Chapter 2

Development of Matched Wavelets for Differential Signals

2.1 Outline

Wavelets are mathematical tool used in different areas of science and engineering. In course of going through different literature on wavelets, it is found that need of time-frequency localization of data, functions, or operators led to development of wavelets. Wavelets are classified according to their basis function characteristics. The characteristics of signal to be analyzed plays an important role in choosing a particular wavelet for analysis. Theoretically, infinite number of basis functions are possible. The wavelets are divided into various families such as orthogonal, non-orthogonal, and bi-orthogonal. Development of matched wavelet directly from the signal adds uniqueness to the application and hence provides better analysis as compared to that obtained using standard wavelets.

This chapter describes the mathematical background for the wavelets and wavelet transform. The significance of matched wavelet and theoretical motivation for developing matched wavelet is also presented in this chapter. The objective of this chapter is to develop a matched mother wavelet for analysis of different signals obtained from a power transformer. This chapter provides details of steps involved in finding optimal matched wavelet using Differential Evolution (DE) for the signal of interest. This chapter also explains the impact of number of coefficients and constraints on the shape and efficacy of designed matched wavelet. The performance of developed matched wavelet and existing wavelets has been compared to establish the efficacy of matched wavelets proposed in the thesis. The purpose of finding matched wavelet is served as its efficacy is superior compared to existing standard wavelets.

This chapter starts with the mathematical background of wavelets and wavelet transforms in relation to the shortcomings of Fourier and STFT in section 2.2. The motivation for finding matched wavelet from signal and using matched wavelet filters is discussed in section 2.3.1. The certain necessary conditions which a wavelet function needs to satisfy are described in section 2.3.2. The least square matching criteria is formulated as an optimization problem in this section. DE algorithm is used for solving this optimization problem to get optimal matched wavelet for the signal is described in section 2.4 of this chapter. In order to get feasible optimal solution, Gradient Repair Method (GRM) is used and the detailed steps of the method are given in section 2.4.2. Section 2.5 of this chapter describes the steps for finding matched wavelets using DE algorithm along with parameters used for the algorithm. The developed matched wavelets obtained for db6 wavelet, inrush signal and in-zone fault signal are discussed in section 2.6. The comparison of detection capability of developed matched wavelets to that of existing standard wavelets is discussed in section 2.6.1.

2.2 Wavelets Families

Wavelet transform of a signal decomposes signal into bricks where each brick corresponds to certain time interval containing set of frequency components within that interval. Engineers, mathematicians, scientists and physicists of the twentieth century came to know the drawback of Fourier analysis. Fourier analysis fails in reproducing signal which are non-stationary having abrupt changes. These signals are practical signals such as speech, rap of snare drum and many others observed in daily life.

In Fourier analysis, signals are decomposed into waves of different frequencies. But, Fourier transformed signal in frequency domain does not contain any information related to instant at which certain frequency component occurs in the signal. Fourier transform is a reversible transform, i.e., the time domain information can be obtained from the frequency domain (transformed signal) and vice versa. But, at an instant, either the time or frequency information is available and not both. This phenomenon of absence of time and frequency information is called **"Uncertainty Principle"** in signal processing literatures. Hence, it is impossible to simultaneously know the exact frequency and the exact time of occurrence of that frequency in a signal. So, the idea of condensing the information in signal in both, time and frequency domains lead to the idea of **"wavelets analysis"**.

A wavelet is a mathematical function which is concentrated in both time and in frequency domains. The first mother wavelet named **"Morlet Wavelet"** was designed for analyzing seismic signals. Morlet wavelet was named as **"wavelet of constant shape"** because of the fact that it maintains the same shape whether its components are dilated, compressed or shifted in time. In 1984, Morlet and Grossman in their paper used the word **"wavelet"** for the first time [73].

The property of **"Orthogonality"** in wavelet were first discussed by Meyer [74]. This property of orthogonality implies that the information captured by wavelet having specific scale and translate is completely independent of the information captured by other scale and translate [1]. A student of Meyer, Stephen Mallat linked theory of wavelets to quadrature mirror filters. Also, Mallat in collaboration with Meyer proved that the process of Multi-Resolution Analysis (MRA) is implicit in wavelets [1]. A signal when analyzed at different frequencies with different resolutions, implies multi-resolution analysis of signal. This fact that each spectral component in MRA is not equally resolved, differentiates wavelet transform from STFT. Since MRA involves different resolution for each frequency component, it provides good frequency resolution with poor time resolution at low frequencies and good time resolution with poor frequency resolution at high frequencies. Fortunately practical signals often have high frequency for short durations, hence MRA is favorable. Based on use of wavelet functions for multi resolution analysis, following communities of wavelets are there.

(i) ***Classical wavelets*** involving dyadic scales and translates of function.

(ii) ***Wavelet packets*** providing better frequency localization as compared to the classical wavelets .

(iii) ***Localized trigonometric bases*** such as sines and cosines functions used at finite intervals.

(iv) ***Multi-wavelets*** involving translates and dilates of finite number of wavelets to get basis function.

(v) ***Second generation wavelets*** are no longer continuing the idea of taking translated and dilated versions of mother and father wavelets. Rather, it utilizes the lifting schemes to obtain Discrete Wavelet Transform (DWT). In case of second generation wavelets, the non-dyadic spacing of data is taken into consideration, instead of using dyadic data set like those of first generation wavelets.

Within wavelet communities, there are different families of wavelets. Each member of family has certain common features that distinguishes one family from other. Some wavelets are suitable for continuous wavelet transform whereas, others support discrete wavelet transform. Some examples of non-orthogonal wavelets for continuous wavelet transform are Gaussian, Morlets, and Mexican Hat. Examples of orthogonal wavelets for discrete wavelet transforms are Daubechies Max- flat Wavelets, Symlets, Coiflets, Bi-orthogonal Splines wavelets, and Complex wavelets.

2.2.1 Wavelet Transforms [1]

STFT is fixed length windowed transform, which uses fixed length window to analyze a signal. STFT gives poor time resolution to low frequency components when window length is short. Also, it gives poor time resolution to high frequency components when window length is long. To overcome this problem, continuous wavelet transform (CWT) [1] can be used, where length of window function is variable. The expression for Windowed Fourier transform or STFT is given as,

$$F(\tau,\omega) = \int_{-\infty}^{\infty} f(t)w(t-\tau)\exp^{(-j\omega t)}\,dt, \tag{2.1}$$

where, *f(t)* is the signal to analyzed, $w(t-\tau)$ is the fixed length window function.

Continuous Wavelet Transform (CWT) [1]

A wavelet function $\psi(t)$ (defined in Eq. (2.26)) is used in CWT. $\psi(t)$ must obey the following admissibility condition.

$$C_\psi = \int_{0}^{\infty} (\frac{\Psi(\omega)}{\omega}) \cdot d\omega < \infty \tag{2.2}$$

Here, $\Psi(\omega)$ is the Fourier transform of $\psi(t)$. By this condition it can be established that $\Psi(\omega)$ goes to zero quickly as ω tends to 0. As a matter of fact, to guarantee that

$C_\psi < \infty$, we must impose $\Psi(0) = 0$, which results in the following condition.

$$\int_{-\infty}^{\infty} \psi(t)dt = 0 \tag{2.3}$$

A second condition imposed on wavelet function is that it should have unit energy described by,

$$\int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1 \tag{2.4}$$

The CWT of a function $f(t)$ over a time interval t, with $\psi(t)$ as a mother wavelet can be calculated as follows,

$$CWT_f^{\psi}(\tau, s) = \Psi_f^{\psi}(\tau, s) = \frac{1}{\sqrt{|s|}} \int f(t)\psi\left(\frac{t-\tau}{s}\right)dt, \tag{2.5}$$

where, parameter s and τ corresponds to scale and translate respectively.

Discrete Wavelet Transform (DWT) [1]

Discrete wavelet transform can be derived from the notion of Continuous wavelet transform in the following manner. Instead of varying the translate and scale parameters continuously, the variations are taken at discrete steps. By putting the scale parameter $s = 2^{-j}$, $j\varepsilon Z$, $k\varepsilon[1, N]$, the DWT of signal $f(t)$ can be written as,

$$\Psi_f^{j,k} = \int f(t)\, 2^{j/2}\psi(2^j t - k)\, dt \tag{2.6}$$

The wavelets obtained at different values of integers j and k yields an orthogonal basis. The so called dyadic wavelets can be written as,

$$\Psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k) \tag{2.7}$$

The mother wavelet is defined by $j = 0$ and $k = 0$, giving $\Psi_{0,0}(t) = \psi(t)$.

2.2.2 Refinement Relation

Multi resolution analysis results into an array of closed subspaces of $L^2(\Re)$, satisfying following properties given in [75] and [76].

1. For all j, array, V_j, is nested which means

$$V_j \subset V_{j+1} \tag{2.8}$$

2. The dyadic spaces can be expressed as:-

$$s(1) \in V_j \Longleftrightarrow s(2) \in V_{j+1} \Longleftrightarrow s(2^{-j}) \in V_0 \tag{2.9}$$

3. Union of the formed spaces are dense, such that for all s in $L^2(\Re)$ it can be given by

$$\lim_{j \longrightarrow -\infty} \parallel s - U_j^0 s \parallel_{L^2} = 0 \tag{2.10}$$

where U_j^0 is the orthonormal projection onto V_j.

4. A null function is obtained when intersection of spaces are taken i.e.

$$\lim_{j \longrightarrow -\infty} \parallel U_j^0 s \parallel_{L^2} = 0 \tag{2.11}$$

5. A function $\phi \in V_0$ exists such that the family

$$\phi(: -k), k \in Z \tag{2.12}$$

is a Riesz basis of V_0 for all k.

A family p_k, $k \in Z$, by definition, is a Riesz basis of Hilbert space H_s, *iff* it spans H_s. That means, finite linear combination of the p_k are densely populated in H_s, and it is assumed that there exist $0 < M_1 \leq M_2$ for all finitely supported sequences, c_k. We have,

$$M_1 \sum_k |c_k|^2 \leq \parallel \sum c_k p_k \parallel_{H_s}^2 \leq M_2 \sum_k \mid c_k \mid^2 \tag{2.13}$$

The equation (2.13) represents the property of "stability" of the basis expansions with respect to the coordinates. The important details regarding Riesz bases can be concluded in the following manner: (i) the series $\sum_{k \in Z} c_k p_k$ convergence in L^2 is unconditional *iff* $\sum_{k \in Z} |c_k|^2$ is finite (ii) with $(c_k)_{k \in Z}$ in $l^2 Z$, any $c \in H_s$ can be decomposed uniquely according to $c = \sum_{k \in Z} c_k p_k$ and (iii) a unique bi-orthogonal Riesz basis $\tilde{p_k}$, $k \in Z$, such that $\langle p_k, \tilde{p_l} \rangle = \delta_{k,l}$ exists and coordinates of c's in the bases are given by $c_k = \langle c, \tilde{p_k} \rangle$. Thus, for all $c \in H_s$ we have,

$$c = \sum_{k \in Z} \langle c, \tilde{p_k} \rangle p_k = \sum_{k \in Z} \langle c, p_k \rangle \tilde{p_k} \tag{2.14}$$

Consequently, the Riesz basis property can be written as:

$$M_1 \parallel c \parallel_{H_s}^2 \leq \sum_{c \in Z} |\langle c, p_k \rangle|^2 \leq M_2 \parallel c \parallel_{H_s}^2 \tag{2.15}$$

The sequence $(p_k)_{k \in Z}$ satisfying equation(2.15) are called frames. As per equation (2.15) the vector c can be reconstructed from the inner product $\langle c, p_k \rangle$. However, the reconstruction from $\langle c, p_k \rangle$ is not unique, because p_k may have redundancy.
On the basis of equation (2.15) expression,

$$\phi_{j,k} = 2^{j/2}\phi(2^j - k), k\epsilon Z \tag{2.16}$$

forms a Riesz basis for V_j. V_j is such that $s_j \in V_j$ can be given by $s_j = 2^{j/2}s_0(2^j)$. As scaled versions of ϕ are used for generating space V_j, ϕ is called scaling function.

As $V_0 \subset V_1$, the scaling function ϕ is expressed as,

$$\phi(t) = \sum_{k=0}^{N-1} h(k)\sqrt{2}\phi(2t - k) \tag{2.17}$$

known as refinement equation.
Here $\phi(2t - k)$ are basis of V_1 and $(h_k)_{k \in Z}$, known as refinement coefficient is in ℓ^2(Z). The above equation, (2.17), is also called as two-scale difference equation.

2.3 Matched Wavelet Theory

For designing a wavelet directly from the signal of interest, an algorithm was developed by Chapa [2]. Such a wavelet matches the signal in shape and hence is called a matched wavelet. Deriving matched wavelet directly from signal of interest has a specific advantage in signal detection/discrimination. The motivation for using matched wavelet for the present purpose is discussed in the next section.

2.3.1 Motivation and Mathematical Foundation

Frequency response of matched filter has the same shape as the frequency spectrum of the signal. When wavelet transform is applied to multi-resolution analyses of signals, theoretically it produces the output similar to those of matched filters. In fact, various wavelet families such as Daubechies, Coiflets, and Morlets were designed or synthesized to capture certain characteristics or phenomena in signal processing applications. However, in later literatures the wavelets were not synthesized according to application but were taken from the existing wavelets families. Designing a matched mother wavelet from a waveform is equivalent to obtaining a basis function which captures characteristic features of the waveform. The matched filter concept is employed in wavelet decomposition to justify above mentioned approach. For an Euclidean space $L^2(\Re)$, a function $\psi \in L^2(\Re)$ is called an orthonormal wavelet if the family $\psi_{j,k}$ is an orthonormal basis of $L^2(\Re)$, which gives [1],

$$\langle \psi_{j,k}, \psi_{l,m} \rangle = \int_{j=-\infty}^{\infty} \int_{l=-\infty}^{\infty} 2^{\frac{j}{2}} \psi(2^j x_i - k) 2^{\frac{l}{2}} \psi(2^l x_i - m). \tag{2.18}$$

Since integral multiplication of orthonormal wavelet bases is equal to 1, therefore above equation can be rewritten as,

$$\langle \psi_{j,k}, \psi_{l,m} \rangle = \delta_{j,l} * \delta_{k,m}, \tag{2.19}$$

where, $\delta_{j,k} = 1$ for $j = k$, otherwise zero. Also every $f \in L^2(\Re)$ can be expressed at different instants x_i as,

$$f(x_i) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} d_k^j 2^{\frac{j}{2}} \psi(2^j x_i - k), \tag{2.20}$$

where, $\psi \in L^2(\Re)$ is an orthonormal wavelet and $i = 1, 2, ..N_s$, N_s is number of samples of wavelet function, and

$$d_k^j = \langle f, \psi_{j,k} \rangle = \int_{j=-\infty}^{\infty} f(x_i) 2^{\frac{j}{2}} \psi(2^j x_i - k) dx. \tag{2.21}$$

The projection equation for the detail function d_k^j, can be rewritten in frequency domain using Parseval's identity as follows.

$$d_k^j = \langle f, \psi_{j,k} \rangle = \langle F(\omega), \Psi_{j,k}(2^j \omega) \rangle, \tag{2.22}$$

where $\Psi_{j,k}(2^j\omega) = 2^{\frac{j}{2}} \exp(-i 2^j k \omega) \Psi(2^j \omega)$, is the Fourier transform of $\psi_{j,k}$. The energy of d_k^j at a particular scale, j_0, and translation k_0 is given by its squared magnitude, i.e.,

$$|d_{k_0}^{j_0}|^2 = |\langle F(\omega), \Psi_{j_0,k_0}(2^{j_0}\omega) \rangle|^2. \tag{2.23}$$

Applying Schwarz inequality to the right side of equation gives,

$$|\langle F(\omega), \Psi_{j_0,k_0}(2^{j_0}\omega) \rangle|^2 \leq \langle F(\omega), F(\omega) \rangle \langle \Psi_{j_0,k_0}(2^{j_0}\omega), \Psi_{j_0,k_0}(2^{j_0}\omega) \rangle. \tag{2.24}$$

In (2.24), equality holds for $F(\omega) = K\Psi_{j_0,k_0}(2^{j_0}\omega)$. Therefore, $|d_{k_0}^{j_0}|^2$ is maximized when $f(x) = K\psi_{j_0,k_0}$. So, to maximize the output of matched filter bank, it is necessary to design a wavelet that matches in shape to signal of interest. In this context the matching is emphasized as far as the values of coefficients are concerned. Wavelet transform of a signal gives maximum outcome (output/response) when the wavelet and signal are matching. It does not mean that for a given signal of interest the obtained matched wavelets will necessarily match the signal in visual sense of the shape. Rather the wavelet obtained based on this concept would yield the highest output and such a wavelet is so called matching wavelet.

In certain cases, the matched wavelet may match in shape in visual sense quite closely as in Fig. 2.1 [2]. Whereas in some other cases the matching wavelets could have an oscillatory shape as in Fig. 2.2 [3]. However, it is worth mentioning that given the number of coefficients and the constraints, the matching wavelet would provide maximum output/response for a matching signal.

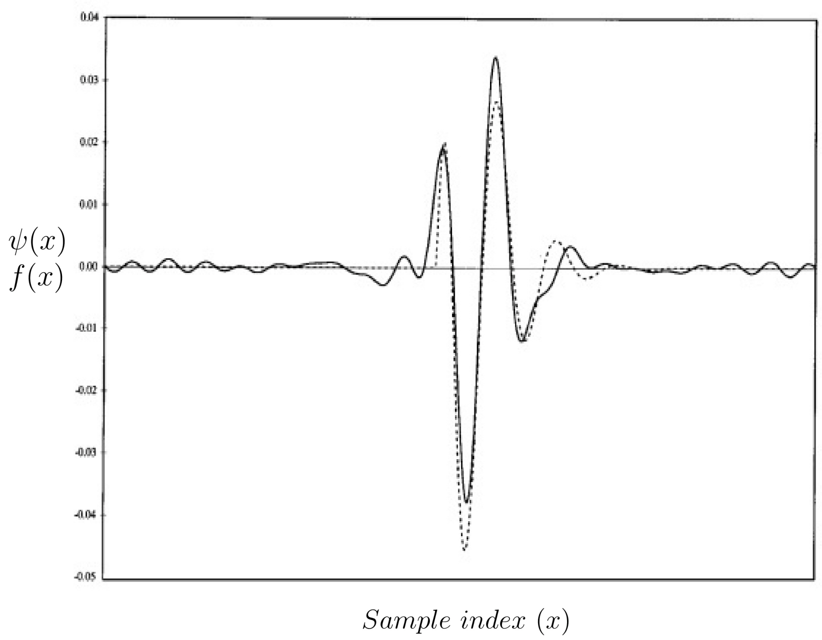

Figure 2.1: Matched wavelet (solid line) versus desired signal(dotted) from ref. [2]. Y-axis: Wavelet coefficient values/signal, X-axis: Sample index.

Rewriting $F(\omega)$ in terms of amplitude and phase gives

$$|F(\omega)| \exp(i\theta_F(\omega)) = K2^{j_0/2}|\Psi(2^{j_0}\omega)| \exp(i\theta_\Psi(2^{j_0}\omega) - 2^{j_0}\omega k_0) \tag{2.25}$$

where, $\theta_F(\omega)$ and $\theta_\Psi(\omega)$ are the phases of $F(\omega)$ and $\Psi(\omega)$, respectively. If $f(x)$ matches exactly with an orthonormal wavelet, then $d_k^j = \delta_{k,k_0}.\delta_{j,j_0}$.

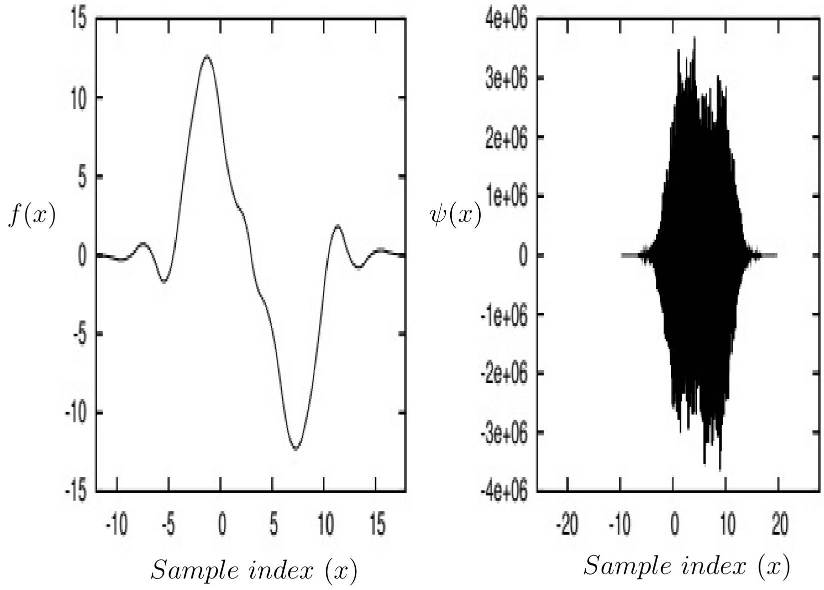

Figure 2.2: A function close to a ramp function (left) and its matched wavelet(right) taken from ref. [3]. Y-axis: Wavelet coefficient values (right)/signal (left), X-axis: Sample index.

2.3.2 Filter Coefficients of Matched Wavelet : Problem Formulation

As discussed earlier in section 2.2.1, for performing wavelet analysis of a signal, the inner product of dilated and translated mother wavelet and signal is taken over specified time interval. Thus, mother wavelet basis selected for purpose of analysis depends primarily on the application at hand. In previous literatures, mother wavelet were taken from the set of libraries of predefined standard wavelet. The correlation of the wavelet with the signal was performed. The selection was made on the basis of minimization of the error function between the signal and the wavelet. In present work matched mother wavelet has been developed using refinement relation in the following manner.

A mother wavelet function and, it's associated scaling function can be defined in the following manner [1].

$$\psi(t) = \sum_{k=0}^{N-1} g(k)\sqrt{2}\phi(2t-k). \tag{2.26}$$

$$\phi(t) = \sum_{k=0}^{N-1} h(k)\sqrt{2}\phi(2t-k). \tag{2.27}$$

Equation (2.27) is also known as refinement relation. In Eq. (2.26), $g(k)$ acts as high pass filter coefficient and in Eq. (2.27), $h(k)$ acts as low pass filter coefficient. These coefficients decide the shape of scaling and wavelet functions and the shape of wavelet function decides its suitability for a given application. Thus, the objective function is given by the following expression.

$$F = \sum [f(t) - \psi(t)]^2, \tag{2.28}$$

where, $f(t)$ is the signal and $\psi(t)$ is the matched wavelet.

In order to get optimal filter coefficients, restrictions are applied on scaling and wavelet functions, in turn, resulting into restrictions on wavelet coefficients. The parameters of wavelet function, $\psi(t)$, are number of wavelet coefficients, N, and number of constraints used to build it. The constraints of the problem are restrictions applied on wavelet function and scaling function. The restrictions or constraints on the wavelet coefficients are discussed as follows.

Condition 1 - Unit area under scaling function: It can be represented as,

$$\int \phi(t)dt = 1. \tag{2.29}$$

This implies,

$$\sum_{k=0}^{N-1} h(k) = \sqrt{2}, \tag{2.30}$$

or

$$h(0) + h(1) + h(2) + ... + h(N-1) = \sqrt{2}. \tag{2.31}$$

Condition 2 - Orthonormality of translates of scaling function: Translates of scaling function are orthonormal. Hence,

$$\int \phi(t)\phi(t-k)dt = \delta_{0,k}. \tag{2.32}$$

Similarly,

$$\int \phi(t)\phi(t+k)dt = \delta_{0,k}. \tag{2.33}$$

This implies,

$$\sum_{l=0}^{N-1} h(l)h(l-2k) = \sum_{l=0}^{N-1} h(l)h(l+2k) = \delta_{0,k}. \tag{2.34}$$

For k=0, i.e., square normalization condition

$$\sum_{l=0}^{N-1} h^2(l) = 1, \quad N \in Z. \tag{2.35}$$

For $k \neq 0$, orthonormal condition,

$$\sum_{l=0}^{N-1} h(l)h(l-2k) = \sum_{l=0}^{N-1} h(l)h(l+2k) = 0. \tag{2.36}$$

Here, k is a positive integer and condition in (2.36) is called double shift orthogonality. When $\phi(t)$ is shifted by one unit, $\phi(2t)$ shifts by 2 units in its own scale.

Condition 3 - Orthonormality of scaling and wavelet functions: This condition can be described as,

$$\int \phi(t)\psi(t)dt = 0, \tag{2.37}$$

which implies that

$$g(k) = (-1)^{N-k-1}h(N-k-1). \tag{2.38}$$

This relation implies that to get the coefficient series $g(k)$, just reverse the coefficient series $h(k)$ and change the sign of the coefficients in alternate positions.

Condition 4 - Approximation condition (smoothness): This condition is described as,

$$\int t^p\psi(t)dt = 0. \tag{2.39}$$

For p=0,

$$\sum_k (-1)^k h(k) = 0. \tag{2.40}$$

For general integer powers $p > 0$,

$$\sum_k (-1)^k k^p h(k) = 0, \tag{2.41}$$

where, p = degree of moment.

Equation (2.35) gives the condition that the sum of squares of all wavelet coefficients must be equal to 1. The constraint equation (2.35), results from the fact that integer translates of scaling function are orthonormal to each other. The condition given by

(2.36), called as double shift orthogonality, ensures that sum of product of doubly shifted wavelet coefficients is equal to zero.

The constraints (2.31), (2.35) and (2.36) are the mandatory conditions for a function to be wavelet. But the constraints, for various values of p, described by (2.41) depend on choice of smoothness in the shape of wavelet. For a specified number of wavelet coefficients, the set of constraints can be different resulting in different wavelets. First the number of coefficients and number of constraints are chosen and accordingly a wavelet function $\psi(t)$ is found.

2.4 Differential Evolution

In many practical optimization problems, objective functions are non-differentiable, non-continuous, and non-linear. Evolutionary algorithms such as Genetic Algorithms, Evolutionary Programming, Evolutionary Strategies, and DE, are useful for handling such optimization problems. These algorithms are inspired by nature. DE gained popularity among researchers because it is simple to code; has fewer number of control parameters; is known for convergence speed and robustness [77]. DE has lower space complexity compared to most of the competitive real parameter optimization methods. The lower space complexity favors DE for application to optimization problems having large number of variables.

The task of optimization algorithm is to find a parameter vector $X^\star$ which minimizes the objective function $f(X)$ ($f : \zeta \subseteq R^D \rightarrow R$) such that $f(\overrightarrow{X^\star}) < f(X)$ for all $X \in \zeta$, where ζ is a non-empty domain of the search.

DE is a search scheme or an algorithm which uses set of variables called parameter vectors as population over several iterations or so called generations. In general, DE generates new parameter vectors by adding the weighted difference between two parameter vectors to a third parameter vector. If the resulting new parameter vector yields a lower objective function value than old one, the newly generated parameter vector replaces the old one. The various processes involved in DE algorithm are depicted in Fig. 2.3.

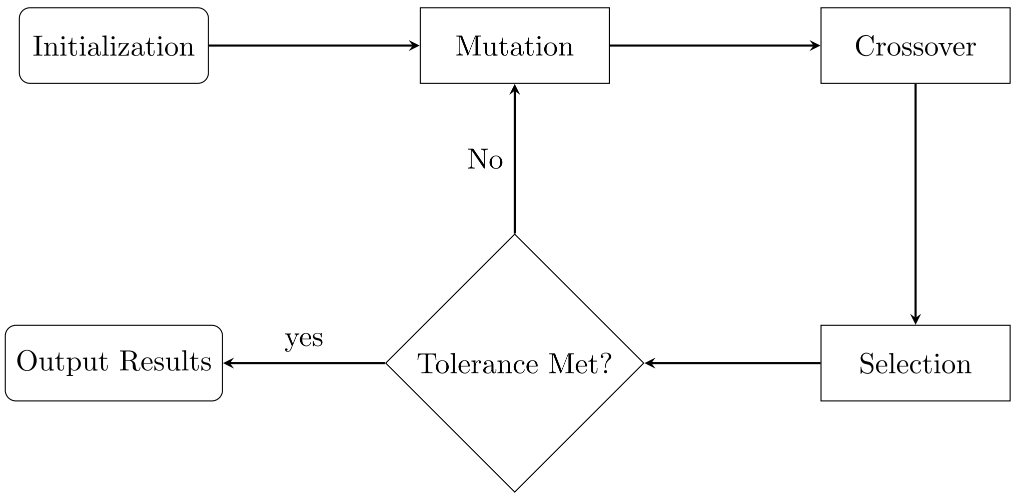

Figure 2.3: Sequence of operations in DE

2.4.1 DE Algorithm

In this work Differential Evolution (DE) algorithm is used to get an optimum matched wavelet in least-square sense. The details of the DE processes are described in the following sections.

Initialization

DE randomly initializes a population of NP parameter vectors. Parameter vectors are D dimensional candidate solution vectors. The subsequent generations of DE are denoted by $G = 0, 1, 2, ...G_{max}$. Hence i^{th} parameter vector for a given generation, G, is denoted by.

$$X_i^G = [x_{i1}^G, x_{i2}^G, x_{i3}^G ... x_{iD}^G]. \tag{2.42}$$

The initial population (at G = 0) is generated in the range of upper and lower bound for each of the variables in the parameter vectors. Hence j^{th} element of the i^{th} vector can be initialized as,

$$x_{ij}^0 = x_{j\ min} + rand[0, 1](x_{j\ max} - x_{j\ min}). \tag{2.43}$$

where $x_{j\ max}$ and $x_{j\ min}$ are the upper and lower bounds for the variable X_j. The values of x_{ij}^0 are generated randomly in the given bounds.

Mutation

The mutation operation in DE algorithm results in a mutant vector V_i^{G+1}. The mutation operation is performed using three parameter vectors taken from the current population. These three participating parameter vectors are called as, target vector (X_{r1}^G) and donor vectors (X_{r2}^G and X_{r3}^G). The mutant vector is obtained using following operations on the these three vectors.

$$V_i^{G+1} = X_{r1}^G + F(X_{r2}^G - X_{r3}^G). \tag{2.44}$$

For a given index i, r_1, r_2, and r_3 are random such that $r_1, r_2, r_3, i \in \{1, 2, 3....NP\}$ are mutually exclusive. Thus, the algorithm calls for having $NP \geq 4$. $F \in [0, 2]$ is a constant factor.

Crossover

The mutant vector obtained after mutation operation undergoes a crossover operation. The crossover operation is performed using mutant vector V_i^{G+1} and a target vector X_i^G to generate a trial vector, U_i^G, in the following manner.

$$u_{j,i}^{G+1} = \begin{cases} v_{j,i}^{G+1} & (r(j) \leq CR) \ or \ (j = ridx(i)), \\ x_{j,i}^G & (r(j) > CR) \ and \ (j \neq ridx(i)). \end{cases} \tag{2.45}$$

Here, $r(j)$ is the random number from uniform distribution between 0 and 1. CR is the crossover probability $\in$ [0,1]. To avoid the condition, wherein all the evaluations of $r(j) > CR$ lead to $U_i^G = x_i^G$ and fixed random index $ridx(i)$ is generated to have at least one of the elements of U_i^G different from X_i^G. Fig. 2.4 depicts the scheme of crossover procedure for the target vector X_i^G, mutant vector V_i^{G+1}, and D = 5. Initially the trial vector $U_{i,G+1}$ is equal to X_i^G and as the crossover process is performed, the elements of target vector X_i^G replace the elements of mutant vector V_i^{G+1}.

Selection

The selection operation decides that a particular trial vector would pass on to the next generation or not. This decision is performed in the following manner.

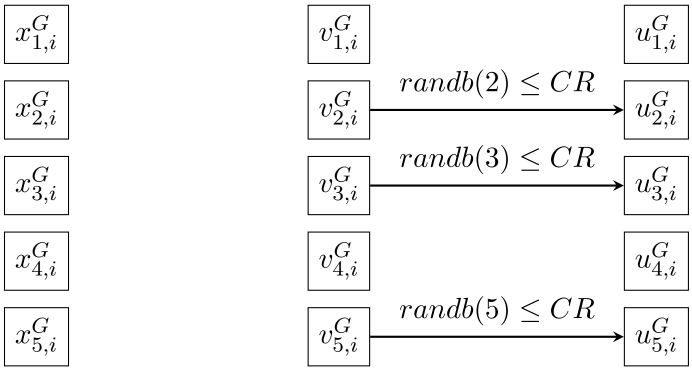

Figure 2.4: Illustration of crossover process for D = 5

$$X_i^{G+1} = \begin{cases} U_i^{G+1} & f(U_i^{G+1}) \leq f(X_i^G), \\ X_i^G & f(U_i^{G+1}) > f(X_i^G). \end{cases} \tag{2.46}$$

Here, $f(\vec{X})$ is the value of objective function for given $\vec{X}$.

2.4.2 Repairing Infeasible Solution in DE

Handling constraints in solving constrained optimization problem through evolutionary methods has been a concern. The solution may be in infeasible region or there may be constraint(s) violation. Infeasible solutions arise during the DE operations also. One way to resolve this problem is to completely remove such infeasible solution and replace the infeasible solution with a randomly generated feasible solution. The process is not only tedious but wasteful too as the removed infeasible solution may be good candidate solution with very small amount of violation [78]. The other way to deal with infeasible solutions is to repair them to make them feasible. In the present context all the constraints are of equality type, there is very high chance that solution generated through DE operation may be infeasible.

In the GRM, the infeasible solutions are repaired, and directed towards the feasible region with help of gradient information derived from the constraint set. Steps involved in GRM are as follows.

1. The solution obtained from the DE algorithm i.e. wavelet coefficients in present case, is put in constraints and checked whether these constraint are satisfied. If all constraints are satisfied, it means the obtained solution is a feasible solution. But if not, then few

more calculations are done to upgrade this solution to feasible region.

2. Let vector V consists of vectors of inequality constraints (g) and equality constraints (h) for the problem.

$$V = [\frac{g_{M*1}}{h_{N*1}}]_{M+N*1} \tag{2.47}$$

3. The gradient is calculated as ∇V, which gives that how far the present solution is violated for given constraints.

4. $\nabla_x V$ is calculated with respect to each variable and can be evaluated as follows.

$$\nabla_x V = [\frac{\nabla_x g}{\nabla_x h}]_{M+N*I}, \tag{2.48}$$

where, $\nabla_x V^+$ is approximated inverse of gradient information $\nabla_x V$.

5. Then the new feasible solution is calculated with the help of equation,

$$x_{t+1} = x_t + \nabla_x V * \nabla_V. \tag{2.49}$$

6. The constraints are checked for this new solution and this is repeated until the solution is obtained in feasible region.

2.5 Optimal Matched Wavelets using DE

With the help of DE algorithm the objective function which is least mean square error between the matched wavelet and signal waveform has been minimized and the optimal matched wavelet has been found for a given number of coefficient and constraints. Different numbers of coefficients investigated for generating matched wavelets were 6, 8, 10, 12, and 16. The number of coefficient corresponds to the dimension of population vector(D) in DE. The values for the various DE parameters used in this study are given in Table. 2.1.

Flowchart given Fig. 2.5 depicts the process of finding the matched wavelet coefficients.

Table 2.1: DE Parameter Values

Parameters	NP ($D < 10$)	NP ($D \geq 10$)	Coefficients(D)	F	CR
Values	80	400	6, 8, 10, 12, 16	0.7	0.9

To initiate the optimization process, the first step is to read variables, constraints, constants (D and NP). Length (L) of wavelet is decided by number of wavelet coefficients chosen (i.e. value of D). Once L is decided, then equidistant L data points are selected from signal for which matched wavelet is to be designed. Third step is to initialize population in uniformly distributed search space. Processes of DE such as mutation, crossover, and selection are run after initializing population vector. The details of DE processes are described in section 2.4.1. The feasibility of the solution is checked after the selection process. If the solution is not feasible, then it is repaired using GRM. For details of GRM refer to section 2.4.2. If the solution is feasible and also stopping criteria is met the obtained solution is taken as optimal solution. The stopping criteria used in the present work is fixed on number of iterations(200) for which the minimum objective function value remains unchanged.

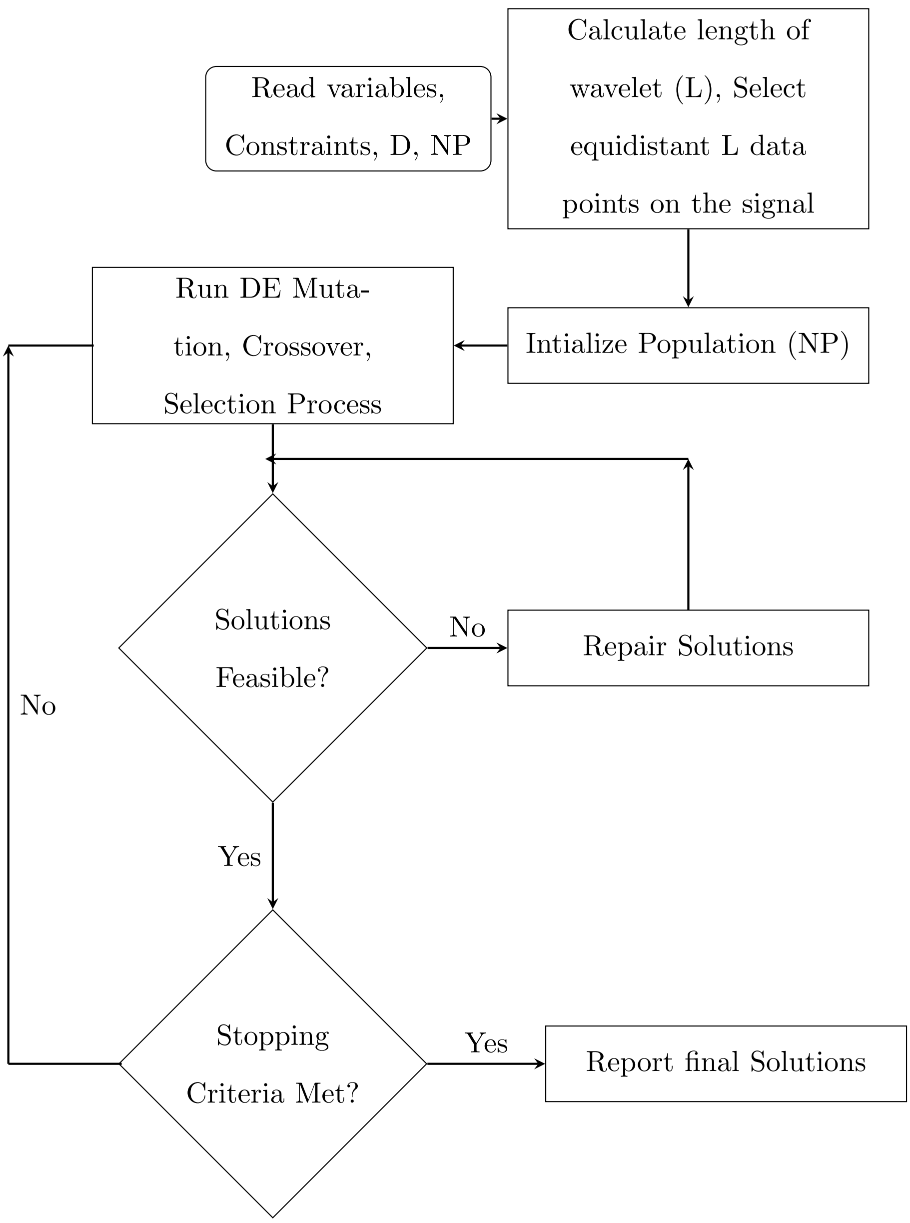

Figure 2.5: Flowchart for DE algorithm

2.6 Results and Discussion

In this section, we discuss various examples of matched wavelets that are obtained for a particular signal. As discussed earlier for finding matching wavelet, there are two significant parameters, number of wavelet coefficient (D) and number of constraints. For a particular signal, quite a large number of wavelet can be designed by selecting different number of wavelet coefficients and constraints. Also, for a specified number of wavelet coefficient one may vary the number of constraints. It is to be noted that, among the constraints there are some necessary constraints (refer to sec.2.3.2) along with constraints which are considered for smoothening of the wavelet.

The signal waveform i.e $f(t)$ and the wavelet i.e $\psi(t)$ are matched in least square sense. Hence the objective function is expressed as

$$F = \sum (f(t) - \psi(t))^2 \tag{2.50}$$

In order to test the algorithm of designing matched wavelet, the matched wavelet is first designed for known Daubechies wavelet. Daubechies wavelet of 6 coefficients (db6) and 6 constraints is taken as signal. The matched wavelet is designed using N = 6, and 6 constraints. Fig.2.6 shows the obtained matched wavelet is the exact replica of the Daubechies wavelet (signal)(db6).

Also, a matched wavelet with 5 coefficients and 2 constraints is designed for chosen signal, i.e. db6. The resulting matched wavelet along with the signal (db6) is depicted in Fig. 2.7. It is observed from the figure that the matched wavelet is not exact replica of the db6 signal as it is in Fig. 2.6. The reason behind this is that the number of coefficients used for designing matched wavelet is less than the coefficients of actual db6 signal with 6 coefficients and 6 constraints. And also due to minimum number (i.e. 2) of constraints used, the matched wavelet is not that smooth. From the above discussion, the method of generating matching wavelet is validated in Fig. 2.6.

Some of the matched wavelets designed for specified number of coefficients and constraints for representative differential waveforms (inrush and fault) are shown in Fig. 2.8 and Fig. 2.9 respectively. These representative differential waveforms for inrush and in-zone fault were obtained through simulations. The inrush waveform corresponds to an unloaded transformer and the in-zone fault waveform corresponds to an L-G fault. The matched wavelets shown in Fig. 2.8 are obtained for inrush waveforms of transformers

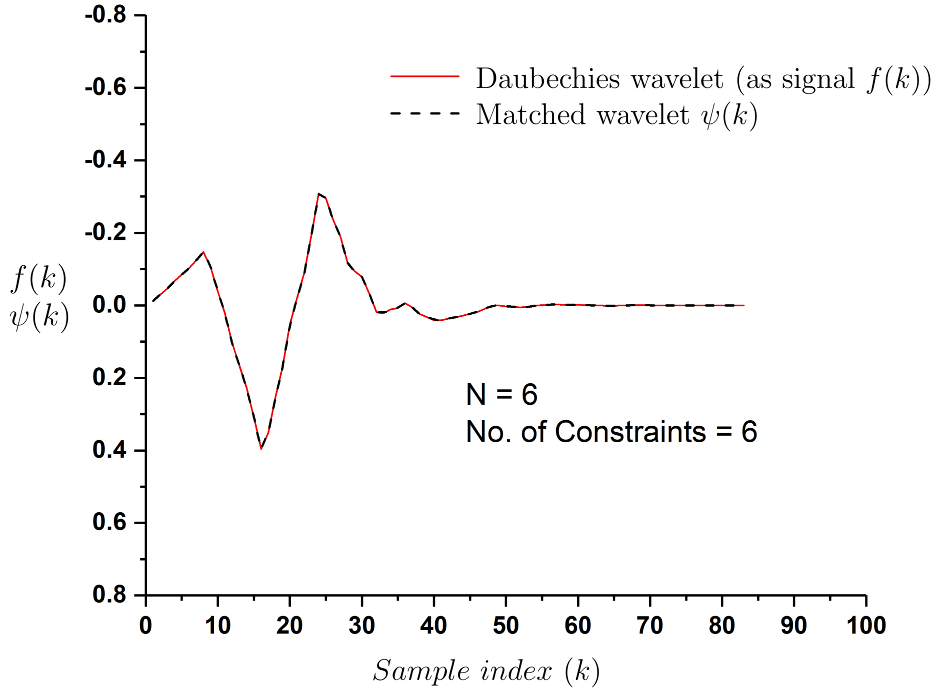

Figure 2.6: Signal(db6) and its corresponding matched wavelet for N = 6

for different number of coefficients such as 6, 8, 10, 12, and 16. Some of the important observations with reference to Fig. 2.8 can be enumerated as follows.

1. Matched wavelets obtained for each of the coefficients i.e. N= 6, 8, 10 ,12 is quite smooth. It can be observed that in each case, number of constraints is much greater than the two necessary constraints. Constraints in addition to the two necessary constraints, make the matched wavelet smooth(refer sec. 2.3.2).

2. The inrush signal is a unipolar signal. as the matched wavelet must satisfy the condition that the area under the wavelet function must be zero, the matched wavelet is bipolar in nature.

The matched wavelets shown in Fig. 2.9 are obtained for a fault waveform of transformer for different number of coefficients like 8, 10, 12, 16. The important points observed from Fig. 2.9 is as follows:

1. The obtained matched wavelet is oscillatory in nature. The matched wavelet is designed with only necessary constraints in this case.

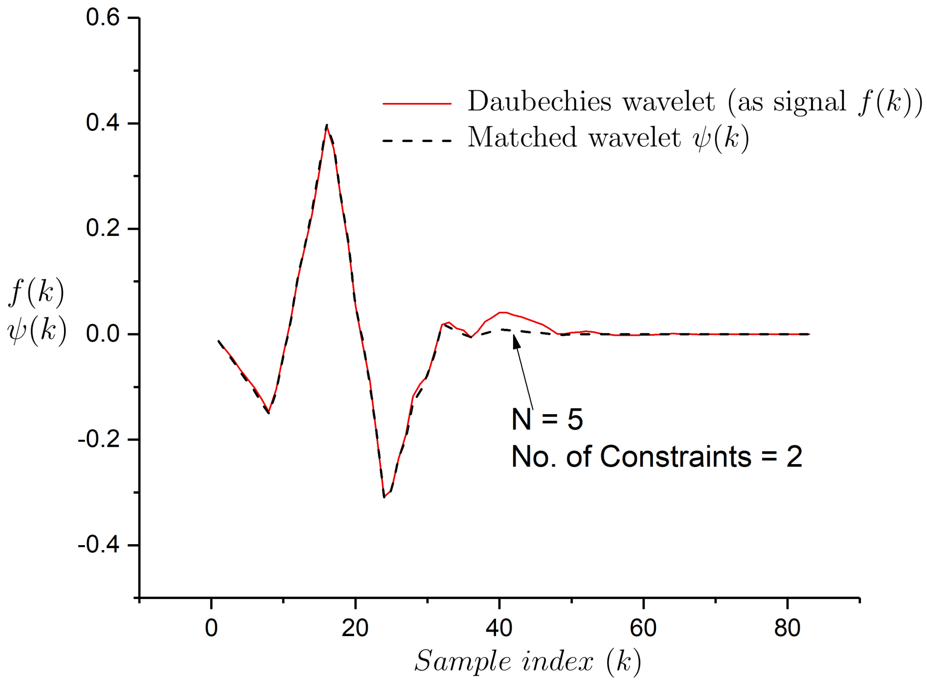

Figure 2.7: Signal(db6) and its corresponding matched wavelet for N = 5 and 2 constraints

2. The matched wavelet for fault waveform is following trajectory of the fault signal.
3. By increasing the number of constraints we can get a smoother wavelet but in that case wavelet looses similarity to the fault waveform.

2.6.1 Comparison with Existing Wavelets

The matched inrush and fault wavelets discussed above and utilized in the present work for detection are shown in Fig. 2.8(a) and Fig. 2.9(a). All the test results and analysis reported in this work pertain to inrush wavelet of Fig. 2.8(a) and fault wavelet of Fig. 2.9(a). The efficacy with which the developed matched wavelet captures the characteristics of waveforms of interest are depicted in Fig. 2.10. Fig. 2.10 (a) shows normalized differential current waveforms for representative inrush and in-zone fault(L-G) conditions of a transformer. The responses of db4 [79] [80] [81], db5 [82] and db7 [53] to an inrush waveform, are depicted in Figs. 2.10(c), 2.10(d), 2.10(e) respectively. The response of inrush wavelet to the matched wavelet is depicted in Fig. 2.10(b). The response of matched inrush wavelet to the representative inrush and in-zone fault signal is depicted in Fig. 2.10

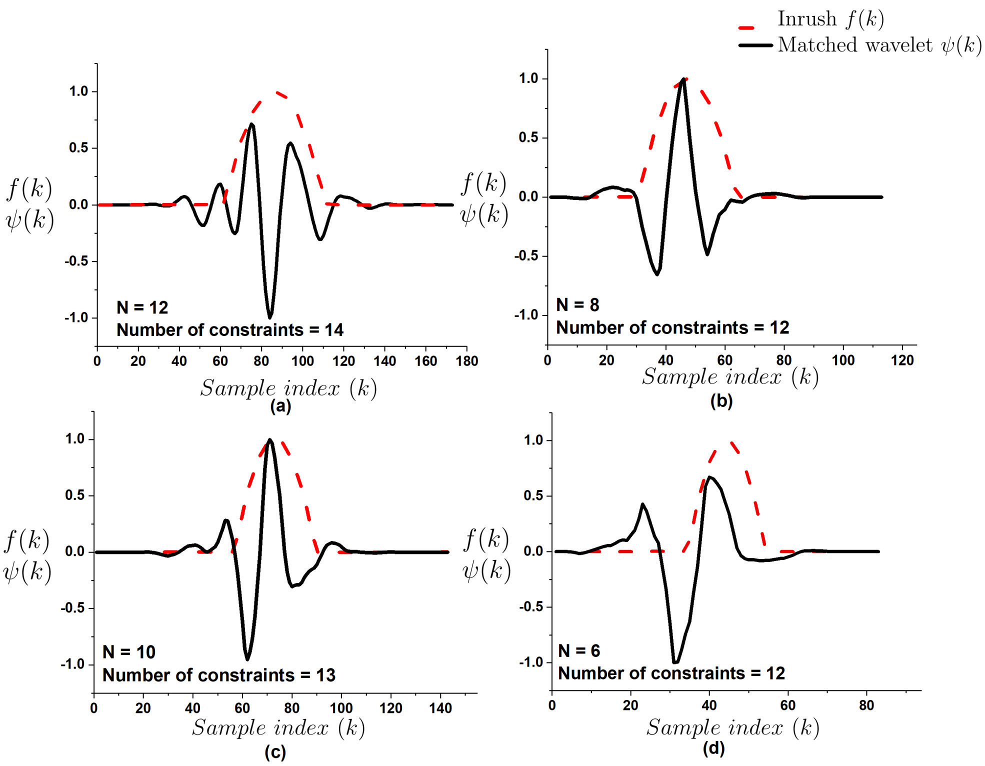

Figure 2.8: Inrush waveform and matched inrush wavelets

(b). It is observed that the matched inrush wavelet gives highest discriminating response, clearly differentiating inrush waveforms from in-zone fault waveform as compared to other wavelets. Also, the matched inrush wavelet gives lowest response (almost negligible) for in-zone fault waveform when compared to other wavelets. It is evident that the matched inrush wavelet out-performs db4, db5 and db7 in discriminating between fault and inrush waveforms. The comparison is made to establish the better discriminating capability of the matched wavelets over existing ones and does not suggest that algorithm developed in the literature are redundant.

It is to be noted that, the wavelets db4, db5 and db7 used in the literature rely on higher level coefficient in their algorithm to detect the inrush waveform. Computing higher level coefficients consumes time and higher level coefficients are also prone to noise in the waveforms.

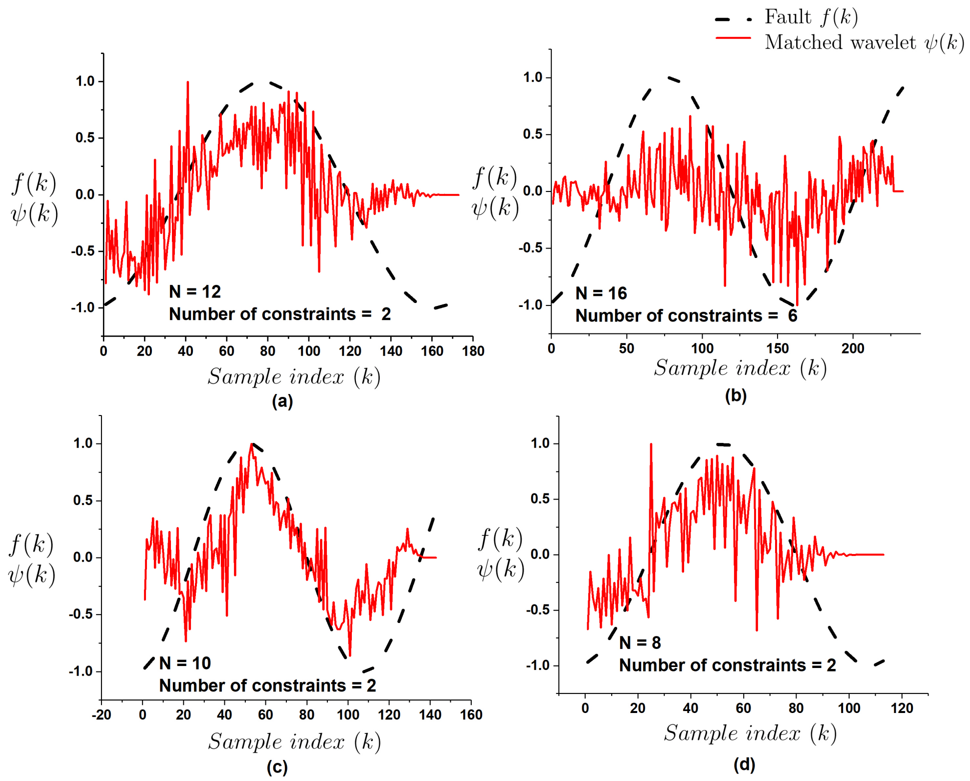

Figure 2.9: Fault waveform (L-G fault) and matched fault wavelets

2.7 Summary

In this chapter matched wavelets are designed for inrush and fault signals. The specific wavelets designed are tested for their efficacy vis-a-vis the standard mother wavelets. The comparison with standard wavelets clearly establishes usefulness of the developed matched wavelets in the discrimination of inrush and fault waveforms. The design of matched wavelets specific to given inrush and fault waveforms for differential protection is major contribution of this chapter.

The matching was done in least square sense. The DE algorithm is used to get an optimal matched wavelets. In course of finding matched wavelet we came across the fact that smoothness of matched wavelets and application for which matched wavelet is designed is independent to each other. This means that the designed matched wavelet for a desired signal can be smooth depending on the smoothening constraints applied. However, if the obtained matched wavelet is not smooth or oscillatory in nature even then it will retain all the qualities of a wavelet.

This chapter discusses about wavelet families and wavelet transform. The theory of

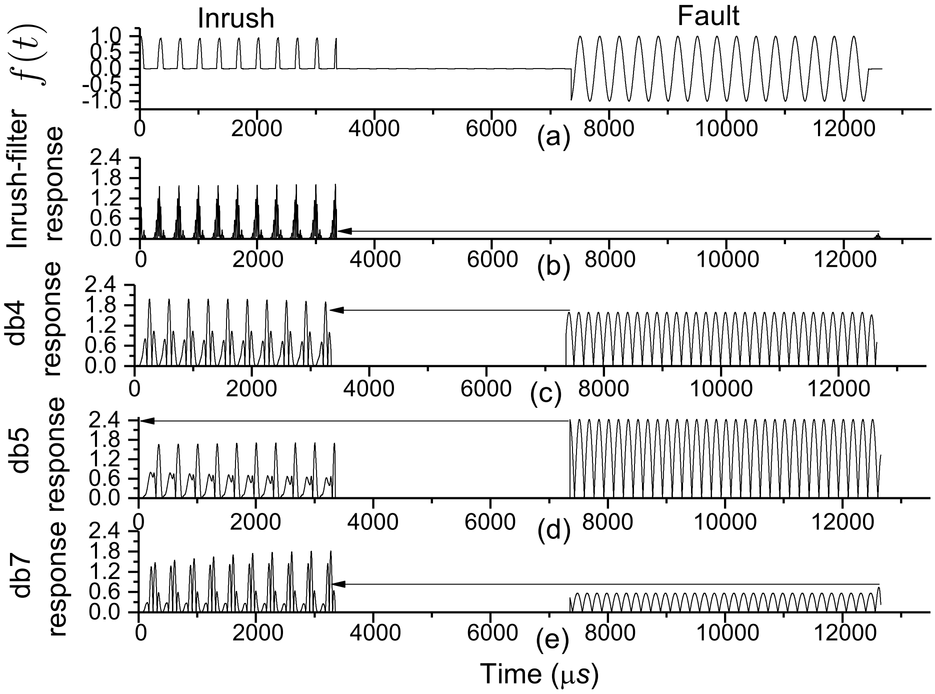

Figure 2.10: Comparison of detection efficiency of wavelets. (a) Normalized differential current waveform (b) Matched wavelet based inrush filter response (c) db4 wavelet based inrush filter response (d) db5 wavelet based inrush filter response (e) db7 based inrush filter wavelet response

matched wavelet and its mathematical relevance with matched filter is presented in this chapter. Finding matched wavelet for a specified signal is formulated as an optimization problem and DE is used to obtain optimal matched wavelet. The matched wavelets obtained for standard wavelets are shown in this chapter to verify the algorithm used for finding matched wavelet. The results of comparison of matched wavelets to that of standard wavelets used in literature are discussed in this chapter.