12}),\\
&\psi_{L_1}^\ell : (+, \omega_4^3, \omega_{14}^{12}), \quad \psi_{L_2}^\ell : (+, \omega_4^3, \omega_{14}^{10}), \quad \psi_{L_3}^\ell : (+, \omega_4^3, \omega_{14}^6),\\
&e_R : (-, \omega_4^3, \omega_{14}^{10}), \quad \mu_R : (+, \omega_4^3, \omega_{14}^{P-1}), \quad \tau_R : (+, \omega_4^3, \omega_{14}),\\
&\chi_1 : (-, 1, \omega_{14}^2), \quad \chi_2 : (+, 1, \omega_{14}^5), \quad \chi_3 : (+, 1, \omega_{14}^2),\\
&\chi_4 : (+, 1, \omega_{14}^{13}), \quad \chi_5 : (+, 1, \omega_{14}^{11}), \quad \chi_6 : (+, 1, \omega_{14}^{10}),
\end{aligned} \tag{5}$$

Table 1. The charges of left- and right-handed fermions and scalar fields under $\mathcal{Z}_2$, $\mathcal{Z}_4$ and $\mathcal{Z}_{14}$ symmetries for the normal mass ordering. The ω_4 is the fourth root and ω_{14} is the 14th root of unity corresponding to the symmetries $\mathcal{Z}_4$ and $\mathcal{Z}_{14}$, respectively.

Fields	$\mathcal{Z}_2$	$\mathcal{Z}_4$	$\mathcal{Z}_{14}$	Fields	$\mathcal{Z}_2$	$\mathcal{Z}_4$	$\mathcal{Z}_{14}$	Fields	$\mathcal{Z}_2$	$\mathcal{Z}_4$	$\mathcal{Z}_{14}$	Fields	$\mathcal{Z}_2$	$\mathcal{Z}_4$	$\mathcal{Z}_{14}$
u_R	$-$	1	ω_{14}^{11}	d_R, s_R, b_R	$+$	1	ω_{14}^{12}	$\psi^q_{L_3}$	$+$	1	ω_{14}^{2}	τ_R	$+$	ω_4^3	ω_{14}
c_R	$+$	1	ω_{14}^{6}	χ_4	$+$	1	ω_{14}^{13}	$\psi^\ell_{L_1}$	$+$	ω_4^3	ω_{14}^{12}	ν_{e_R}	$+$	ω_4	ω_{14}^{8}
t_R	$+$	1	ω_{14}^{4}	χ_5	$+$	1	ω_{14}^{11}	$\psi^\ell_{L_2}$	$+$	ω_4^3	ω_{14}^{10}	ν_{μ_R}	$-$	ω_4	ω_{14}^{3}
χ_1	$-$	1	ω_{14}^{2}	χ_6	$+$	1	ω_{14}^{10}	$\psi^\ell_{L_3}$	$+$	ω_4^3	ω_{14}^{6}	ν_{τ_R}	$-$	ω_4	ω_{14}^{3}
χ_2	$+$	1	ω_{14}^{5}	$\psi^q_{L_1}$	$+$	1	ω_{14}^{13}	e_R	$-$	ω_4^3	ω_{14}^{10}	χ_7	$-$	ω_4^2	ω_{14}^{8}
χ_3	$+$	1	ω_{14}^{2}	$\psi^q_{L_2}$	$+$	1	ω_{14}	μ_R	$+$	ω_4^3	ω_{14}^{13}	φ	$+$	1	1

where ω_4 is the fourth root and ω_{14} is the 14th root of unity corresponding to the symmetries $\mathcal{Z}_4$ and $\mathcal{Z}_{14}$, respectively. Moreover, $N = 2$, $M \geq 4$, and $P \geq 14$ are needed. For recovering the desired flavor structure of the charged fermions in the standard HVM, we must have $N = 2$.

It turns out that for the flavor structure of the standard HVM assuming normal neutrino mass ordering, we must have $N = 2$, $M \geq 4$, and $P \geq 14$. For instance, we use the $\mathcal{Z}_2 \times \mathcal{Z}_4 \times \mathcal{Z}_{14}$ symmetry, and assign charges to different fermionic and scalar fields under this symmetry, as shown in Table 1.

The Lagrangian providing masses to charged fermions is now given as

$$\begin{aligned}
\mathcal{L}_f = \frac{1}{\Lambda}[&y^u_{11}\bar{\psi}^q_{L_1}\tilde{\varphi}\psi^u_{R_1}\chi_1 + y^u_{13}\bar{\psi}^q_{L_1}\tilde{\varphi}\psi^u_{R_3}\chi_2^\dagger + y^u_{22}\bar{\psi}^q_{L_2}\tilde{\varphi}\psi^u_{R_2}\chi_2^\dagger + y^u_{23}\bar{\psi}^q_{L_2}\tilde{\varphi}\psi^u_{R_3}\chi_5 \\
&+ y^u_{32}\bar{\psi}^q_{L_3}\tilde{\varphi}\psi^u_{R_2}\chi_6 + y^u_{33}\bar{\psi}^q_{L_3}\tilde{\varphi}\psi^u_{R_3}\chi_3^\dagger + y^d_{11}\bar{\psi}^q_{L_1}\varphi\psi^d_{R_1}\chi_4^\dagger + y^d_{12}\bar{\psi}^q_{L_1}\varphi\psi^d_{R_2}\chi_4^\dagger \\
&+ y^d_{13}\bar{\psi}^q_{L_1}\varphi\psi^d_{R_3}\chi_4^\dagger + y^d_{21}\bar{\psi}^q_{L_2}\varphi\psi^d_{R_1}\chi_5^\dagger + y^d_{22}\bar{\psi}^q_{L_2}\varphi\psi^d_{R_2}\chi_5^\dagger + y^d_{23}\bar{\psi}^q_{L_2}\varphi\psi^d_{R_3}\chi_5^\dagger \\
&+ y^d_{31}\bar{\psi}^q_{L_3}\varphi\psi^d_{R_1}\chi_6^\dagger + y^d_{32}\bar{\psi}^q_{L_3}\varphi\psi^d_{R_2}\chi_6^\dagger + y^d_{33}\bar{\psi}^q_{L_3}\varphi\psi^d_{R_3}\chi_6^\dagger + y^\ell_{11}\bar{\psi}^\ell_{L_1}\varphi\psi^\ell_{R_1}\chi_1 \\
&+ y^\ell_{12}\bar{\psi}^\ell_{L_1}\varphi\psi^\ell_{R_2}\chi_4 + y^\ell_{13}\bar{\psi}^\ell_{L_1}\varphi\psi^\ell_{R_3}\chi_5 + y^\ell_{22}\bar{\psi}^\ell_{L_2}\varphi\psi^\ell_{R_2}\chi_5 + y^\ell_{23}\bar{\psi}^\ell_{L_2}\varphi\psi^\ell_{R_3}\chi_2^\dagger \\
&+ y^\ell_{33}\bar{\psi}^\ell_{L_3}\varphi\psi^\ell_{R_3}\chi_2 + \text{H.c.}].
\end{aligned} \tag{6}$$

The fermionic mass spectrum can be explained in terms of the VEVs pattern $\langle\chi_4\rangle > \langle\chi_1\rangle$, $\langle\chi_2\rangle >> \langle\chi_5\rangle$, $\langle\chi_3\rangle >> \langle\chi_6\rangle$, $\langle\chi_3\rangle >> \langle\chi_2\rangle >> \langle\chi_1\rangle$, and $\langle\chi_6\rangle >> \langle\chi_5\rangle >> \langle\chi_4\rangle$. The mass matrices of up, down quarks and leptons turn out to be

$$\begin{aligned}
&\mathcal{M}_{\mathcal{U}} = \frac{v}{\sqrt{2}}\begin{pmatrix} y^u_{11}\epsilon_1 & 0 & y^u_{13}\epsilon_2 \\ 0 & y^u_{22}\epsilon_2 & y^u_{23}\epsilon_5 \\ 0 & y^u_{32}\epsilon_6 & y^u_{33}\epsilon_3 \end{pmatrix}, \quad \mathcal{M}_{\mathcal{D}} = \frac{v}{\sqrt{2}}\begin{pmatrix} y^d_{11}\epsilon_4 & y^d_{12}\epsilon_4 & y^d_{13}\epsilon_4 \\ y^d_{21}\epsilon_5 & y^d_{22}\epsilon_5 & y^d_{23}\epsilon_5 \\ y^d_{31}\epsilon_6 & y^d_{32}\epsilon_6 & y^d_{33}\epsilon_6 \end{pmatrix}, \\
&\mathcal{M}_\ell = \frac{v}{\sqrt{2}}\begin{pmatrix} y^\ell_{11}\epsilon_1 & y^\ell_{12}\epsilon_4 & y^\ell_{13}\epsilon_5 \\ 0 & y^\ell_{22}\epsilon_5 & y^\ell_{23}\epsilon_2 \\ 0 & 0 & y^\ell_{33}\epsilon_2 \end{pmatrix},
\end{aligned} \tag{7}$$

where $\epsilon_i = \frac{\langle\chi_i\rangle}{\Lambda}$ and $\epsilon_i < 1$.

The masses of charged fermions are

$$\begin{aligned} & m_t = |y_{33}^u|\epsilon_3 v/\sqrt{2}, \quad m_c = \left| y_{22}^u\epsilon_2 - \frac{y_{23}^u y_{32}^u \epsilon_5\epsilon_6}{y_{33}^u\epsilon_3}\right| v/\sqrt{2}, \quad m_u = |y_{11}^u|\epsilon_1 v/\sqrt{2}, \\ & m_b \approx |y_{33}^d|\epsilon_6 v/\sqrt{2}, \quad m_s \approx \left| y_{22}^d - \frac{y_{23}^d y_{32}^d}{y_{33}^d}\right|\epsilon_5 v/\sqrt{2}, \\ & m_d \approx \left| y_{11}^d - \frac{y_{12}^d y_{21}^d}{y_{22}^d - \frac{y_{23}^d y_{32}^d}{y_{33}^d}} - \frac{y_{13}^d(y_{31}^d y_{22}^d - y_{21}^d y_{32}^d) - y_{31}^d y_{12}^d y_{23}^d}{\left(y_{22}^d - \frac{y_{23}^d y_{32}^d}{y_{33}^d}\right) y_{33}^d}\right| \epsilon_4 v/\sqrt{2}, \\ & m_\tau \approx |y_{33}^\ell|\epsilon_2 v/\sqrt{2}, \quad m_\mu \approx \ |y_{22}^\ell|\epsilon_5 v/\sqrt{2},\ m_e = |y_{11}^\ell|\epsilon_1 v/\sqrt{2}. \end{aligned} \tag{8}$$

The quark mixing angles are[64]

$$\sin\theta_{12} \simeq \left|\frac{y_{12}^d\epsilon_4}{y_{22}^d\epsilon_5}\right| = \frac{\epsilon_4}{\epsilon_5}, \quad \sin\theta_{23} \simeq \left|\frac{y_{23}^d\epsilon_5}{y_{33}^d\epsilon_6} - \frac{y_{23}^u\epsilon_5}{y_{33}^u\epsilon_3}\right|, \quad \sin\theta_{13} \simeq \left|\frac{y_{13}^d\epsilon_4}{y_{33}^d\epsilon_6} - \frac{y_{13}^u\epsilon_2}{y_{33}^u\epsilon_3}\right|. \tag{9}$$

We assume all $|y_{ij}^d| = 1$, and write the quark mixing angles as

$$\begin{aligned} \sin\theta_{12} &\simeq \frac{\epsilon_4}{\epsilon_5}, \\ \sin\theta_{23} &\simeq \frac{\epsilon_5}{\epsilon_6}, \\ \sin\theta_{13} &\simeq \frac{\epsilon_4}{\epsilon_6} - \left|\frac{y_{13}^u}{y_{33}^u}\right|\frac{\epsilon_2}{\epsilon_3}. \end{aligned} \tag{10}$$

The above result leads to the prediction of the quark mixing angle θ_{13} given by

$$\sin\theta_{13} \simeq \sin\theta_{12}\sin\theta_{23} - \frac{\epsilon_2}{\epsilon_3}, \tag{11}$$

for $|y_{ij}^u| = 1$. The standard HVM is capable of predicting the quark mixing angle $\sin\theta_{13}$ for $\epsilon_3 = 0.55$, which is an allowed value of this parameter.

In general, the values of the ϵ_i for reproducing masses and mixing parameters are

$$\begin{aligned} & \epsilon_1 = 3.16\times 10^{-6}, \quad \epsilon_2 = 0.0031, \quad \epsilon_3 = 0.87, \quad \epsilon_4 = 0.000061, \\ & \epsilon_5 = 0.000270, \quad \epsilon_6 = 0.0054, \quad \epsilon_7 = 7.18\times 10^{-10}. \end{aligned} \tag{12}$$

3. Neutrino Masses and Mixing

Now we address the issue of neutrino masses and mixing in the SHVM. The neutrino masses are recovered by introducing the three right-handed neutrinos $\nu_{eR}, \nu_{\mu R}, \nu_{\tau R}$ to the SM.

3.1. *Normal mass ordering*

We write the following Lagrangian to produce the neutrino masses:

$$-\mathcal{L}_{\text{Yukawa}}^{\nu} = y_{ij}^{\nu}\bar{\psi}_{L_i}^{\ell}\tilde{\varphi}\nu_{f_R}\left[\frac{\chi_i\chi_j}{\Lambda^2}\right] + \text{H.c.} \tag{13}$$

The requirement that the neutrino masses originates from Eq. (13) constrains the symmetry $\mathcal{Z}_P$ allowing only $P = 14$. Thus, $P = 14$ is a magic number whose origin may lie in the UV theory. The Lagrangian providing masses to neutrinos reads

$$\mathcal{L}_f = \frac{1}{\Lambda^2}[y_{11}^\nu \bar{\psi}_{L_1}^\ell \tilde{\varphi} \psi_{R_1}^\nu \chi_1^\dagger \chi_7^\dagger + y_{12}^\nu \bar{\psi}_{L_1}^\ell \tilde{\varphi} \psi_{R_2}^\nu \chi_4^\dagger \chi_7 + y_{13}^\nu \bar{\psi}_{L_1}^\ell \tilde{\varphi} \psi_{R_3}^\nu \chi_4^\dagger \chi_7 + y_{22}^\nu \bar{\psi}_{L_2}^\ell \tilde{\varphi} \psi_{R_2}^\nu \chi_4 \chi_7 + y_{23}^\nu \bar{\psi}_{L_2}^\ell \tilde{\varphi} \psi_{R_3}^\nu \chi_4 \chi_7 + y_{32}^\nu \bar{\psi}_{L_3}^\ell \tilde{\varphi} \psi_{R_2}^\nu \chi_5 \chi_7 + y_{33}^\nu \bar{\psi}_{L_3}^\ell \tilde{\varphi} \psi_{R_3}^\nu \chi_5 \chi_7 + \text{H.c.}]. \tag{14}$$

The Dirac mass matrix for neutrinos reads

$$\mathcal{M}_{\mathcal{D}} = \frac{v}{\sqrt{2}} \begin{pmatrix} y_{11}^\nu \epsilon_1 \epsilon_7 & y_{12}^\nu \epsilon_4 \epsilon_7 & y_{13}^\nu \epsilon_4 \epsilon_7 \\ 0 & y_{22}^\nu \epsilon_4 \epsilon_7 & y_{23}^\nu \epsilon_4 \epsilon_7 \\ 0 & y_{32}^\nu \epsilon_5 \epsilon_7 & y_{33}^\nu \epsilon_5 \epsilon_7 \end{pmatrix} \tag{15}$$

and neutrino masses can be written as

$$m_3 \approx |y_{33}^\nu| \epsilon_5 \epsilon_7 v/\sqrt{2}, \quad m_2 \approx \left| y_{22}^\nu - \frac{y_{23}^\nu y_{32}^\nu}{y_{33}^\nu} \right| \epsilon_4 \epsilon_7 v/\sqrt{2}, \quad m_1 \approx |y_{11}^\nu| \epsilon_1 \epsilon_7 v/\sqrt{2}. \tag{16}$$

The masses of neutrinos turn out to be $\{m_3, m_2, m_1\} = \{5.05 \times 10^{-2}, 8.67 \times 10^{-3}, 2.67 \times 10^{-4}\}$ eV for the values of y_{ij}^ν given in Appendix A.

The remarkable predictions are the neutrino mixing angles,[64]

$$\begin{aligned} \sin\theta_{12}^\ell &\simeq \left| \frac{y_{12}^\ell \epsilon_4}{y_{22}^\ell \epsilon_5} - \frac{y_{12}^\nu}{y_{22}^\nu} + \frac{y_{23}^{\ell *} y_{13}^\nu \epsilon_4}{y_{33}^\ell y_{33}^\nu \epsilon_5} \right|, \quad \sin\theta_{23}^\ell \simeq \left| \frac{y_{23}^\ell}{y_{33}^\ell} - \frac{y_{23}^\nu \epsilon_4}{y_{33}^\nu \epsilon_5} \right|, \\ \sin\theta_{13}^\ell &\simeq \left| \frac{y_{13}^\ell \epsilon_5}{y_{33}^\ell \epsilon_2} - \frac{y_{13}^\nu \epsilon_4}{y_{33}^\nu \epsilon_5} \right|. \end{aligned} \tag{17}$$

Assuming all the couplings of the order one, we further predict

$$\begin{aligned} \sin\theta_{12}^\ell &\simeq \left| -\frac{y_{12}^\nu}{y_{22}^\nu} + \frac{y_{12}^\ell \epsilon_4}{y_{22}^\ell \epsilon_5} + \frac{y_{23}^{\ell *} y_{13}^\nu \epsilon_4}{y_{33}^\ell y_{33}^\nu \epsilon_5} \right| \geq \left| -\frac{y_{12}^\nu}{y_{22}^\nu} \right| - \left| \frac{y_{12}^\ell}{y_{22}^\ell} + \frac{y_{23}^{\ell *} y_{13}^\nu}{y_{33}^\ell y_{33}^\nu} \right| \frac{\epsilon_4}{\epsilon_5} \approx 1 - 2\sin\theta_{12}, \\ \sin\theta_{23}^\ell &\simeq \left| \frac{y_{23}^\ell}{y_{33}^\ell} - \frac{y_{23}^\nu \epsilon_4}{y_{33}^\nu \epsilon_5} \right| \geq \left| \frac{y_{23}^\ell}{y_{33}^\ell} \right| - \left| \frac{y_{23}^\nu}{y_{33}^\nu} \right| \frac{\epsilon_4}{\epsilon_5} \approx 1 - \sin\theta_{12}, \\ \sin\theta_{13}^\ell &\simeq \left| -\frac{y_{13}^\nu \epsilon_4}{y_{33}^\nu \epsilon_5} + \frac{y_{13}^\ell \epsilon_5}{y_{33}^\ell \epsilon_2} \right| \geq \left| -\frac{y_{13}^\nu}{y_{33}^\nu} \right| \frac{\epsilon_4}{\epsilon_5} - \left| \frac{y_{13}^\ell}{y_{33}^\ell} \right| \frac{\epsilon_5}{\epsilon_2} \approx \sin\theta_{12} - \frac{m_s}{m_c}, \end{aligned} \tag{18}$$

where $m_s/m_c = \epsilon_5/\epsilon_2$. Thus, we conclude that the leptonic mixing angles are predicted by standard HVM in terms of the Cabibbo angle and masses of strange and charm quarks.

We can now predict the leptonic mixing angles precisely using the measurement of the Cabibbo angle given in Eq. (2). For the first two mixing angles, we predict $\sin\theta_{12}^\ell = 0.55 \pm 0.00134$ and $\sin\theta_{23}^\ell = 0.775 \pm 0.00067$. The theoretical precision of the $\sin\theta_{12}^\ell$ directly depends on the experimental precision of the Cabibbo angle, and the theoretical precision of the $\sin\theta_{23}^\ell$ is identical to that of the Cabibbo angle.

The mixing angle $\sin\theta^{\ell}_{13}$ can be predicted using the values of running masses of strange and charm quarks given at 1 TeV as $m_c = 0.532^{+0.074}_{-0.073}$ GeV and $m_s = 4.7^{+1.4}_{-1.3} \times 10^{-2}$ GeV.[65] Using these values, we predict $\sin\theta^{\ell}_{13}$ to be $(0.1169 - 0.1509)$. However, we observe that the lower end of the experimental measurement eliminates the range $(0.1169 - 0.1413)$ of the theoretical prediction. On the other side, the upper end of the theoretical prediction rules out the range $(0.1509 - 0.1550)$ of the experimental observation. This leads to the final conclusive range of the angle $\sin\theta^{\ell}_{13} = 0.1413 - 0.1509$.

3.2. *Inverted mass ordering*

We note that neutrino masses may be inverted in order if the sign of the Δm^2_{31} turns out to be negative. This scenario can be realized in the standard HVM by assigning the charges to the right-handed neutrinos under the $\mathcal{Z}_2 \times \mathcal{Z}_4 \times \mathcal{Z}_{14}$ symmetry as: $\nu_{e_R} : (-, \omega, \omega^3)$, $\nu_{\mu_R} : (-, \omega, \omega)$, $\nu_{\tau_R} : (+, \omega, \omega^{10})$. The mass matrix of the Dirac neutrinos becomes

$$\mathcal{M}_{\text{Dirac}} = \frac{v}{\sqrt{2}} \begin{pmatrix} y^{\nu}_{11}\epsilon_4\epsilon_7 & y^{\nu}_{12}\epsilon_5\epsilon_7 & 0 \\ y^{\nu}_{21}\epsilon_4\epsilon_7 & y^{\nu}_{22}\epsilon_5\epsilon_7 & 0 \\ y^{\nu}_{31}\epsilon_2\epsilon_7 & y^{\nu}_{32}\epsilon_4\epsilon_7 & y^{\nu}_{33}\epsilon_1\epsilon_7. \end{pmatrix}. \tag{19}$$

The masses of neutrinos are

$$m_3 \approx |y^{\nu}_{33}|\epsilon_1\epsilon_7 v/\sqrt{2}, \quad m_2 \approx |y^{\nu}_{22}|\epsilon_5\epsilon_7 v/\sqrt{2}, \quad m_1 \approx \left| y^{\nu}_{11} - \frac{y^{\nu}_{12}y^{\nu}_{21}}{y^{\nu}_{33}} \right| \epsilon_4\epsilon_7 v/\sqrt{2}. \tag{20}$$

The neutrino mixing angles now become

$$\sin\theta^{\ell}_{12} \simeq \left| \frac{y^{\ell}_{12}\epsilon_4}{y^{\ell}_{22}\epsilon_5} - \frac{y^{\nu}_{12}}{y^{\nu}_{22}} \right|, \quad \sin\theta^{\ell}_{23} \simeq \left| \frac{y^{\ell}_{23}}{y^{\ell}_{33}} \right|, \quad \sin\theta^{\ell}_{13} \simeq \left| \frac{y^{\ell}_{13}\epsilon_2}{y^{\ell}_{33}\epsilon_6} \right|. \tag{21}$$

In this case, our main predictions for the neutrino mixing angle are

$$\sin\theta^{\ell}_{12} \geq 1 - \sin\theta^{q}_{12}, \quad \sin\theta^{\ell}_{13} \geq \frac{m_c}{m_b}, \tag{22}$$

where $\epsilon_7 \approx 10^{-10}$, $m_c/m_b = \epsilon_2/\epsilon_6$.

We predict the range $\sin\theta^{\ell}_{13} \leq 0.18519 - 0.25958$ using the mass of b-quark at 1 TeV $m_b = 2.43 \pm 0.08$ GeV.[65] The current global fit provides $\sin\theta^{\ell}_{13} = 0.14206 - 0.15569$. We observe that the equality sign of this prediction is ruled out by the present data. Thus, we conclude that the inverted mass hierarchy of neutrinos is disfavored in the SHVM.

3.3. *Majorana neutrinos*

There is a possibility of the Majorana masses for neutrinos in this paper. We can write the following Weinberg-type operators for the Majorana mass:

$$-\mathcal{L}^{\ell}_{\text{Weinberg}} = h^{\nu}_{ij} \frac{\bar{\tilde{\psi}}^{\ell}_{L_i}\varphi\tilde{\varphi}^{\dagger}\psi^{\ell}_{L_j}}{\Lambda} + h^{\nu\prime}_{ij} \frac{\bar{\tilde{\psi}}^{\ell}_{L_i}\varphi\tilde{\varphi}^{\dagger}\psi^{\ell}_{L_j}}{\Lambda} \left[\frac{\chi_i(\text{or } \chi^{\dagger}_i)}{\Lambda} \right] + \text{H.c.}, \tag{23}$$

where $\tilde{\psi}^{\ell}_{L_i} = i\sigma_2\psi^c_{L_i}$. It turns out that these operators are not allowed by the charge assignments given in Table 1. We may also have the right-handed neutrino mass operators $\mathcal{L}_{M_R}$ given by

$$\mathcal{L}_{\mathrm{M\dot{R}}} = c_{ij}\chi_i\bar{\nu^c_{f_R}}\nu_{f_R} + c'_{ij}\chi_i\bar{\nu^c_{f_R}}\nu_{f_R}\left[\frac{\chi_i(\text{or } \chi_i^\dagger)}{\Lambda}\right] + c''_{ij}\chi_i\bar{\nu^c_{f_R}}\nu_{f_R}\left[\frac{\chi_i\chi_j(\text{or } \chi_i\chi_j^\dagger)}{\Lambda^2}\right] + \text{H.c.} \tag{24}$$

However, they are forbidden in the standard HVM. Therefore, neutrinos in the standard HVM, are Dirac type.

4. Scalar Potential of the SHVM

The scalar potential is written as

$$\begin{aligned} V = {} & \mu^2\varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi)^2 + \mu^2_{\chi_1}|\chi_1|^2 + \mu^2_{\chi_2}|\chi_2|^2 + \mu^2_{\chi_3}|\chi_3|^2 + \mu^2_{\chi_4}|\chi_4|^2 + \mu^2_{\chi_5}|\chi_5|^2 \\ & + \mu^2_{\chi_6}|\chi_6|^2 + \mu^2_{\chi_7}|\chi_7|^2 + \lambda_{\chi_1}|\chi_1|^4 + \lambda_{\chi_2}|\chi_2|^4 + \lambda_{\chi_3}|\chi_3|^4 + \lambda_{\chi_4}|\chi_4|^4 \\ & + \lambda_{\chi_5}|\chi_5|^4 + \lambda_{\chi_6}|\chi_6|^4 + \lambda_{\chi_7}|\chi_7|^4 + \lambda_{\varphi\chi}\varphi^\dagger\varphi\chi_i^\dagger\chi_j + \lambda_{\chi_{12}}|\chi_1|^2|\chi_2|^2 \\ & + \lambda_{\chi_{13}}|\chi_1|^2|\chi_3|^2 + \lambda_{\chi_{14}}|\chi_1|^2|\chi_4|^2 + \lambda_{\chi_{15}}|\chi_1|^2|\chi_5|^2 + \lambda_{\chi_{16}}|\chi_1|^2|\chi_6|^2 \\ & + \lambda_{\chi_{17}}|\chi_1|^2|\chi_7|^2 + \lambda_{\chi_{23}}|\chi_2|^2|\chi_3|^2 + \lambda_{\chi_{24}}|\chi_2|^2|\chi_4|^2 + \lambda_{\chi_{25}}|\chi_2|^2|\chi_5|^2 \\ & + \lambda_{\chi_{26}}|\chi_2|^2|\chi_6|^2 + \lambda_{\chi_{27}}|\chi_2|^2|\chi_7|^2 + \lambda_{\chi_{34}}|\chi_2|^3|\chi_3|^4 + \lambda_{\chi_{35}}|\chi_2|^3|\chi_5|^2 \\ & + \lambda_{\chi_{36}}|\chi_3|^2|\chi_6|^2 + \lambda_{\chi_{37}}|\chi_3|^2|\chi_7|^2 + \lambda_{\chi_{45}}|\chi_4|^2|\chi_5|^2 + \lambda_{\chi_{46}}|\chi_4|^2|\chi_6|^2 \\ & + \lambda_{\chi_{47}}|\chi_4|^2|\chi_7|^2 + \lambda_{\chi_{56}}|\chi_5|^2|\chi_6|^2 + \lambda_{\chi_{57}}|\chi_5|^2|\chi_7|^2 + \lambda_{\chi_{67}}|\chi_6|^2|\chi_7|^2 \\ & + \lambda_{1122}\chi_1^2\chi_2^2 + \lambda_{1133}\chi_1^2\chi_3^{\dagger 2} + \lambda_{2233}\chi_2^2\chi_3^2 + \lambda_{2224}\chi_2^3\chi_4 + \lambda_{4436}\chi_4^2\chi_3^\dagger\chi_6^\dagger \\ & + \lambda_{2555}\chi_2\chi_5^{\dagger 3} + \sigma_{16}\chi_1^2\chi_6 + \sigma_{26}\chi_2^2\chi_6^\dagger + \sigma_{36}\chi_3^2\chi_6 + \sigma_{43}\chi_4^2\chi_3 + \sigma_{73}\chi_7^2\chi_3^\dagger \\ & + \sigma_{345}\chi_3\chi_4^\dagger\chi_5 + \sigma_{235}\chi_2\chi_3^\dagger\chi_5 + \sigma_{456}\chi_4\chi_5\chi_6^\dagger + \sigma_{246}\chi_2\chi_4\chi_6 + \text{H.c.} \end{aligned} \tag{25}$$

We can parametrize the scalar fields as

$$\chi_i(x) = \frac{v_i + s_i(x) + ia_i(x)}{\sqrt{2}}, \quad \varphi = \begin{pmatrix} G^+ \\ \dfrac{v + h + iG^0}{\sqrt{2}} \end{pmatrix}. \tag{26}$$

4.1. *Emergence of a multiple ALPs scenario*

The pseudoscalars a_i can achieve their masses through the potential,[66]

$$V_\chi = -\lambda'_i\frac{\chi_i^{\tilde{\mathrm{N}}}}{\Lambda^{\tilde{\mathrm{N}}-4}} + \text{H.c.}, \tag{27}$$

Table 2. The masses of pseudoscalars a_i for $|\lambda'| = 1$ and $\tilde{N} = 14$.

Pseudoscalars	a_1	a_2	a_3	a_4	a_5	a_6	a_7
Mass (GeV) at $v_i = 10^2$ GeV	1.6×10^{-25}	1.4×10^{-10}	246.71	4.18×10^{-19}	7.10×10^{-16}	2.27×10^{-9}	9.45×10^{-44}
Mass (GeV) at $v_i = 10^{11}$ GeV	1.6×10^{-16}	0.14	2.47×10^{11}	4.18×10^{-10}	7.10×10^{-7}	2.27	9.45×10^{-35}
Mass (GeV) at $v_i = 10^{19}$ GeV	1.6×10^{-8}	1.42×10^{7}	2.47×10^{19}	0.04	71.02	2.27×10^{8}	9.45×10^{-27}

where $\tilde{N}$ is the least common multiple of the N, M and P in the $\mathcal{Z}_N \times \mathcal{Z}_M \times \mathcal{Z}_P$ flavor symmetry. The masses of pseudo-scalar particles are now given by

$$m_{a_i}^2 = \frac{1}{8}|\lambda_i'|\ \tilde{N}^{\ 2}\epsilon_i^{\tilde{N}\ -4}v_i^2. \tag{28}$$

From these, we can calculate the masses of the pseudoscalars a_i. For instance, for $|\lambda'| = 1$ and $\tilde{N} = 14$, the masses are given for three different values of v_i in Table 2. $\tilde{N} = 14$ corresponds to the symmetry $\mathcal{Z}_2 \times \mathcal{Z}_M \times \mathcal{Z}_{14}$ where $M = 7$ or 14. Thus, we see that except the pseudoscalar a_3, all the other pseudoscalars are very light below the scale 10^{11} GeV, and are acting like ALPS. The pseudoscalar a_6 becomes heavier at the scale above 10^{11} GeV.

We notice that the pseudoscalars $a_{4,5,6}$ contribute to the neutral meson mixing (K, B_d, B_s) parameters above the scale of the mixing processes. However, the contribution of the pseudoscalars $a_{4,5,6}$ to the neutral meson mixing (K, B_d, B_s) parameters is proportional to $1/v_i^2$. Therefore, this contribution is hugely suppressed, and does not enhance the SM contribution.[67] Thus, there are no observable effects of the particles $a_{4,5,6}$ in the quark and leptonic flavor physics. However, the pseudoscalar a_3 may reveal itself through the top-quark interactions such as the decays $a_3 \to \gamma\gamma, t\bar{t}$ providing a smoking gun signature of the SHVM.[67]

4.2. *Masses of scalars*

We assume $\lambda_{\varphi\chi} = 0$ and $\lambda_{\chi_i} = \lambda_\chi$ for simplicity, and diagonalize the mass matrix square of the scalar particles numerically by writing $v_i = \sqrt{2}\epsilon_i\Lambda$, where ϵ_i can be found in Eq. (12). Thus, we have

$$U^T\mathcal{M}_s^2U = \mathcal{M}_{dia}^2, \tag{29}$$

where $\mathcal{M}_s^2$ denotes the mass matrix square of the scalars, $\mathcal{M}_{dia}$ shows the diagonalized scalar mass matrix, and U represents an orthogonal matrix. These matrices can be found in Appendix A.

The physical scalar states can be defined by the following transformation:

$$\begin{pmatrix} s_7 \\ s_1 \\ s_4 \\ s_5 \\ s_2 \\ s_6 \\ s_3 \end{pmatrix} = U^T \begin{pmatrix} h_7 \\ h_1 \\ h_4 \\ h_5 \\ h_2 \\ h_6 \\ h_3 \end{pmatrix}. \tag{30}$$

The masses of scalars turn out to be

$$\begin{aligned} &m^2_{h_7} = 4.72 \times 10^{-18}\lambda_\chi\Lambda^2, \quad m^2_{h_1} = 9.33 \times 10^{-11}\lambda_\chi\Lambda^2, \quad m^2_{h_4} = 3.57 \times 10^{-8}\lambda_\chi\Lambda^2, \\ &m^2_{h_5} = 7.3 \times 10^{-7}\lambda_\chi\Lambda^2, \quad m^2_{h_2} = 9.45 \times 10^{-5}\lambda_\chi\Lambda^2, \\ &m^2_{h_6} = 3.67 \times 10^{-4}\lambda_\chi\Lambda^2, \quad m^2_{h_3} = 12\lambda_\chi\Lambda^2. \end{aligned} \tag{31}$$

5. A Possible Dark Matter Candidate

The SHVM framework may provide a new class of dark matter candidate named as "neutrinic dark matter". This can be found by observing that if the ALPS a_i are dark matter candidates, the most powerful bounds on the scale Λ come from the FCNC process $K^+ \to \pi^+ a_{4,5}$. The FCNC process $K^+ \to \pi^+ a_{4,5}$ implies that $v_{4,5} \gtrsim 7 \times 10^{11} V^d_{21}$GeV where $V^d_{21} \approx 0.225$.[68] This corresponds to $\Lambda = v_i/\sqrt{2}\epsilon_i \approx 10^{17}$ GeV. We do not consider this scenario since it will lead to the extreme suppression of the all quark and leptonic FCNC processes of the SHVM. Thus, there will be no signature of the SHVM in flavor and collider physics. Hence, we do not consider the pseudoscalar $a_i (i = 1-6)$ to be ALPs and dark matter candidates. This can be achieved by making the pseudoscalars $a_i (i = 1-6)$ very heavy through the following soft symmetry breaking potential:

$$V^{a_i}_{\text{soft}} = \rho_i(\chi_i^2 + \chi_i^{*2}), \tag{32}$$

where $i = 1-6$. This term can be generated by the strong dynamics of the dark-technicolor, whose the fields χ_i $(i = 1-6)$ are the bound states, discussed in Appendix A. The mass of the pseudoscalars a_i now reads

$$m_{a_i} = \sqrt{\rho_i}, \tag{33}$$

where $i = 1-6$. Now the masses m_{a_i} are free parameters, they can be anywhere between the eletroweak and the Planck scale.

The only particle which maybe a dark matter candidate is the pseudoscalar a_7. This is a bound state of the dark-QCD in the dark technicolor model as discussed in Appendix A. The interaction between the dark technicolor whose bound states are χ_i $(i = 1-6)$ and the dark-QCD are extremely suppressed in the UV completion. Therefore, we assume that there is zero interaction, effectively, between the scalar fields χ_i and χ_7. For the VEV $v_7 = 1$ GeV, the scale Λ should be of the order 10^9 GeV. The mass of the ALP a_7 from Eq. (28) is $\approx 10^{-38}$ eV for $\lambda_7' = 1$ and $\lambda = 10^9$ GeV. The lower bound on the mass of cold dark matter particle is 10^{-21} eV.[70] We observe that for the scale $\Lambda = 10^{19}$ GeV, the mass of the ALP a_7 is of the order 10^{-26} eV. Thus, the mass of the ALP a_7 is far below the lower bound of the dark matter. Therefore, the ALP a_7 cannot be a dark matter candidate for the mass given by Eq. (28).

However, the ALP a_7 may be a novel class of dark matter if we introduce its mass through the soft symmetry breaking potential given by

$$V_{\text{soft}}^{a_7} = \rho_7(\chi_7^2 + \chi_7^{*2}). \tag{34}$$

The mass of the ALP a_7 is now given as

$$m_{a_7} = \sqrt{\rho_7}, \tag{35}$$

which is a free parameter. The only decay channel of the ALP a_7 is to a pair of neutrinos. The decay rate is

$$\Gamma(a_7 \to \nu\nu) = \frac{1}{8\pi} g_{a_7\nu\nu}^2 m_{a_7}\beta_\nu, \tag{36}$$

where $\beta_f = (1 - 4m_\nu^2/m_{a_7}^2)^{1/2}$ and $g_{a_7\nu\nu} = \frac{v}{v_7\sqrt{2}}\epsilon_5\epsilon_7 = \frac{v}{2\Lambda}\epsilon_5$. For $m_{a_7} = 10^{-8}$ GeV, the $\Gamma(a_7 \to \nu\nu) = 8.8 \times 10^{-45}$ GeV where the scale Λ is 10^{16} GeV. However, the scale Λ can be as low as the eletroweak scale if $m_{a_7} < 2m_\nu$. In this scenario, a_7 may still be a possible dark-matter particle if its mass is greater than 10^{-21} eV.[70]

The possible neutrinic dark matter may reveal itself either through the Higgs-χ_7 portal (the term $\varphi^\dagger\varphi\chi_7^\dagger\chi_7$), or χ_i-χ_7 portal (the term $\chi_i^\dagger\chi_i\chi_7^\dagger\chi_7$). The crucial difference between the possible neutrinic dark matter and other ALP dark matter is that the possible neutrinic dark matter does not have coupling to electromagnetic field at leading-order. Therefore, the operator $\frac{1}{4}g_{a_7\gamma\gamma}\varphi F^{\mu\nu}\tilde{F}_{\mu\nu}$ effectively does not exist. This keeps the possible neutrinic dark matter apart from other ALPs dark matter candidates.

6. Summary

In this paper, we have derived a framework to address, in particular, the leptonic flavor mixing based on the VEVs hierarchy. We name it the standard HVM. The standard HVM predicts its unique and defining signatures in the form of inequalities $\sin\theta_{12}^\ell \geq 1 - 2\sin\theta_{12}$, $\sin\theta_{23}^\ell \geq 1 - \sin\theta_{12}$, and $\sin\theta_{13}^\ell \geq \sin\theta_{12} - \frac{m_s}{m_c}$ for the normal mass ordering. This results in very precise predictions of the three leptonic mixing angles given by $\sin\theta_{12}^\ell = 0.55 \pm 0.00134$, $\sin\theta_{23}^\ell = 0.775 \pm 0.00067$, and $\sin\theta_{13}^\ell = 0.1413 - 0.1509$. Inverted neutrino mass ordering is disfavored by the SHVM.

The next generation neutrino experiments such as DUNE, Hyper-Kamiokande, and JUNO[69] may probe these predictions. Furthermore, we have predicted the quark mixing angle $\sin\theta_{13}$ to be $\approx \sin\theta_{12}^3 - 2\frac{m_c}{m_t}$, which is perfectly in agreement with the observed value. However, for this prediction, we need to accept a reasonable assumption that $|y_{13}^u/y_{33}^u| \approx 2$. An important implication of the SHVM is a new class of possible dark matter candidate, named as neutrinic dark matter. The flavor physics and collider signatures of the SHVM are investigated in Ref. 67.

6.1. *UV completion in the dark-QCD paradigm: A minimal model*

A UV completion of the SHVM is possible in the DTC paradigm discussed in Ref. 2. In this section, we propose a new Dark-QCD (DQCD) paradigm, which provides a UV completion in a minimal framework. The DQCD paradigm is based on a minimal symmetry $\mathcal{G}_{\rm DQCD} = {\rm SU}(N_{\rm TC}) \times {\rm SU}(N_{\rm DQ})$, where D$Q$ shows the DQCD dynamics of vector-like fermions. The difference between the DTC and the DQCD paradigm is the absence of the ${\rm SU}(N_{\rm DTC})$ dynamics in the later. The multi-fermions chiral condensates, which play the role of the VEVs $\langle \chi_r \rangle$, are formed by the TC fermions $D_{L,R}$.

The transformations of TC fermions under the ${\rm SU}(3)_c \times {\rm SU}(2)_L \times U(1)_Y \times \mathcal{G}_{\rm DQCD}$ are

$$\begin{aligned} &T^i \equiv \begin{pmatrix} T \\ B \end{pmatrix}_L : (1,2,0,N_{\rm TC},1), \quad T^i_R : (1,1,1,{\rm N}_{\rm TC},1), B^i_R : (1,1,-1,N_{\rm TC},1), \\ &D_{L,R} \equiv C_{i,L,R} : (1,1,1,N_{\rm TC},1), S_{i,L,R} : (1,1,-1,N_{\rm TC},1), \end{aligned} \tag{37}$$

where $i = 1,2,3,\ldots$, and electric charges $+\frac{1}{2}$ for T and $\mathcal{C}$, and $-\frac{1}{2}$ for B and $\mathcal{S}$.

The ${\rm SU}(N_{\rm DQ})$ dynamics has vector-like fermions transforming as

$$\begin{aligned} &F_{L,R} \equiv U^i_{L,R} \equiv \left(3,1,\frac{4}{3},1,N_F\right), D^i_{L,R} \equiv \left(3,1,-\frac{2}{3},1,N_F\right), \\ &N^i_{L,R} \equiv (1,1,0,1,N_F), E^i_{L,R} \equiv (1,1,-2,1,N_F). \end{aligned} \tag{38}$$

The multi-fermion condensate is paramerized as[73]

$$\langle (\bar{\psi}_L \psi_R)^n \rangle \sim (\Lambda {\rm exp}\,(k\Delta\chi))^{3n}, \tag{39}$$

where $\Delta\chi$ is the chirality of the corresponding operator, k is a constant, and Λ denotes the scale of the gauge dynamics.

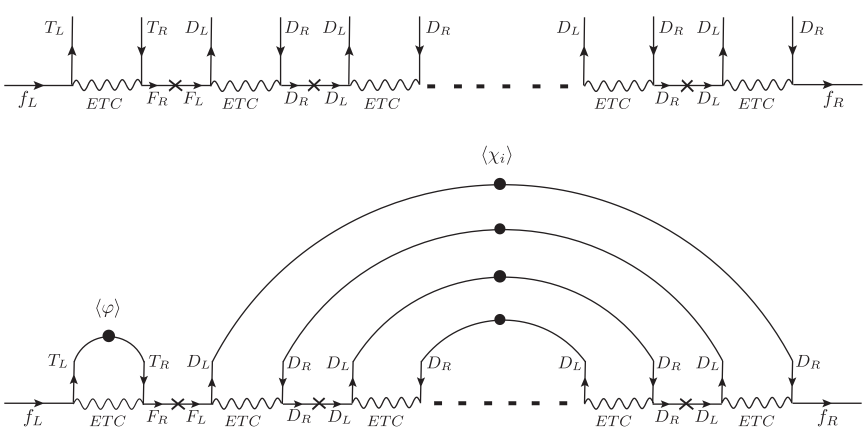

Fig. 1. The Feynman diagrams for the masses of the charged fermions in the DQCD paradigm.

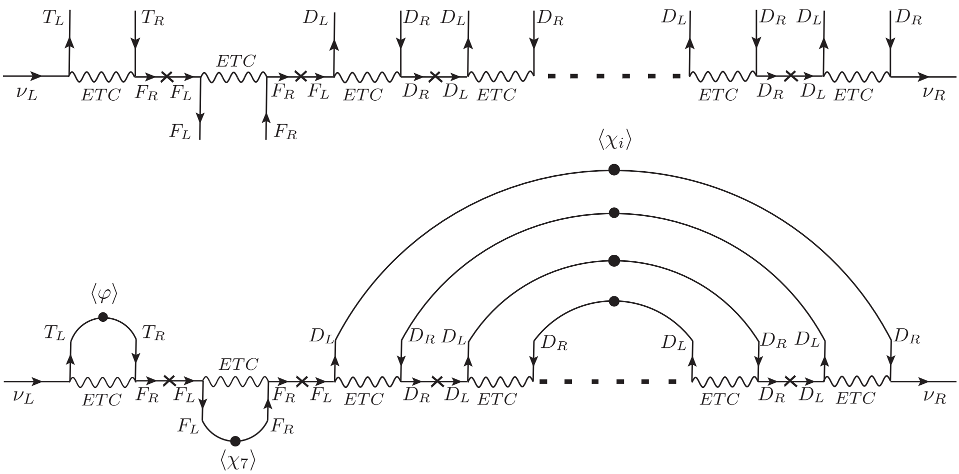

Fig. 2. The Feynman diagrams generating dimension-6 operators, responsible for neutrino masses, given in Eq. (43). On the top, the generic interactions involving the SM, TC, DTC gauge sectors mediated by ETC are shown. In the bottom, we show the formation of the fermionic condensate $\langle \bar{F}_L F_R \rangle$ which is the VEV $\langle \chi_7 \rangle$ in the SHVM.

In this model, we have three axial $U(1)_A^{\text{QCD, TC, DQCD}}$ symmetries. They are broken to a discrete cyclic group by instantons as[72]

$$U(1)_A^{\text{QCD, TC, DQCD}} \to \mathcal{Z}_{2\mathrm{K}_{\text{QCD, TC, DQCD}}}, \tag{40}$$

where $K_{\text{QCD, TC, DQCD}}$ are number of fundamental massless flavors of the QCD, TC, and DQCD gauge dynamics in the N-dimensional representation of the gauge group $\mathrm{SU}(N)_{\text{QCD, TC, DQCD}}$. This results in a generic $\mathcal{Z}_N \times \mathcal{Z}_M \times \mathcal{Z}_P$ flavor symmetry, where $N = 2K_{\text{QCD}}$, $M = 2K_{\text{TC}}$ and $P = 2K_{\text{DQCD}}$.

We assume that TC and the SM fermions can be accommodated in an Extended TC (ETC) sector. In the DQCD paradigm, the masses of fermions of SHVM originate from the generic interactions given in Fig. 1 on the top. The formation of multi-fermion chiral consdensates (denoted by blob) and resulting mass of the SM fermion are shown in Fig. 1.

We write the charged fermion masses as

$$m_f = y_f \frac{\Lambda_{\text{TC}}^3}{\Lambda_{\text{ETC}}^2} \frac{1}{\Lambda_F} \frac{\Lambda_{\text{TC}}^{n_i+1}}{\Lambda_{\text{ETC}}^{n_i}} [\exp(n_i k)]^{n_i/2}, \tag{41}$$

where $n_i = 2, 4, \ldots, 2n$ denotes the number of fermions in a multi-quark chiral condensate which play the role of the VEV $\langle \chi_i \rangle$ in the SHVM.

The form of the ϵ_i parameters appearing in the SHVM can be written as

$$\epsilon_i \propto \frac{1}{\Lambda_F} \frac{\Lambda_{\text{DTC}}^{n_i+1}}{\Lambda_{\text{EDTC}}^{n_i}} [\exp(n_i k)]^{n_i/2}. \tag{42}$$

The creation of the dimension-6 operators responsible for neutrino masses, given in Eq. (43), is shown in Fig. 2. The chiral condensate $\langle \bar{F}_L F_R \rangle$ plays the role of the VEV $\langle \chi_7 \rangle$.

The neutrinos masses are recovered as

$$m_\nu = y_f \frac{\Lambda_{\rm TC}^3}{\Lambda_{\rm ETC}^2} \frac{1}{\Lambda_{\rm F}} \frac{\Lambda_{\rm DTC}^{n_i+1}}{\Lambda_{\rm ETC}^{n_i}} \exp{(n_i k)} \frac{1}{\Lambda_F} \frac{\Lambda_{\rm F}^3}{\Lambda_{\rm ETC}^2} \exp{(2k)}, \tag{43}$$

where

$$\epsilon_7 \propto \frac{1}{\Lambda_{\rm F}} \frac{\Lambda_{\rm F}^3}{\Lambda_{\rm ETC}^2} \exp{(2k)}. \tag{44}$$

Appendix A.

A.1. *Benchmark couplings*

For reproducing the fermion masses and mixing from the standard HVM, the values of the fermion masses at 1 TeV, given in Ref. 65, are used. The quark mixing parameters are taken from Ref. 46. The neutrino oscillation data for the normal hierarchy are taken from the global fit results given in Ref. 45. The coefficients $y_{ij}^{u,d,\ell,\nu} = |y_{ij}^{u,d,\ell,\nu}| e^{i\phi_{ij}^{q,\ell,\nu}}$ are scanned in the ranges $|y_{ij}^{u,d,\ell,\nu}| \in [0.9, 3]$ and $\phi_{ij}^{q,\ell,\nu} \in [0, 2\pi]$. The fitting results are

$$y_{ij}^u = \begin{pmatrix} 1.0 + 1.73i & 0 & 0.58 + 1.95i \\ 0 & -0.89 + 0.45i & -0.98 - 0.14i \\ 0 & -0.32 + 1.60i & 1 - 0.02i \end{pmatrix},$$

$$y_{ij}^d = \begin{pmatrix} -1.77 + 2.19i & -1.30 + 2.29i & 1.08 - 2.16i \\ 0.76 - 0.95i & 1.12 - 2.36i & 2.09 + 0.24i \\ 0.31 + 1.98i & -2.25 + 0.16i & -1.65 - 2.04i \end{pmatrix}.$$

The Dirac CP phase in the standard parametrization is $\delta_{\rm CP}^q \approx 1.144$.

$$y_{ij}^\ell = \begin{pmatrix} 1.64 - 0.20i & 0.52 - 0.89i & 0.36 - 1.04i \\ 0 & 0.07 + 1.01i & 0.42 + 1.02i \\ 0 & 0 & 1.09 \end{pmatrix}.$$

For normal mass ordering the neutrino couplings are

$$y_{ij}^\nu = \begin{pmatrix} 0.6 - 0.88i & 0.87 - 0.41i & 1.5 - 0.00004i \\ 0 & 0.96 + 0.12i & -0.52 - 1.41i \\ 0 & -1.43 + 0.28i & 1.5 + 0.00004i \end{pmatrix}$$

and the leptonic Dirac CP phase is $\delta_{\rm CP}^\ell \approx 3.14$.

A.2. *Scalar mass matrix diagonalization*

For simplification, we impose an extra $\mathcal{Z}_2'$ symmetry on the right-handed fermions such that right-handed fermions transform as $\psi_R : -$, the singlet scalar fields χ as $\chi_i : -$ where $i = 1 - 6$, and the field χ_7 as $\chi_7 : +$ under this new $\mathcal{Z}_2'$ symmetry. This will forbid all cubic terms in the scalar potential without affecting the flavor structure of the model.

$$
m_s^2 = \lambda_\chi \Lambda^2 \begin{pmatrix}
16\epsilon_7^2 & 8\epsilon_1\epsilon_7 & 8\epsilon_4\epsilon_7 & 8\epsilon_5\epsilon_7 & 8\epsilon_2\epsilon_7 & 8\epsilon_6\epsilon_7 & 8\epsilon_3\epsilon_7 \\
8\epsilon_1\epsilon_7 & 16\epsilon_1^2 & 8\epsilon_1\epsilon_4 & 8\epsilon_1\epsilon_5 & 8\epsilon_1\epsilon_2 & 8\epsilon_1\epsilon_6 & 8\epsilon_1\epsilon_3 \\
8\epsilon_4\epsilon_7 & 8\epsilon_1\epsilon_4 & 16\epsilon_4^2 & 8\epsilon_4\epsilon_5 & 8\epsilon_2\epsilon_4 & 8\epsilon_4\epsilon_6 & 8\epsilon_3\epsilon_4 \\
8\epsilon_5\epsilon_7 & 8\epsilon_1\epsilon_5 & 8\epsilon_4\epsilon_5 & 16\epsilon_5^2 & 8\epsilon_2\epsilon_5 & 8\epsilon_5\epsilon_6 & 8\epsilon_3\epsilon_5 \\
8\epsilon_2\epsilon_7 & 8\epsilon_1\epsilon_2 & 8\epsilon_2\epsilon_4 & 8\epsilon_2\epsilon_5 & 16\epsilon_2^2 & 8\epsilon_2\epsilon_6 & 8\epsilon_2\epsilon_3 \\
8\epsilon_6\epsilon_7 & 8\epsilon_1\epsilon_6 & 8\epsilon_4\epsilon_6 & 8\epsilon_5\epsilon_6 & 8\epsilon_2\epsilon_6 & 16\epsilon_6^2 & 8\epsilon_3\epsilon_6 \\
8\epsilon_3\epsilon_7 & 8\epsilon_1\epsilon_3 & 8\epsilon_3\epsilon_4 & 8\epsilon_3\epsilon_5 & 8\epsilon_2\epsilon_3 & 8\epsilon_3\epsilon_6 & 16\epsilon_3^2
\end{pmatrix},
$$

$$
U = \begin{pmatrix}
1 & -3.24\times10^{-4} & -1.94\times10^{-6} & -5.34\times10^{-7} & -4.88\times10^{-8} & 4.69\times10^{-8} & 4.14\times10^{-10} \\
-3.24\times10^{-4} & -0.999 & -8.56\times10^{-3} & -2.35\times10^{-3} & -2.15\times10^{-4} & 2.06\times10^{-4} & 1.82\times10^{-6} \\
-1.68\times10^{-6} & 8.66\times10^{-3} & -0.999 & -4.73\times10^{-2} & -4.18\times10^{-3} & 3.98\times10^{-3} & 3.52\times10^{-4} \\
-3.79\times10^{-7} & 1.95\times10^{-2} & 4.75\times10^{-2} & -0.998 & -1.85\times10^{-2} & 1.77\times10^{-2} & 1.56\times10^{-4} \\
-3.37\times10^{-8} & 1.73\times10^{-4} & 3.95\times10^{-3} & 2.25\times10^{-2} & -0.968 & 2.49\times10^{-1} & 1.76\times10^{-3} \\
-1.89\times10^{-8} & 9.75\times10^{-5} & 2.23\times10^{-3} & 1.26\times10^{-2} & 2.499\times10^{-1} & 0.97 & 3.12\times10^{-3} \\
-1.18\times10^{-10} & 6.08\times10^{-7} & 1.39\times10^{-5} & 7.84\times10^{-4} & 9.27\times10^{-4} & -3.46\times10^{-3} & 0.999
\end{pmatrix}
$$

$$
M_{dia}^2 = \lambda_\chi \Lambda^2 \mathrm{dia}(4.71\times10^{-18}, 9.33\times10^{-11}, 3.56\times10^{-8}, 7.30\times10^{-7}, 9.45\times10^{-5}, 3.67\times10^{-4}, 12.01).
$$

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