Chapter 5

Existence, uniqueness and Ulam–Hyers stability results for advection-diffusion-reaction predator-prey systems of variable order and their numerical solution

5.1 Introduction

An advection-diffusion-reaction predator-prey system is a mathematical model that can be represented by a system of partial differential equations (PDEs) used to describe the dynamics of two interacting populations, typically referred to as the predator and prey. The model combines elements of advection, diffusion, and reaction processes to simulate the spatial and temporal dynamics of the populations. Advection represents the movement or transport of the predator and prey populations due to some external forces or environmental factors. Diffusion accounts for the spreading or dispersal of the predator and prey populations due to random movements or dispersal patterns. This term considers the tendency of individuals to move from areas of high population density to areas of lower density. The reaction term describes the interactions between the predator and prey populations. It includes the growth and decline rates of both populations based on their interactions. This term describes how the predator population grows by consuming prey and how the prey population is reduced due to predation. This type of model finds applications in various fields, such as ecology, epidemiology, population dynamics, and environmental sciences. For more details about the advection-diffusion-reaction predator-prey system, see the references [132–137] .

In this chapter, the generalized variable order advection-diffusion-reaction predator-prey system of the following forms has been considered.

$$\begin{aligned} D_t^{\alpha_1(x,t)}u(x,t) &= d_1u_{xx}(x,t) - b_1u_x(x,t) + u(x,t)[r_1 - b_{11}u(x,t) + b_{12}v(x,t)] + f_1(x,t), \\ D_t^{\alpha_2(x,t)}v(x,t) &= d_2v_{xx}(x,t) - b_2v_x(x,t) + v(x,t)[r_2 - b_{21}v(x,t) + b_{22}u(x,t)] + f_2(x,t), \end{aligned} \tag{5.1}$$

subject to the initial conditions (ICs)

$$u(x,0) = u_0(x), \quad v(x,0) = \tilde{u}_0(x), \tag{5.2}$$

and boundary conditions (BCs)

$$\begin{aligned} u(0,t) &= u_1(t), \quad v(0,t) = \tilde{u}_1(t), \\ u(L,t) &= u_2(t), \quad v(L,t) = \tilde{u}_2(t), \end{aligned} \tag{5.3}$$

where $\alpha_1(x,t)$ and $\alpha_2(x,t)$ are the spatial and time dependent variable order derivatives such that $0 < \alpha_1(x,t), \alpha_2(x,t) < 1$. $u(x,t)$ and $v(x,t)$ denote the prey and predator densities at position x and time t, respectively. d_1 and d_2 are diffusion constants, the advection rates are b_1 and b_2. The parameters d_1, d_2, r_1, and b_{ij} $(i,j = 1,2)$ are positive constants, while b_1, b_2 and r_2 are real numbers. $f_1(x,t)$ and $f_2(x,t)$ are the source terms.

The primary goal of this study is to deliver an extremely efficient and effective method, which is the combination of a second kind of shifted airfoil collocation and the operational matrix method to obtain the approximate solution of the variable order advection-diffusion-reaction predator-prey equations. The fundamental concept behind this approach is that we derive operational matrices based on the shifted airfoil polynomial of the second kind. The considered model is transformed using operational matrices into the products of several dependent matrices, which can be seen as an algebraic system of equations by utilising collocation points. As a result of solving the algebraic system of equations, a numerical solution is obtained. Since airfoil polynomial of the second kind is orthogonal, the operational matrices based on airfoil polynomial significantly reduce the size of computational work while accurately providing the approximate solution. We can observe from a few numerical examples that our results are in good agreement with the analytical solution, proving the effectiveness of this method.

5.2 Airfoil polynomials and their shifted forms

The second kind of airfoil polynomials and some of their properties have been addressed in this subsection. Using the following recursive identity, the second kind of airfoil polynomials can be created as [138]

$$\mathbb{A}_{n+1}(z) = 2z\mathbb{A}_n(z) - \mathbb{A}_{n-1}(z), \quad n = 1, 2, ..., \tag{5.4}$$

with the initial values

$$\mathbb{A}_0(z) = 1, \qquad \mathbb{A}_1(z) = 2z + 1.$$

The second kind of airfoil polynomials $\mathbb{A}_n(z)$ has the following explicit expansion form as

$$\mathbb{A}_n(z) = \frac{1}{2^n}\sum_{j=0}^{n}(-1)^j\binom{2n+1}{2j+1}(1-z)^j(1+z)^{n-j}, \qquad n = 0, 1, ..., \tag{5.5}$$

These polynomials $\mathbb{A}_n(z)$ are orthogonal to (-1,1) w.r.t. the weight function $\omega(z) = \sqrt{\frac{1-z}{1+z}}$ as follows

$$\int_{-1}^{1}\mathbb{A}_m(z)\mathbb{A}_n(z)\omega(z)dz = \begin{cases}\pi, & m = n,\\ 0, & m \neq n.\end{cases} \tag{5.6}$$

The shifted form of the aforesaid polynomials on the interval $[0, 1]$ are defined by

$$\mathbb{A}^*_n(z) = \mathbb{A}_n(2z-1).$$

By using the aforementioned transformation in equations (5.4) and (5.5), we generated the following recurrence and explicit formula, respectively

$$\mathbb{A}^*_{n+1}(z) = 2(2z-1)\mathbb{A}^*_n(z) - \mathbb{A}^*_{n-1}(z), \quad n = 1, 2, ...,$$

with the initial values

$$\mathbb{A}^*_0(z) = 1, \qquad \mathbb{A}^*_1(z) = 4z-1.$$

and

$$\mathbb{A}^*_n(z) = \sum_{j=0}^{n}(-1)^j\binom{2n+1}{2j+1}(1-z)^j z^{n-j}, \qquad n = 0, 1, ..., \tag{5.7}$$

or

$$\mathbb{A}^*_n(z) = \sum_{j=0}^{n}(-1)^{n-j}\frac{2^{2j}\Gamma(n+j+1)}{\Gamma(n-j+1)\Gamma(2j+1)}z^j, \qquad n = 0, 1, ..., \tag{5.8}$$

the polynomials $\mathbb{A}^*_n(z)$ are orthogonal to $(0,1)$ w.r.t. the weight function $\omega^*(z) = \sqrt{\frac{1-z}{z}}$ as follows

$$\int_0^1 \mathbb{A}^*_m(z)\mathbb{A}^*_n(z)\omega^*(z)dz = \begin{cases} \frac{\pi}{2}, & m = n, \\ 0, & m \neq n. \end{cases} \tag{5.9}$$

5.3 Function approximation

Let the function $u(z) \in L^2[0,1]$, then the function $u(z)$ can be written as a power series of the second kind of shifted airfoil polynomials as

$$u(z) = \sum_{j=0}^{\infty} c_j \mathbb{A}^*_J(z), \tag{5.10}$$

where the coefficients c_j are unknown.

In general, the series in equation (5.10) can be approximated by a finite sum of $(n+1)$ terms of the second kind of shifted airfoil polynomials as

$$u(z) \approx \sum_{j=0}^{n} c_j \mathbb{A}^*_J(z) = C^T \Pi(z), \tag{5.11}$$

where

$$C^T = [c_0, c_1, ..., c_n], \qquad \Pi(z) = [\mathbb{A}^*_0(z), \mathbb{A}^*_1(z), ..., \mathbb{A}^*_n(z)]^T, \tag{5.12}$$

and $c_j, j = 0, 1, ..., n$ are coefficients that are unknown which can be obtained by the following expression

$$c_j = \frac{2}{\pi} \int_0^1 u(z)\mathbb{A}^*_j(z)\omega^*(z)dz. \tag{5.13}$$

Suppose an arbitrary function $u(x,t) \in L^2[0,1] \times [0,1]$, it can be expanded in terms of the shifted airfoil polynomials as follows

$$u(x,t)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}u_{jk}\mathbb{A}_j^*(x)\mathbb{A}_k^*(t), \tag{5.14}$$

where

$$u_{jk}=\frac{4}{\pi^2}\int_0^1\int_0^1 u(x,t)\mathbb{A}_j^*(x)\mathbb{A}_k^*(t)\omega^*(x)\omega^*(t)dxdt, \qquad j,k=0,1,2,... \tag{5.15}$$

The truncated series of equation (5.14), we have

$$u(x,t)\approx\sum_{j=0}^{n}\sum_{k=0}^{m}u_{jk}\mathbb{A}_j^*(x)\mathbb{A}_k^*(t)=\Pi^T(x)U\Pi(t), \tag{5.16}$$

where

$$\Pi(x)=[\mathbb{A}_0^*(x),\mathbb{A}_1^*(x),...,\mathbb{A}_n^*(x)]^T,\ \ \Pi(t)=[\mathbb{A}_0^*(t),\mathbb{A}_1^*(t),...,\mathbb{A}_m^*(t)]^T, U=(u_{jk})_{j,k=0}^{n,m}. \tag{5.17}$$

Theorem 1. The derivative of the shifted airfoil vector $\Pi(x)$ can be presented as follows

$$\frac{d\Pi(x)}{dx}=M^{(1)}\Pi(x), \tag{5.18}$$

where $M^{(1)}=(m_{ij})_{(n+1)\times(n+1)}$ is an operational matrix of the first order derivative, such that

$$m_{ij}=\begin{cases}2(i+j+1), & for\ \ i>j,\ (i+j)\text{ is odd},\\ 2(j-i), & for\ \ i>j,\ (i+j)\text{ is even},\\ 0, & for\ \ i<j.\end{cases} \tag{5.19}$$

The k^{th} order derivatives of vector $\Pi(x)$ is given as

$$\frac{d^k\Pi(x)}{dx^k} = (M^{(1)})^k\Pi(x) = M^{(k)}\Pi(x),$$

where k is a natural number.

Proof. We omit the proof since it is straightforward.

5.4 Operational matrix of variable order derivative of $\Pi(x)$

Theorem 2. Let $\Pi(x)$ be the shifted airfoil vector defined in equation (5.12) and $\beta(x,t) > 0$, then

$$D_x^{\beta(x,t)}(\Pi(x)) \simeq M^{(\beta(x,t))}\Pi(x), \tag{5.20}$$

where $M^{(\beta(x,t))}$ is the $(n+1)\times(n+1)$ operational matrix of the Caputo variable derivative of order $\beta(x,t)$ and it is defined by

$$M^{(\beta(x,t))} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{k=q}^{q} Q_{q,0,k} & \sum_{k=q}^{q} Q_{q,1,k} & \cdots & \sum_{k=q}^{q} Q_{q,n,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=q}^{r} Q_{r,0,k} & \sum_{k=q}^{r} Q_{r,1,k} & \cdots & \sum_{k=q}^{r} Q_{r,n,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=q}^{n} Q_{n,0,k} & \sum_{k=q}^{n} Q_{n,1,k} & \cdots & \sum_{k=q}^{n} Q_{n,n,k} \end{pmatrix}$$

where

$$\begin{aligned} Q_{i,j,k} =& x^{-\beta(x,t)} \frac{(-1)^{i-k} 2^{2k} \Gamma(i+k+1)\Gamma(k+1)}{\sqrt{\pi}\Gamma(i-k+1)\Gamma(2k+1)\Gamma(k+1-\beta(x,t))} \\ & \times \sum_{l=0}^{j} \frac{(-1)^{j-l} 2^{2l} \Gamma(j+l+1)\Gamma(k+l+\frac{1}{2})}{\Gamma(j-l+1)\Gamma(2l+1)\Gamma(2+k+l)}. \end{aligned}$$

Proof. Using the Caputo derivative in equation (5.8), we get

$$D_x^{\beta(x,t)}(\mathbb{A}_i^*(x)) = D_x^{\beta(x,t)} \left(\sum_{k=0}^{i} (-1)^{i-k} \frac{2^{2k}\Gamma(i+k+1)}{\Gamma(i-k+1)\Gamma(2k+1)} x^k \right). \tag{5.21}$$

By linearity property of the Caputo derivative, we have

$$D_x^{\beta(x,t)}(\mathbb{A}_i^*(x)) = \sum_{k=0}^{i} (-1)^{i-k} \frac{2^{2k}\Gamma(i+k+1)}{\Gamma(i-k+1)\Gamma(2k+1)} D_x^{\beta(x,t)} x^k. \tag{5.22}$$

From equation (1.19) we can write

$$D_x^{\beta(x,t)}(\mathbb{A}_i^*(x)) = 0, \quad i = 0, 1, ..., q-1, \tag{5.23}$$

and

$$D_x^{\beta(x,t)}(\mathbb{A}_i^*(x)) = x^{-\beta(x,t)} \sum_{k=q}^{i} (-1)^{i-k} \frac{2^{2k}\Gamma(i+k+1)\Gamma(k+1)}{\Gamma(i-k+1)\Gamma(2k+1)\Gamma(k+1-\beta(x,t))} x^k, \ i = q, ..., n, \tag{5.24}$$

Now, approximate x^k by $(n+1)$ terms of the shifted airfoil polynomials, we have

$$x^k \simeq \sum_{j=0}^{n} c_{kj} \mathbb{A}_j^*(x), \tag{5.25}$$

where

$$c_{kj} = \frac{2}{\pi}\int_0^1 x^k \mathbb{A}_j^*(x)\omega^*(x)dx. \tag{5.26}$$

Now, putting the values of $\mathbb{A}_j^*(x)$ and $\omega^*(x)$ into equation (5.26), we get

$$\begin{aligned} c_{kj} =&\frac{2}{\pi}\int_0^1 x^k\sqrt{\frac{1-x}{x}}\sum_{l=0}^{j}(-1)^{j-l}\frac{2^{2l}\Gamma(j+l+1)}{\Gamma(j-l+1)\Gamma(2l+1)}x^l dx,\\ =&\frac{2}{\pi}\sum_{l=0}^{j}(-1)^{j-l}\frac{2^{2l}\Gamma(j+l+1)}{\Gamma(j-l+1)\Gamma(2l+1)}\int_0^1 x^{k+l}\sqrt{\frac{1-x}{x}}dx,\\ =&\frac{1}{\sqrt{\pi}}\sum_{l=0}^{j}(-1)^{j-l}\frac{2^{2l}\Gamma(j+l+1)\Gamma(k+l+\frac{1}{2})}{\Gamma(j-l+1)\Gamma(2l+1)\Gamma(2+k+l)}, \end{aligned} \tag{5.27}$$

therefore, equation (5.25) becomes

$$x^k \simeq \sum_{j=0}^{n}\frac{1}{\sqrt{\pi}}\sum_{l=0}^{j}(-1)^{j-l}\frac{2^{2l}\Gamma(j+l+1)\Gamma(k+l+\frac{1}{2})}{\Gamma(j-l+1)\Gamma(2l+1)\Gamma(2+k+l)}\mathbb{A}_j^*(x), \tag{5.28}$$

with the help of equations (5.24) and (5.28), we obtain

$$\begin{aligned} D_x^{\beta(x,t)}(\mathbb{A}_i^*(x)) \simeq& \sum_{j=0}^{n}\sum_{k=q}^{i} x^{-\beta(x,t)}\frac{(-1)^{i-k}2^{2k}\Gamma(i+k+1)\Gamma(k+1)}{\sqrt{\pi}\Gamma(i-k+1)\Gamma(2k+1)\Gamma(k+1-\beta(x,t))}\\ &\times\sum_{l=0}^{j}\frac{(-1)^{j-l}2^{2l}\Gamma(j+l+1)\Gamma(k+l+\frac{1}{2})}{\Gamma(j-l+1)\Gamma(2l+1)\Gamma(2+k+l)}\mathbb{A}_j^*(x), \quad i=q,...,n, \end{aligned} \tag{5.29}$$

$$D_x^{\beta(x,t)}(\mathbb{A}_i^*(x)) \simeq \sum_{j=0}^{n}\sum_{k=q}^{i} Q_{i,j,k}\mathbb{A}_j^*(x), \quad i=q,...,n, \tag{5.30}$$

where

$$\begin{aligned} Q_{i,j,k} =&x^{-\beta(x,t)}\frac{(-1)^{i-k}2^{2k}\Gamma(i+k+1)\Gamma(k+1)}{\sqrt{\pi}\Gamma(i-k+1)\Gamma(2k+1)\Gamma(k+1-\beta(x,t))}\\ &\times\sum_{l=0}^{j}\frac{(-1)^{j-l}2^{2l}\Gamma(j+l+1)\Gamma(k+l+\frac{1}{2})}{\Gamma(j-l+1)\Gamma(2l+1)\Gamma(2+k+l)}, \end{aligned}$$

Equation (5.30) can be expressed in vector form as

$$D_x^{\beta(x,t)}(\mathbb{A}_i^*(x)) \simeq \left[\sum_{k=q}^{i} Q_{i,0,k}, \sum_{k=q}^{i} Q_{i,1,k}, \ldots \sum_{k=q}^{i} Q_{i,n,k}\right] \Pi(x). \tag{5.31}$$

Also, the equation (5.23) can be written as

$$D_x^{\beta(x,t)}(\mathbb{A}_i^*(x)) \simeq [0, 0, \ldots, 0]\, \Pi(x), \qquad i = 0, 1, \ldots, q-1. \tag{5.32}$$

The combination of equations (5.31) and (5.32) prove the desired results.

5.5 Existence, uniqueness and Ulam–Hyers stability

In this part, it is first demonstrated the existence and uniqueness of the solution to equations (5.1)-(5.3) and then discussed its Ulam–Hyers stability.

5.5.1 Existence and uniqueness analysis

By using the Riemann-Liouville integral operator of variable order in equation (5.1), we have

$$\begin{aligned} u(x,t) - u(x,0) = I_t^{\alpha_1(x,t)}\Big(& d_1 u_{xx}(x,t) - b_1 u_x(x,t) + r_1 u(x,t) \\ & - b_{11} u^2(x,t) + b_{12} u(x,t) v(x,t) + f_1(x,t)\Big), \\ = \frac{1}{\Gamma(\alpha_1(x,t))} \int_0^t & (t-s)^{\alpha_1(x,t)-1}\Big(d_1 u_{xx}(x,s) - b_1 u_x(x,s) \\ & + r_1 u(x,s) - b_{11} u^2(x,s) + b_{12} u(x,s) v(x,s) + f_1(x,s)\Big) ds, \end{aligned} \tag{5.33}$$

and

$$v(x,t)-v(x,0)=\frac{1}{\Gamma(\alpha_2(x,t))}\int_0^t (t-s)^{\alpha_2(x,t)-1}\Big(d_2v_{xx}(x,s)-b_2v_x(x,s)$$
$$+r_2v(x,s)-b_{21}v^2(x,s)+b_{22}v(x,s)u(x,s)+f_2(x,s)\Big)ds. \tag{5.34}$$

Suppose that

$$K\left(t,u(x,t)\right)=d_1u_{xx}(x,t)-b_1u_x(x,t)+r_1u(x,t)-b_{11}u^2(x,t)+b_{12}u(x,t)v(x,t)+f_1(x,t), \tag{5.35}$$

and

$$\tilde{K}\left(t,v(x,t)\right)=d_2v_{xx}(x,t)-b_2v_x(x,t)+r_2v(x,t)-b_{21}v^2(x,t)+b_{22}v(x,t)u(x,t)+f_2(x,t), \tag{5.36}$$

and for continuous functions $u(x,t),\ u_1(x,t),\ v(x,t)$ and $v_1(x,t)\ \in\ L^2\Big((0,1)\times(0,1)\Big)$, there exist some constants $\lambda_1,\ \tilde{\lambda}_1>0$ and $\lambda_2,\ \tilde{\lambda}_2>0$, such that

$$\begin{aligned}&\|u_{xx}-(u_1)_{xx}\|\le\lambda_1\|u-u_1\|, \quad \|u_x-(u_1)_x\|\le\lambda_2\|u-u_1\|,\\ &\|v_{xx}-(v_1)_{xx}\|\le\tilde{\lambda}_1\|v-v_1\|, \quad \|v_x-(v_1)_x\|\le\tilde{\lambda}_2\|v-v_1\|.\end{aligned} \tag{5.37}$$

Also, here $|d_1|\le m_1,\ |b_1|\le m_2,\ |r_1|\le m_3,\ |b_{11}|\le m_4,\ |b_{12}|\le m_5,\ |d_2|\le\tilde{m}_1,\ |b_2|\le\tilde{m}_2,\ |r_2|\le\tilde{m}_3,\ |b_{21}|\le\tilde{m}_4,$ and $|b_{12}|\le\tilde{m}_5$ such that $m_i,\tilde{m}_i>0,\ i=1,2,...,5.$ Now, it is the time to prove that $K\left(t,u(x,t)\right)$ and $\tilde{K}\left(t,v(x,t)\right)$ satisfies Lipschitz condition. For this purpose, we have

$$
\begin{aligned}
\|K(t,u)-K(t,u_1)\| &= ||d_1u_{xx}-b_1u_x+r_1u-b_{11}u^2+b_{12}uv-d_1(u_1)_{xx} \\
&\quad +b_1(u_1)_x-r_1u_1+b_{11}u_1^2-b_{12}u_1v||, \\
&\leq m_1\lambda_1\|u-u_1\|+m_2\lambda_2\|u-u_1\|+m_3\|u-u_1\|+m_4\left\|u^2-u_1^2\right\| \\
&\quad +m_5\|v\|\|u-u_1\| \\
&\leq (m_1\lambda_1+m_2\lambda_2+m_3+m_4(\gamma_1+\gamma_2)+m_5\tilde{\gamma}_1)\|u-u_1\|.
\end{aligned} \tag{5.38}
$$

Setting $\sigma = m_1\lambda_1+m_2\lambda_2+m_3+m_4(\gamma_1+\gamma_2)+m_5\tilde{\gamma}_1$, where $u,\ u_1,\ v$ and v_1 are bounded functions such that $\|u\|\leq\gamma_1$, $\|u_1\|\leq\gamma_2$, $\|v\|\leq\tilde{\gamma}_1$ and $\|v_1\|\leq\tilde{\gamma}_2$, then we have

$$\|K(t,u)-K(t,u_1)\|\leq\sigma\|u-u_1\|. \tag{5.39}$$

Similarly

$$\left\|\tilde{K}(t,v)-\tilde{K}(t,v_1)\right\|\leq\tilde{\sigma}\|v-v_1\|, \tag{5.40}$$

where, $\tilde{\sigma}=\tilde{m}_1\tilde{\lambda}_1+\tilde{m}_2\tilde{\lambda}_2+\tilde{m}_3+\tilde{m}_4(\tilde{\gamma}_1+\tilde{\gamma}_2)+\tilde{m}_5\gamma_1$.

Hence, the Lipschitz condition is fulfilled for $K(t,u)$ and $\tilde{K}(t,v)$ and if an additionally $0\leq\sigma,\tilde{\sigma}\leq 1$, then it is also a contraction. From the equations (5.33)-(5.34), we consider the following recursive relations as

$$u_{n+1}(x,t)=\frac{1}{\Gamma(\alpha_1(x,t))}\int_0^t(t-s)^{\alpha_1(x,t)-1}K(s,u_n)ds, \tag{5.41}$$

and

$$v_{n+1}(x,t)=\frac{1}{\Gamma(\alpha_2(x,t))}\int_0^t(t-s)^{\alpha_2(x,t)-1}\tilde{K}(s,v_n)ds, \tag{5.42}$$

with $u_0(x,t) = u(x,0)$ and $v_0(x,t) = v(x,0)$.

Let us define the following differences as

$$\begin{aligned}\theta_{n+1}(x,t) &= u_{n+1}(x,t) - u_n(x,t),\\ &= \frac{1}{\Gamma(\alpha_1(x,t))}\int_0^t (t-s)^{\alpha_1(x,t)-1}\Big(K(s,u_n) - K(s,u_{n-1})\Big)ds, \qquad (5.43)\end{aligned}$$

and

$$\begin{aligned}\tilde{\theta}_{n+1}(x,t) &= v_{n+1}(x,t) - v_n(x,t),\\ &= \frac{1}{\Gamma(\alpha_2(x,t))}\int_0^t (t-s)^{\alpha_2(x,t)-1}\Big(\tilde{K}(s,v_n) - \tilde{K}(s,v_{n-1})\Big)ds, \qquad (5.44)\end{aligned}$$

Applying norm on both side of equation (5.43), we get

$$\begin{aligned}\|\theta_{n+1}(x,t)\| &= \left\|\frac{1}{\Gamma(\alpha_1(x,t))}\int_0^t (t-s)^{\alpha_1(x,t)-1}\Big(K(s,u_n) - K(s,u_{n-1})\Big)ds\right\|,\\ &= \frac{1}{\Gamma(\alpha_1(x,t))}\int_0^t (t-s)^{\alpha_1(x,t)-1}\|K(s,u_n) - K(s,u_{n-1})\|ds, \qquad (5.45)\end{aligned}$$

and

$$\left\|\tilde{\theta}_{n+1}(x,t)\right\| = \frac{1}{\Gamma(\alpha_2(x,t))}\int_0^t (t-s)^{\alpha_2(x,t)-1}\left\|\tilde{K}(s,v_n) - \tilde{K}(s,v_{n-1})\right\|ds. \qquad (5.46)$$

Now using the above result, we prove the following theorem.

Theorem 3. The advection-diffusion-reaction predator-prey system of variable order given by the equation (5.1) has a solution if there exists t_0 satisfying the following inequalities hold.

$$\frac{\sigma t_0^{\alpha_1(x,t)}}{\Gamma(\alpha_1(x,t)+1)} < 1, \tag{5.47}$$

$$\frac{\tilde{\sigma} t_0^{\alpha_2(x,t)}}{\Gamma(\alpha_2(x,t)+1)} < 1. \tag{5.48}$$

Proof. Let the functions $\Psi_n(x,t) = u_{n+1} - u + u(x,0)$ and $\tilde{\Psi}_n(x,t) = v_{n+1} - v + v(x,0)$, then by equation (5.45), we have

$$\begin{aligned}\|\Psi_n(x,t)\| &= \frac{1}{\Gamma(\alpha_1(x,t))}\int_0^t (t-s)^{\alpha_1(x,t)-1}\|K(s,u_n) - K(s,u)\|ds,\\ &\leq \frac{\sigma t^{\alpha_1(x,t)}}{\Gamma(\alpha_1(x,t)+1)}\|u_n - u\|.\end{aligned} \tag{5.49}$$

Recursively applying the same process results in

$$\|\Psi_n(x,t)\| \leq \left(\frac{\sigma t^{\alpha_1(x,t)}}{\Gamma(\alpha_1(x,t)+1)}\right)^n \|u - u_1\|, \tag{5.50}$$

at $t = t_0$ above equation becomes

$$\|\Psi_n(x,t)\| \leq \left(\frac{\sigma t_0^{\alpha_1(x,t)}}{\Gamma(\alpha_1(x,t)+1)}\right)^n \|u - u_1\|. \tag{5.51}$$

Similarly

$$\left\|\tilde{\Psi}_n(x,t)\right\| \leq \left(\frac{\tilde{\sigma} t_0^{\alpha_2(x,t)}}{\Gamma(\alpha_2(x,t)+1)}\right)^n \|v - v_1\|. \tag{5.52}$$

Thus, $\|\Psi_n(x,t)\| \to 0$ and $\left\|\tilde{\Psi}_n(x,t)\right\| \to 0$ as $n \to \infty$ for $\frac{\sigma t_0^{\alpha_1(x,t)}}{\Gamma(\alpha_1(x,t)+1)} < 1$ and $\frac{\tilde{\sigma} t_0^{\alpha_2(x,t)}}{\Gamma(\alpha_2(x,t)+1)} < 1$, respectively, which implise that $\underset{n\to\infty}{lim}\, u_n(x,t) = u(x,t)$ and $\underset{n\to\infty}{lim}\, v_n(x,t) = v(x,t)$. Then solution of the given equation (5.1) exist.

Next, we analyse the uniqueness of a solution to the given equation (5.1). To do this work, let us consider that there exist solutions $\zeta(x,t)$ and $\xi(x,t)$ of the system (5.1), so that

$$u(x,t)-\zeta(x,t)=\frac{1}{\Gamma(\alpha_1(x,t))}\int_0^t (t-s)^{\alpha_1(x,t)-1}\Big(K(s,u(x,s))-K(s,\zeta(x,s))\Big)ds. \tag{5.53}$$

Taking the norm of equation (5.53), we get

$$\begin{aligned}\|u(x,t)-\zeta(x,t)\| &= \left\|\frac{1}{\Gamma(\alpha_1(x,t))}\int_0^t (t-s)^{\alpha_1(x,t)-1}\Big(K(s,u(x,s))-K(s,\zeta(x,s))\Big)ds\right\|,\\ &\le \frac{\sigma t^{\alpha_1(x,t)}}{\Gamma(\alpha_1(x,t)+1)}\|u(x,t)-\zeta(x,t)\|, \end{aligned} \tag{5.54}$$

which gives

$$\|u(x,t)-\zeta(x,t)\|\left(1-\frac{\sigma t^{\alpha_1(x,t)}}{\Gamma(\alpha_1(x,t)+1)}\right)\le 0. \tag{5.55}$$

Then, using equation (5.55), we can obtain $\|u(x,t)-\zeta(x,t)\| = 0$. Consequently, $u(x,t)=\zeta(x,t)$.

Similarly, for solution $v(x,t)$, we can find out $v(x,t)=\xi(x,t)$. Thus, the solution of equation (5.1) is unique.

5.5.2 Ulam–Hyers stability

Here, Ulam–Hyers stability is demonstrated for the advection-diffusion-reaction predator-prey system which is provided in equation (5.1).

Definition The presented equation (5.1) is said to be Ulam–Hyers stable if for any $\delta_1, \delta_2 > 0$ and any u and v satisfying the following relations

$$|D_t^{\alpha_1(x,t)}u(x,t)-d_1u_{xx}(x,t)+b_1u_x(x,t)-r_1u(x,t)+b_{11}u^2(x,t)-b_{12}u(x,t)v(x,t)-f_1(x,t)| < \delta_1, \tag{5.56}$$

and

$$|D_t^{\alpha_2(x,t)}v(x,t)-d_2v_{xx}(x,t)+b_2v_x(x,t)-r_2v(x,t)+b_{21}v^2(x,t)-b_{22}v(x,t)u(x,t)-f_2(x,t)| < \delta_2, \tag{5.57}$$

there exist solutions u^* and v^*, such that

$$|u-u^*| < a_1\delta_1, \qquad a_1 \in \mathbb{R}, \tag{5.58}$$

and

$$|v-v^*| < a_2\delta_2, \qquad a_2 \in \mathbb{R}. \tag{5.59}$$

When u satisfy equation (5.56), then there exists a function $q_1(x,t)$ satisfying $|q_1| < \delta_1$ such that

$$\begin{aligned} D_t^{\alpha_1(x,t)}u(x,t) - d_1u_{xx}(x,t) + b_1u_x(x,t) - r_1u(x,t) + b_{11}u^2(x,t) \\ - b_{12}u(x,t)v(x,t) - f_1(x,t) = q_1(x,t). \end{aligned} \tag{5.60}$$

By applying the Riemann–Liouville variable order integral operator to both sides of equation (5.60), we obtain

$$\begin{aligned} u(x,t) - u(x,0) + I_t^{\alpha_1(x,t)}\Big(- d_1u_{xx}(x,t) + b_1u_x(x,t) - r_1u(x,t) + b_{11}u^2(x,t) \\ - b_{12}u(x,t)v(x,t) - f_1(x,t)\Big) = I_t^{\alpha_1(x,t)}q_1(x,t). \end{aligned} \tag{5.61}$$

Then

$$\begin{aligned}\Big|u(x,t)-u(x,0)+I_t^{\alpha_1(x,t)}\Big(-d_1u_{xx}(x,t)+b_1u_x(x,t)-r_1u(x,t)+b_{11}u^2(x,t)\\ -b_{12}u(x,t)v(x,t)-f_1(x,t)\Big)\Big| &= |I_t^{\alpha_1(x,t)}q_1(x,t)|\\ &\le |q_1|I_t^{\alpha_1(x,t)}(1)\\ &\le \frac{t^{\alpha_1(x,t)}\delta_1}{\mathit{\Gamma}(\alpha_1(x,t)+1)}.\end{aligned} \tag{5.62}$$

Suppose that the function $u^*(x,t)$ be the solution of system (5.1) with $u(x,0)=u^*(x,0)=u_0(x)$, we have

$$\begin{aligned}u^*(x,t)=u(x,0)+I_t^{\alpha_1(x,t)}\Big(d_1u^*_{xx}(x,t)-b_1u^*_x(x,t)+r_1u^*(x,t)\\ -b_{11}u^{*2}(x,t)+b_{12}u^*(x,t)v(x,t)+f_1(x,t)\Big).\end{aligned} \tag{5.63}$$

Then

$$\begin{aligned}\|u-u^*\| &= \Big\|u(x,t)-u(x,0)-I_t^{\alpha_1(x,t)}\Big(d_1u^*_{xx}(x,t)-b_1u^*_x(x,t)+r_1u^*(x,t)-b_{11}u^{*2}(x,t)\\ &\quad +b_{12}u^*(x,t)v(x,t)+f_1(x,t)\Big)\Big\|\\ &= \Big\|u(x,t)-u(x,0)+I_t^{\alpha_1(x,t)}\Big(-d_1u_{xx}(x,t)+b_1u_x(x,t)-r_1u(x,t)+b_{11}u^2(x,t)\\ &\quad -b_{12}u(x,t)v(x,t)-f_1(x,t)\Big)-I_t^{\alpha_1(x,t)}\Big(-d_1u_{xx}(x,t)+b_1u_x(x,t)-r_1u(x,t)\\ &\quad +b_{11}u^2(x,t)-b_{12}u(x,t)v(x,t)-f_1(x,t)\Big)-I_t^{\alpha_1(x,t)}\Big(d_1u^*_{xx}(x,t)-b_1u^*_x(x,t)\\ &\quad +r_1u^*(x,t)-b_{11}u^{*2}(x,t)+b_{12}u^*(x,t)v(x,t)+f_1(x,t)\Big)\Big\|\\ &\le \frac{t^{\alpha_1(x,t)}\delta_1}{\mathit{\Gamma}(\alpha_1(x,t)+1)}+\Big(m_1\lambda_1+m_2\lambda_2+m_3+m_4(\gamma_1+\gamma_2)+m_5\tilde{\gamma}_1\Big)\Big(I_t^{\alpha_1(x,t)}\Big(\|u-u^*\|\Big)\Big).\end{aligned} \tag{5.64}$$

In [139], the Gronwall relation is introduced for the Riemann-Liouville integral operator of variable order.

Lemma Let $\alpha(x,t) > 0$ and $w_1(x,t)$ be a nonnegative, nondecreasing and locally integrable function on the interval (c,d). Also, let $w_2(x,t)$ is bounded by some constant and $u(x,t)$ be a nonnegative and locally integrable on the interval (c,d) such that

$$u(x,t) \leq w_1(x,t) + w_2(x,t)\Big(I_t^{\alpha(x,t)}u(x,t)\Big), \tag{5.65}$$

then

$$u(x,t) \leq w_1(x,t)E_{\alpha(x,t)}\Big(w_2(x,t)(t-c)^{\alpha(x,t)}\Big), \tag{5.66}$$

where $E_{\alpha(x,t)}(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(\alpha(x,t)i+1)}$.

We obtain the following result by applying the Gronwall relation to equation (5.64) with $w_1(x,t) = \frac{t^{\alpha_1(x,t)}\delta_1}{\Gamma(\alpha_1(x,t)+1)}$ and $w_2(x,t) = m_1\lambda_1 + m_2\lambda_2 + m_3 + m_4(\gamma_1+\gamma_2) + m_5\tilde{\gamma}_1$.

$$\|u-u^*\| \leq \frac{t^{\alpha_1(x,t)}\delta_1}{\Gamma(\alpha_1(x,t)+1)}E_{\alpha_1(x,t)}\Big((m_1\lambda_1+m_2\lambda_2+m_3+m_4(\gamma_1+\gamma_2)+m_5\tilde{\gamma}_1)t^{\alpha_1(x,t)}\Big), \tag{5.67}$$

which implies $\|u-u^*\| \leq a_1\delta_1$, with

$$a_1 = \frac{t^{\alpha_1(x,t)}}{\Gamma(\alpha_1(x,t)+1)}E_{\alpha_1(x,t)}\Big((m_1\lambda_1+m_2\lambda_2+m_3+m_4(\gamma_1+\gamma_2)+m_5\tilde{\gamma}_1)t^{\alpha_1(x,t)}\Big).$$

Similarly, for the solution $v(x,t)$, we can calculate $\|v-v^*\| < a_2\delta_2$, with

$$a_2 = \frac{t^{\alpha_2(x,t)}}{\Gamma(\alpha_2(x,t)+1)}E_{\alpha_2(x,t)}\Big((\tilde{m}_1\tilde{\lambda}_1+\tilde{m}_2\tilde{\lambda}_2+\tilde{m}_3+\tilde{m}_4(\tilde{\gamma}_1+\tilde{\gamma}_2)+\tilde{m}_5\gamma_1)t^{\alpha_1(x,t)}\Big).$$

As a result, the solution to equation (5.1) is Ulam-Hyers stable.

5.6 Application of the presented technique

In this section, the presented technique has been discussed for obtaining numerical solutions of the variable order advection-diffusion-reaction predator-prey model using the operational matrix derived in the previous section. Let us approximate $u(x,t)$ and $v(x,t)$ by shifted airfoil polynomials as

$$\begin{aligned} u(x,t) &\approx \Pi^T(x)U\Pi(t), \\ v(x,t) &\approx \Pi^T(x)V\Pi(t), \end{aligned} \tag{5.68}$$

where $U = [u_{jk}]$ and $V = [\tilde{u}_{jk}]$ are unknown $(n+1)\times(n+1)$ matrices and $\Pi(x) = [\mathbb{A}_0^*(x), \mathbb{A}_1^*(x), ..., \mathbb{A}_n^*(x)]^T$ is the column vector. Now applying the Caputo variable operator $D_t^{\beta(x,t)}$ on the equation (5.68) and using Theorem (2), we get

$$\begin{aligned} D_t^{\alpha_1(x,t)}u(x,t) \approx & D_t^{\alpha_1(x,t)}\left(\Pi^T(x)U\Pi(t)\right) \\ =&\Pi^T(x)UD_t^{\alpha_1(x,t)}\Pi(t) \\ =&\Pi^T(x)UM^{(\alpha_1(x,t))}\Pi(t). \end{aligned} \tag{5.69}$$

Similarly

$$D_t^{\alpha_2(x,t)}v(x,t) \approx \Pi^T(x)VM^{(\alpha_2(x,t))}\Pi(t), \tag{5.70}$$

and

$$\begin{aligned} u_x(x,t) &\approx \Pi^T(x)(M^1)^TU\Pi(t), \\ v_x(x,t) &\approx \Pi^T(x)(M^1)^TV\Pi(t), \end{aligned} \tag{5.71}$$

and

$$\begin{aligned} u_{xx}(x,t) &\approx \Pi^T(x)(M^2)^T U\Pi(t), \\ v_{xx}(x,t) &\approx \Pi^T(x)(M^2)^T V\Pi(t). \end{aligned} \tag{5.72}$$

Using equations (5.68)-(5.72), the residuals $R_1(x,t)$ and $R_2(x,t)$ for equation (5.1) can be written as

$$\begin{aligned} R_1(x,t) = {} & \Pi^T(x)UM^{(\alpha_1(x,t))}\Pi(t) - d_1\Pi^T(x)(M^2)^T U\Pi(t) + b_1\Pi^T(x)(M^1)^T U\Pi(t) \\ & - \big(\Pi^T(x)U\Pi(t)\big)\left[r_1 - b_{11}\Pi^T(x)U\Pi(t) + b_{12}\Pi^T(x)V\Pi(t)\right] - f_1(x,t) \simeq 0, \end{aligned} \tag{5.73}$$

$$\begin{aligned} R_2(x,t) = {} & \Pi^T(x)VM^{(\alpha_2(x,t))}\Pi(t) - d_2\Pi^T(x)(M^2)^T V\Pi(t) + b_2\Pi^T(x)(M^1)^T V\Pi(t) \\ & - \big(\Pi^T(x)V\Pi(t)\big)\left[r_2 - b_{21}\Pi^T(x)V\Pi(t) + b_{22}\Pi^T(x)U\Pi(t)\right] - f_2(x,t) \simeq 0, \end{aligned} \tag{5.74}$$

now using equation (5.68), we approximate the initial and boundary conditions, we get

$$\Pi^T(x)U\Pi(0) = u_0(x), \quad \Pi^T(x)V\Pi(0) = \tilde{u}_0(x), \tag{5.75}$$

and

$$\begin{aligned} \Pi^T(0)U\Pi(t) = u_1(t), &\quad \Pi^T(0)V\Pi(t) = \tilde{u}_1(t), \\ \Pi^T(L)U\Pi(t) = u_2(t), &\quad \Pi^T(L)V\Pi(t) = \tilde{u}_2(t). \end{aligned} \tag{5.76}$$

Now to find the unknown matrices U and V, we collocate the equations (5.73)-(5.76) at the collocation points $x_r = \frac{2r+1}{2(n+1)}$ and $t_{\tilde{r}} = \frac{2\tilde{r}+1}{2(n+1)}$, we have

$$R_1(x_r, t_{\tilde{r}}) = 0, \quad r = 1, 2, ..., n-1, \quad \tilde{r} = 1, 2, ..., n, \tag{5.77}$$

$$R_2(x_r, t_{\tilde{r}}) = 0, \quad r = 1, 2, ..., n-1, \quad \tilde{r} = 1, 2, ..., n, \tag{5.78}$$

the prescribed ICs and BCs become

$$\Pi^T(x_r)U\Pi(0) = u_0(x_r), \quad \Pi^T(x_r)V\Pi(0) = \tilde{u}_0(x_r), \quad r = 1, 2, ..., n+1, \tag{5.79}$$

and

$$\begin{aligned} \Pi^T(0)U\Pi(t_{\tilde{r}}) = u_1(t_{\tilde{r}}), \quad \Pi^T(0)V\Pi(t_{\tilde{r}}) = \tilde{u}_1(t_{\tilde{r}}), \quad \tilde{r} = 1, 2, ..., n, \\ \Pi^T(L)U\Pi(t_{\tilde{r}}) = u_2(t_{\tilde{r}}), \quad \Pi^T(L)V\Pi(t_{\tilde{r}}) = \tilde{u}_2(t_{\tilde{r}}), \quad \tilde{r} = 1, 2, ..., n. \end{aligned} \tag{5.80}$$

Hence from equations (5.77)-(5.80), we have a $2(n+1)^2$ system of algebraic equations that can be solved by Newton's method. By solving this algebraic system, we obtain the required approximate solutions.

5.7 Numerical results

This section includes some particular cases of the considered models for validation of the presented technique. This is done by comparing our numerical results with the exact ones. The feasibility of the method can be seen through the tables and error graphs. The accuracy of the numerical solution is measured by using the L_2

and L_∞ error norms, which are defined as follows

$$L_2 = \sqrt{\sum_{i=0}^{n}\sum_{j=0}^{n} \left|u(x_i,t_j) - \tilde{u}(x_i,t_j)\right|^2}, \tag{5.81}$$

$$L_\infty = \max_{i,j} \left|u(x_i,t_j) - \tilde{u}(x_i,t_j)\right|, \tag{5.82}$$

where $u(x_i,t_j)$ and $\tilde{u}(x_i,t_j)$ are the exact and approximate solutions, respectively, at points (x_i,t_j).

Example 1. Consider system (5.1) by assuming $d_1 = d_2 = 1.5,\ b_1 = b_2 = 1,\ r_1 = r_2 = 0.5,\ b_{11} = b_{21} = 1,$ and $b_{12} = b_{22} = 0.75$ with

$$\begin{aligned} f_1(x,t) =& t^{1-\alpha_1(x,t)} cosh(x) E_{2,2-\alpha_1(x,t)}(-t^2) + b_1 sinh(x) sin(t) \\ &- cosh(x) sin(t)\Big(d_1 + r_1 - b_{11} cosh(x) sin(t) + b_{12} sinh(x) cos(t)\Big), \end{aligned} \tag{5.83}$$

$$\begin{aligned} f_2(x,t) =& - t^{2-\alpha_2(x,t)} sinh(x) E_{2,3-\alpha_2(x,t)}(-t^2) - d_2 sinh(x) cos(t) \\ &- sinh(x) cos(t)\Big(r_2 - b_2 - b_{21} sinh(x) cos(t) + b_{22} cosh(x) sin(t)\Big). \end{aligned} \tag{5.84}$$

The ICs and BCs are obtained by using the exact solution $u(x,t) = cosh(x)sin(t)$ and $v(x,t) = sinh(x)cos(t)$. We apply the suggested technique to this example and demonstrate the numerical results in Tables 5.1, 5.2 and Figures 5.1, 5.2. A comparison of L_2 and L_∞ errors is presented for the solution $u(x,t)$ and $v(x,t)$ with various values of n and two different values of $\alpha_1(x,t)$ and $\alpha_2(x,t)$ in Table 5.1 and Table 5.2, respectively. From these tables, we can see that the errors decrease rapidly as the value of n increases. Figure 5.1 and Figure 5.2 display the numerical and absolute error results for $n = 11$ with $\alpha_1(x,t) = 0.50 + 0.2e^{-xt}$ and $\alpha_2(x,t) = 0.65 + 0.2sin(xt)$. The achieved results confirm that the proposed technique provides approximate solutions with high accuracy.

TABLE 5.1: For Example 1, a comparison of L_2 and L_∞ errors of the solution $u(x,t)$

	$\alpha_1(x,t) = 0.50 + 0.2e^{-xt}$ $\alpha_2(x,t) = 0.65 + 0.2sin(xt)$		$\alpha_1(x,t) = 0.70 + 0.2e^{-xt}$ $\alpha_2(x,t) = 0.85 + 0.2sin(xt)$	
n	L_2	L_∞	L_2	L_∞
3	3.419×10^{-3}	1.333×10^{-3}	3.471×10^{-3}	1.367×10^{-3}
5	2.672×10^{-5}	7.122×10^{-6}	2.663×10^{-5}	7.107×10^{-6}
7	1.015×10^{-7}	2.066×10^{-8}	1.023×10^{-7}	2.089×10^{-8}
9	2.106×10^{-10}	3.463×10^{-11}	2.107×10^{-10}	3.468×10^{-11}
11	2.891×10^{-13}	4.041×10^{-14}	2.962×10^{-13}	3.997×10^{-14}

TABLE 5.2: For Example 1, a comparison of L_2 and L_∞ errors of the solution $v(x,t)$

	$\alpha_1(x,t) = 0.50 + 0.2e^{-xt}$ $\alpha_2(x,t) = 0.65 + 0.2sin(xt)$		$\alpha_1(x,t) = 0.70 + 0.2e^{-xt}$ $\alpha_2(x,t) = 0.85 + 0.2sin(xt)$	
n	L_2	L_∞	L_2	L_∞
3	2.620×10^{-3}	8.788×10^{-4}	2.734×10^{-3}	9.076×10^{-4}
5	1.927×10^{-5}	4.374×10^{-6}	1.948×10^{-5}	4.518×10^{-6}
7	7.967×10^{-8}	1.233×10^{-8}	8.307×10^{-8}	1.296×10^{-8}
9	1.598×10^{-10}	2.040×10^{-11}	1.609×10^{-10}	2.059×10^{-11}
11	2.211×10^{-13}	2.354×10^{-14}	2.485×10^{-13}	4.366×10^{-14}

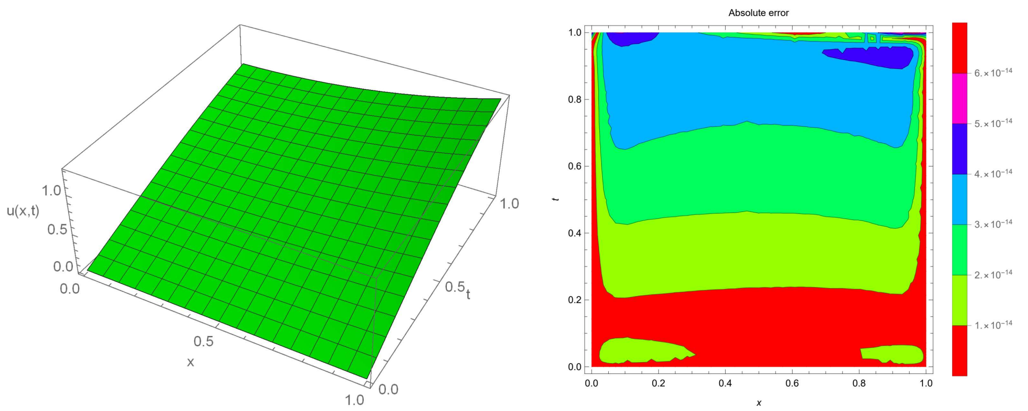

FIGURE 5.1: The graph of approximate solution and absolute error result of $u(x,t)$ for $n = 11$ with $\alpha_1(x,t) = 0.50 + 0.2e^{-xt}$ and $\alpha_2(x,t) = 0.65 + 0.2sin(xt)$ for Example 1

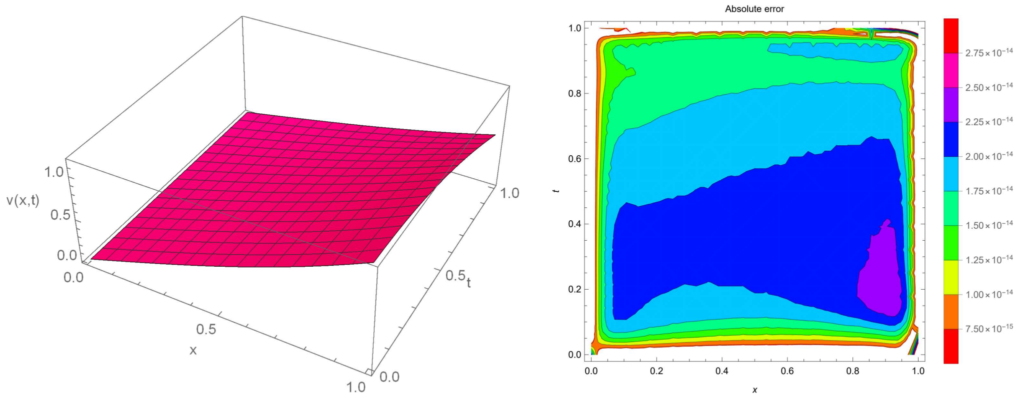

FIGURE 5.2: The graph of approximate solution and absolute error result of $v(x,t)$ for $n = 11$ with $\alpha_1(x,t) = 0.50 + 0.2e^{-xt}$ and $\alpha_2(x,t) = 0.65 + 0.2sin(xt)$ for Example 1

Example 2. Consider the system given in equation (5.1) with $d_1 = d_2 = 1,\ b_1 = b_2 = 0.5,\ r_1 = r_2 = 0.75,\ b_{11} = b_{21} = 1,$ and $b_{12} = b_{22} = 1$ and

$$\begin{aligned} f_1(x,t) =& t^{1-\alpha_1(x,t)}cos(x)E_{2,2-\alpha_1(x,t)}(-t^2) - t^{2-\alpha_1(x,t)}sin(x)E_{2,3-\alpha_1(x,t)}(-t^2) \\ & + b_1cos(x+t) + sin(x+t)\Big(d_1 - r_1 + b_{11}sin(x+t) - b_{12}cos(x+t)\Big), \end{aligned} \tag{5.85}$$

Table 5.3: For Example 2, a comparison of L_2 and L_∞ errors of the solution $u(x,t)$

	$\alpha_1(x,t) = 0.60 + 0.25sin(xt)$ $\alpha_2(x,t) = 0.80 - 0.2(x^2+t^2)$		$\alpha_1(x,t) = 0.80 + 0.25sin(xt)$ $\alpha_2(x,t) = 1 - 0.2(x^2+t^2)$	
n	L_2	L_∞	L_2	L_∞
3	4.768×10^{-3}	1.521×10^{-3}	5.003×10^{-3}	1.597×10^{-3}
5	3.860×10^{-5}	8.191×10^{-6}	4.025×10^{-5}	8.447×10^{-6}
7	1.428×10^{-7}	2.273×10^{-8}	1.487×10^{-7}	2.324×10^{-8}
9	2.996×10^{-10}	3.822×10^{-11}	3.121×10^{-10}	3.891×10^{-11}
11	4.050×10^{-13}	4.408×10^{-14}	4.039×10^{-13}	4.563×10^{-14}

Table 5.4: For Example 2, a comparison of L_2 and L_∞ errors of the solution $v(x,t)$

	$\alpha_1(x,t) = 0.60 + 0.25sin(xt)$ $\alpha_2(x,t) = 0.80 - 0.2(x^2+t^2)$		$\alpha_1(x,t) = 0.80 + 0.25sin(xt)$ $\alpha_2(x,t) = 1 - 0.2(x^2+t^2)$	
n	L_2	L_∞	L_2	L_∞
3	3.379×10^{-3}	1.216×10^{-3}	3.699×10^{-3}	1.360×10^{-3}
5	2.815×10^{-5}	6.759×10^{-6}	3.104×10^{-5}	7.526×10^{-6}
7	1.052×10^{-7}	1.878×10^{-8}	1.172×10^{-7}	2.119×10^{-8}
9	2.218×10^{-10}	3.155×10^{-11}	2.499×10^{-10}	3.654×10^{-11}
11	2.893×10^{-13}	3.486×10^{-14}	2.794×10^{-13}	4.419×10^{-14}

$$\begin{aligned} f_2(x,t) = &- t^{1-\alpha_2(x,t)} sin(x) E_{2,2-\alpha_2(x,t)}(-t^2) - t^{2-\alpha_2(x,t)} cos(x) E_{2,3-\alpha_2(x,t)}(-t^2) \\ &- b_2 sin(x+t) + cos(x+t)\Big(d_2 - r_2 + b_{21} cos(x+t) - b_{22} sin(x+t)\Big). \end{aligned} \tag{5.86}$$

ICs and BCs can be extracted from exact solution $u(x,t) = sin(x+t)$ and $v(x,t) = cos(x+t)$. The root-mean-square errors and maximum absolute errors of solution $u(x,t)$ and $v(x,t)$ for various choices of n with two values of $\alpha_1(x,t)$ and $\alpha_2(x,t)$ are exhibited in Table 5.3 and Table 5.4, respectively. From the outcomes, it is clear that the approximate solution approaches to the exact solution as the number of basis functions increases. In addition, to further illustrate the effectiveness and accuracy of the proposed method, we show the numerical results and absolute error for $u(x,t)$ in Figure 5.3 and for $v(x,t)$ in Figure 5.4.

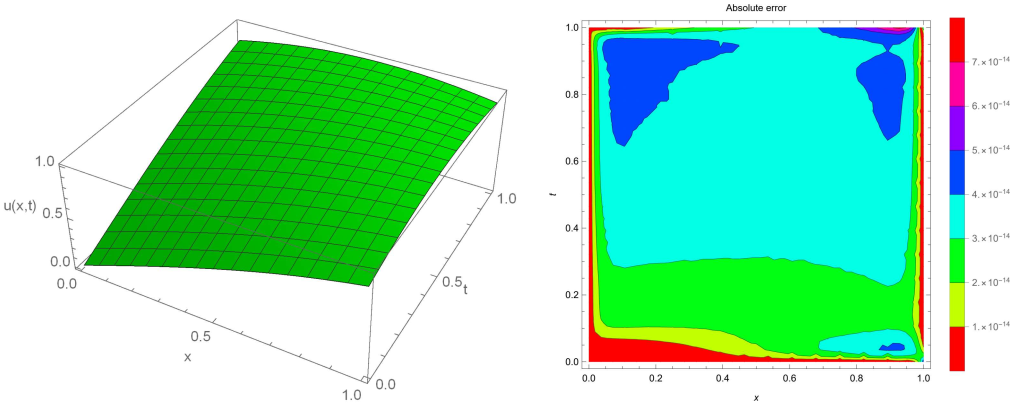

FIGURE 5.3: The graph of approximate solution and absolute error result of $u(x,t)$ for $n = 11$ with $\alpha_1(x,t) = 0.60 + 0.25 sin(xt)$ and $\alpha_2(x,t) = 0.80 - 0.2(x^2 + t^2)$ for Example 2

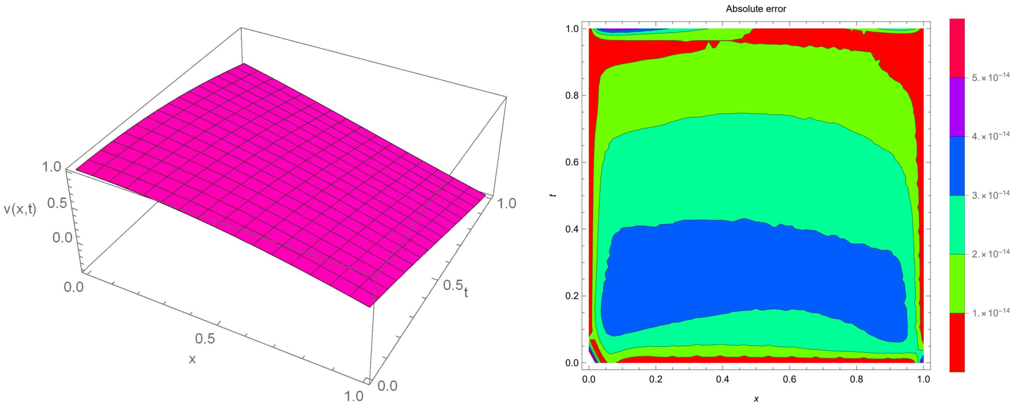

FIGURE 5.4: The graph of approximate solution and absolute error result of $v(x,t)$ for $n = 11$ with $\alpha_1(x,t) = 0.60 + 0.25 sin(xt)$ and $\alpha_2(x,t) = 0.80 - 0.2(x^2 + t^2)$ for Example 2

Example 3. By assuming $d_1 = d_2 = 0.5,\ b_1 = b_2 = 0.75,\ r_1 = r_2 = 0.35,\ b_{11} = b_{21} = 2,$ and $b_{12} = b_{22} = 1.5$ in model (5.1) with

$$
\begin{aligned}
f_1(x,t) = &- t^{1-\alpha_1(x,t)} E_{1,2-\alpha_1(x,t)}(-t) - 2d_1 + 2b_1 x \\
&- (x^2 - e^{-t})\Big(r_1 - b_{11}(x^2 - e^{-t}) + b_{12}(t^2 - e^{-x})\Big), \qquad (5.87)
\end{aligned}
$$

TABLE 5.5: For Example 3, a comparison of L_2 and L_∞ errors of the solution $u(x,t)$

	$\alpha_1(x,t) = 0.80 - 0.3e^{-xt}$ $\alpha_2(x,t) = 0.80 - 0.2sin(xe^{-t})$		$\alpha_1(x,t) = 1 - 0.3e^{-xt}$ $\alpha_2(x,t) = 1 - 0.2sin(xe^{-t})$	
n	L_2	L_∞	L_2	L_∞
3	7.951×10^{-4}	4.030×10^{-4}	1.062×10^{-3}	5.827×10^{-4}
5	4.513×10^{-6}	1.753×10^{-6}	7.026×10^{-6}	3.349×10^{-6}
7	1.481×10^{-8}	4.471×10^{-9}	2.434×10^{-8}	1.009×10^{-8}
9	2.934×10^{-11}	7.172×10^{-12}	4.860×10^{-11}	1.787×10^{-11}
11	7.729×10^{-14}	2.420×10^{-14}	1.043×10^{-13}	1.987×10^{-14}

$$f_2(x,t) = \frac{2t^{2-\alpha_2(x,t)}}{\mathit{\Gamma}(3-\alpha_2(x,t))} + e^{-x}(b_2+d_2) - (t^2 - e^{-x})\Big(r_2 - b_{21}(t^2 - e^{-x}) + b_{22}(x^2 - e^{-t})\Big). \tag{5.88}$$

With the help of exact solution $u(x,t) = x^2 - e^{-t}$ and $v(x,t) = t^2 - e^x$, initial and boundary conditions can be obtained. In this example, we use the method mentioned above for $n = 3, 5, 7, 9, 11$ and two different choices, $\alpha_1(x,t)$ and $\alpha_2(x,t)$. The graphs of the numerical solution and absolute error for $n = 11$ with $\alpha_1(x,t) = 1 - 0.3e^{-xt}$ and $\alpha_2(x,t) = 1 - 0.2sin(xe^{-t})$ are shown in Figures 5.5 and 5.6. As can be seen from these figures, the numerical result agrees well with the exact solution. Moreover, Table 5.5 illustrates the comparison of L_2 and L_∞ errors for $u(x,t)$ and Table 5.6 illustrates the same comparison for $v(x,t)$. From the obtained results, it is clear that the presented method has a higher efficiency and accuracy.

TABLE 5.6: For Example 3, a comparison of L_2 and L_∞ errors of the solution $u(x,t)$

	$\alpha_1(x,t) = 0.80 - 0.3e^{-xt}$ $\alpha_2(x,t) = 0.80 - 0.2sin(xe^{-t})$		$\alpha_1(x,t) = 1 - 0.3e^{-xt}$ $\alpha_2(x,t) = 1 - 0.2sin(xe^{-t})$	
n	L_2	L_∞	L_2	L_∞
3	3.734×10^{-3}	1.308×10^{-3}	3.839×10^{-3}	1.379×10^{-3}
5	3.174×10^{-5}	6.961×10^{-6}	3.215×10^{-5}	7.331×10^{-6}
7	1.183×10^{-7}	1.892×10^{-8}	1.190×10^{-7}	1.985×10^{-8}
9	2.487×10^{-10}	3.149×10^{-11}	2.493×10^{-10}	3.277×10^{-11}
11	4.534×10^{-13}	7.250×10^{-14}	6.821×10^{-13}	1.298×10^{-13}

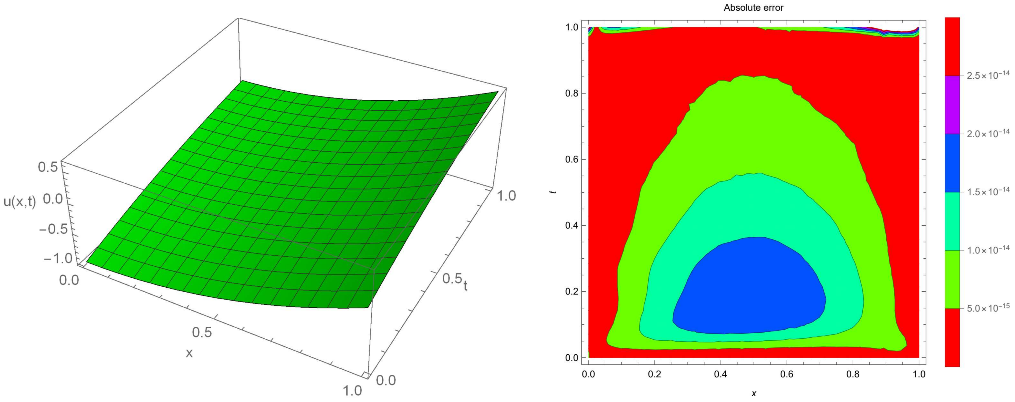

FIGURE 5.5: The graph of approximate solution and absolute error result of $u(x,t)$ for $n = 11$ with $\alpha_1(x,t) = 1 - 0.3e^{-xt}$ and $\alpha_2(x,t) = 1 - 0.2sin(xe^{-t})$ for Example 3

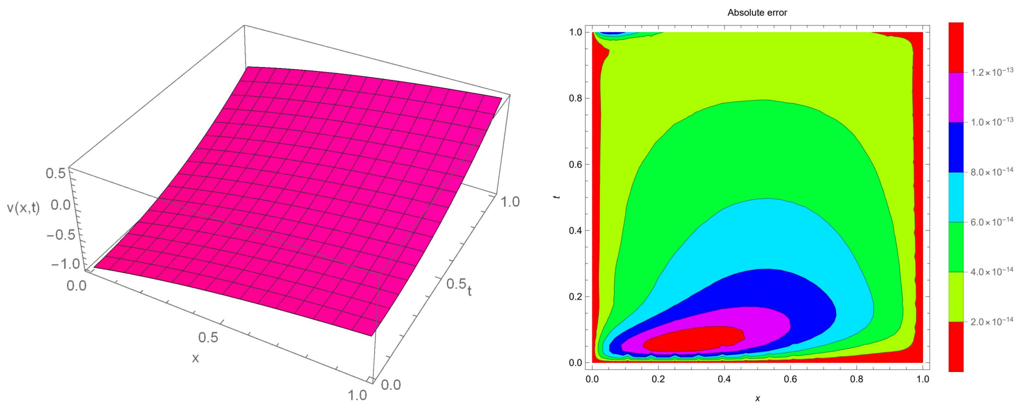

FIGURE 5.6: The graph of approximate solution and absolute error result of $v(x,t)$ for $n = 11$ with $\alpha_1(x,t) = 1 - 0.3e^{-xt}$ and $\alpha_2(x,t) = 1 - 0.2sin(xe^{-t})$ for Example 3

5.8 Conclusion

In this chapter, a numerical technique has been analysed to find the approximate solution of variable order advection-diffusion-reaction predator-prey equations in which variable order derivatives are in the Caputo sense. A discussion is given about the existence and uniqueness of a solution for the considered model. Additionally, the Ulam–Hyers stability of the solution has been taken into account. The shifted airfoil polynomials of the second kind are assumed to be the basis functions in the spectral method, which are used to approximate the solution in space and the temporal directions. The operational matrices for both integer and variable order differential operators have been derived in order to find the approximate solution to the aforementioned problem. The validity and applicability of the presented scheme are exhibited through a few examples, and it is found that the proposed method is effective and adequately accurate.

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