Chapter 2

An operational matrix for solving time-fractional order Cahn-Hilliard equation

2.1 Introduction

In this chapter, a sincere attempt has been taken to find the numerical solution of Cahn–Hilliard equation with given initial and boundary conditions. The historical model related to diffusion of fluid of the mathematical physics known as Cahn–Hilliard equation (C-H equation), which had been developed by J.W. Cahn and J. Hilliard [66]. The equation is related to the phenomena like phase ordering dynamics, spinodal decomposition and also with process of phase separation towards separation of components of binary fluid. The equation describes the time evolution of concerned fields as the interfaces between the phases have finite thickness in which gradual changes of composition are found. C.M. Elliott and Z. Songmu have discussed the Cahn-Hillard equation with the neural boundary conditions [67]. C.G. Gal and A. Miranville consider the case of a model of non-isothermal phase separation C-H equation with dynamic boundary conditions [68]. The energy functional for a particular configuration is given by Berti and Bochicchio [69].

$$F(u(x,t)) = V(u) - \frac{\gamma}{2}|\nabla u|^2, \tag{2.1}$$

where the potential function $V(u) = \frac{1}{4}(1-u^2)^2$ and γ is a constant given to penalize phase boundary.

Now from the equation of continuity and Fick's first law, we get

$$\frac{\partial u}{\partial t} = D\frac{\partial^2}{\partial x^2}(u^3 - u - \gamma\frac{\partial^2 u}{\partial x^2}). \tag{2.2}$$

2.2 Proposed Cahn-Hilliard model and its chemical behavior

Due to the property of diverse phenomena of the equation (2.2) in phase transition and its application in soft matter to complex fields the researchers are applying the equation in Navier-Stokes equation of fluid flow. This has motivated the author to study the physical behavior of the model (2.2) in fractional order system, which being non-Markovian in nature will generate Brownian motion and will have long term memory.

Thus the nonlocal C-H equation for fractional order system with the presence of advection and reaction terms is described in the following manner as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = D\frac{\partial^2}{\partial x^2}(u^3 - u - \gamma\frac{\partial^2 u}{\partial x^2}) + v\frac{\partial u}{\partial x} + k(1-u)u, 0 < \alpha \leq 1 \tag{2.3}$$

where $0 \leq x \leq 1$, and $0 \leq t \leq 1$ with the following initial and boundary conditions

$$\begin{aligned} u(x,0) &= (1-x)x, \\ u(1,t) &= 0, \\ u(0,t) &= 0. \end{aligned} \tag{2.4}$$

In the above expressions, $u(x,t)$ is the concentration of solute, D denotes the diffusion coefficient, v be the advection coefficient, k denotes the reaction coefficient.

The models of phase separation based on non-local C-H equation [66] describe the nucleation dynamics and coarsening dynamics [70] in a smooth manner. J. W. Cahn and J. E. Hilliard have developed a free energy function based on fundamental thermodynamics assuming that local free energy depends on the concentration of immediate environment and local concentration, which means they did not neglect the gradients of higher order of the concentration. From the variance of free energy with respect to density, they have calculated chemical potential. In a linear driving force approach, chemical potential is the driving force. They have developed a model based on phenomenological observations and statistical thermodynamics which describe binary phase decomposition by all means. Free energy of the system is given in terms of internal energy U as $F = U - TS$, where T and

S are temperature and entropy, respectively. Free energy of the system with volume V is given by

$$F = \int_V \phi(u, \nabla u, \nabla^2 u, \cdots) dV. \tag{2.5}$$

Assuming an isobaric and isothermal system, free energy density ϕ only depends on the concentrations and their derivatives. Here we can write results for free energy density from [66] as

$$\phi(u, \nabla^2 u) = \phi_0(u) + \frac{1}{2}\kappa(\nabla u)^2. \tag{2.6}$$

Thus equation (2.5) can be written as

$$F = \int_V (\phi_0(u) + \frac{1}{2}\kappa(\nabla u)^2) dV, \tag{2.7}$$

where ϕ_0 is the free energy density for a homogeneous mixture which depends only on the local concentration. Cahn and Hilliard [66] used it as

$$\phi_0(u) = \prod_n (\omega(T, u_n)u_n) + \kappa_B T \sum_n u_n \ln(u_n), \tag{2.8}$$

where T and κ_B are the temperature and Boltzmann constants, respectively. In the above equation there are two parts which represent energy U. Again $\omega(T)$ and κ include the contribution of internal energy and interface energy, respectively. Later part of above equation is the entropy.

2.3 Laguerre polynomials and its some properties

Laguerre polynomials, nowadays are widely focused in various branches of mathematics like complex analytic number theory, homotopy analysis, basic number theory, basic differential topology and different aspects of physics like quantum physics, etc.

Further, suppose I be an interval and the weight function $r(x) = e^{-x}$ in ordinary manner. Now, Define $L_r^2(I) = \{\phi | \phi$ is measurable on I and $\|\phi\|_r < \infty\}$, which is equipped by the following mentioned inner product operation and corresponding norm as $(\psi, \phi)_r = \int_I \psi(x)\phi(x)r(x)dx$, and $\|\phi\|_r = (\psi, \phi)_r^{\frac{1}{2}}$.

Now, the Laguerre polynomials of degree q is denoted and defined by

$$L_q(x) = \frac{1}{q!} e^x \partial_x^q (x^q e^{-x}), \quad q = 0, 1, \cdots; \tag{2.9}$$

The following recurrence relations are satisfied by Laguerre polynomial as

$$\begin{aligned} &\partial_x(xe^{-x}\partial_x L_q(x)) + qe^{-x}L_q(x) = 0, \quad x \in I, \\ &L_q(x) = \partial_x L_q(x) - \partial_x L_{q+1}(x), \quad q \geq 0, \\ &L_{q+1}(x) = \frac{(2q+1-x)L_q(x) - qL_{q-1}(x)}{q+1}, \ \ \forall q \geq 1. \end{aligned} \tag{2.10}$$

The collection of discussed Laguerre polynomials form the $L_r^2(I)$-orthogonal system, i.e.,

$$\int_I L_l(x)L_m(x)r(x)dx = \delta_{lm}, \quad \forall l, m \geq 0, \tag{2.11}$$

where δ_{lm} is the kronecher delta function.

Now, analytical outline of the Laguerre polynomials of degree n on the semi-half interval $I \equiv (0, \infty)$ is defined by

$$L_n(x) = \sum_{\rho=0}^{n} \frac{n!(-1)^\rho}{(-\rho+n)!(\rho!)^2} x^n, \quad n = 0, 1, \cdots ; \tag{2.12}$$

From above, the special value can be calculated in following manner, which can be used later.

$$D^\alpha L_m(0) = (-1)^\alpha \sum_{i=0}^{m-\alpha} \frac{(-i-1+m)!}{(m-i-\alpha)!(\alpha-1)!} \ , \tag{2.13}$$

where α is a natural number.

2.4 Laguerre operational matrix for fractional differentiation

Consider $u(x) \in L_r^2(I)$ can be written in the form of famous Laguerre polynomials as

$$u(x) = \sum_{m=0}^{\infty} u_m L_m(x), \tag{2.14}$$

where the constant coefficients are given by the following equation.

$$u_m = \int_0^\infty u(x)L_m(x)r(x)dx, \quad m = 0, 1, 2, \cdots ; \tag{2.15}$$

In customary, let us use mainly the starting $(N+1)-$terms of Laguerre polynomials. Thus

$$u(x) \equiv u_N(x) = \sum_{n=0}^{N} u_n L_n(x) = C^T \psi(x). \tag{2.16}$$

The coefficient of Laguerre vector C and the mentioned Laguerre vector in the above expressions are given as

$$C^T = [c_0, c_1, \cdots, c_N], \quad \psi(x) = [L_0(x), L_1(x), \cdots, L_N(x)]^T. \tag{2.17}$$

Similarly, we can approximate an arbitrary function $u(x,t)$ from $L_r^2(I) \times L_r^2(I)$ of two variables in the form of Laguerre polynomials as

$$u(x,t) = \sum_{l=0}^{N} \sum_{m=0}^{N} u_{lm} L_l(x) L_m(t), \tag{2.18}$$

where $V = [u_{lm}]$ and $u_{lm} = (L_l(x), (u(x,t), L_m(t)))$. Further, the differentiation of the Laguerre vector $\psi(x)$ can be written as

$$\frac{d\psi(x)}{dx} = G^{(1)} \psi(x). \tag{2.19}$$

In above $G^{(1)}$is an operational matrix of order $(N+1) \times (N+1)$ for the differentiation, which is given as

$$G^{(1)} = -\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 0 \end{bmatrix}. \tag{2.20}$$

Now, by the use of equation (2.19), we observe that

$$\frac{d^n \psi(x)}{dx^n} = (G^{(1)})^n \psi(x), \tag{2.21}$$

where n be the natural number and the superscript in $G^{(1)}$ denotes the powers of matrix. Hence, we have

$$G^{(m)} = (G^{(1)})^m, \ \ m = 1, 2, \cdots; \tag{2.22}$$

Theorem 2.1. *Let us suppose that $L_m(x)$ be the Laguerre polynomial of order m; then we have*

$$G^{\alpha}L_m(x) = 0, \quad m = 0, 1, \cdots, \lceil\alpha\rceil - 1, \quad \alpha > 0, \tag{2.23}$$

where $\lceil\alpha\rceil$ be the ceiling function.

Proof. This result can be easily proved by using the basic properties of fractional Caputo derivative given in the equation (1.9) of Chapter 1. □

In the following theorem, a step is taken to generalize an operational matrix for derivatives of Laguerre polynomials discussed in equation (2.19) for fractional order derivatives.

Theorem 2.2. *Let us suppose that $\psi(x)$ be the Laguerre vector which is defined in equation (2.17) and $\alpha > 0$; then we have*

$$G^{\alpha}\psi(x) = G^{(\alpha)}\psi(x), \tag{2.24}$$

where $G^{(\alpha)}$ is the operational matrix of order $(N+1)\times(N+1)$ for differentiation of order α in Caputo sense, and is constructed as

$$G^{(\alpha)} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ S_\alpha(\lceil\alpha\rceil,0) & S_\alpha(\lceil\alpha\rceil,1) & S_\alpha(\lceil\alpha\rceil,2) & \cdots & S_\alpha(\lceil\alpha\rceil,N) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ S_\alpha(i,0) & S_\alpha(i,1) & S_\alpha(i,2) & \cdots & S_\alpha(i,N) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ S_\alpha(N,0) & S_\alpha(N,1) & S_\alpha(N,2) & \cdots & S_\alpha(N,N) \end{bmatrix}, \tag{2.25}$$

where

$$S_\alpha(a,m) = \sum_{\nu=\lceil\alpha\rceil}^{a}\sum_{l=0}^{m}\frac{(-1)^{\nu+l}a!m!\Gamma(\nu-\alpha+l+1)}{(a-\nu)!\Gamma(\nu-\alpha+1)\nu!(m-l)!(l!)^2}. \tag{2.26}$$

Proof. We are going to derive the operational matrix of orthogonal Laguerre polynomials. On solving the equations (2.14)-(2.15) and equation (2.12), we get the following expression as

$$G^{\alpha}L_a(x) = \sum_{\nu=0}^{a}\frac{(-1)^{\nu}a!G^{\alpha}x^{\nu}}{(a-\nu)!(\nu!)^2} = \sum_{\nu=\lceil\alpha\rceil}^{a}\frac{(-1)^{\nu}a!x^{\nu-\alpha}}{(a-\nu)!\Gamma(\nu-\alpha+1)\nu!}, \quad a = \lceil\alpha\rceil, \cdots, N. \tag{2.27}$$

Now, writing the term $x^{\nu-\alpha}$ in the form of Laguerre polynomials as

$$x^{\nu-\alpha} = \sum_{b=0}^{N} k_m L_m(x), \tag{2.28}$$

where k_m is determined from the equation (2.16) by taking $u(x) = x^{\nu-\alpha}$ as

$$k_m = \sum_{l=0}^{m} \frac{(-1)^l m!\Gamma(\nu-\alpha+l+1)}{(m-l)!(l!)^2} \tag{2.29}$$

Now, employing the equations (2.27) – (2.29), we have

$$G^{\alpha}L_a(x) = \sum_{m=0}^{N} S_\alpha(a,m)L_m(x), \quad a = \lceil\alpha\rceil, \cdots, N, \tag{2.30}$$

where $S_\alpha(a,m) = \sum_{\nu=\lceil\alpha\rceil}^{a}\sum_{l=0}^{m}\frac{(-1)^{\nu+l}a!m!\Gamma(\nu-\alpha+l+1)}{(a-\nu)!\Gamma(\nu-\alpha+1)\nu!(m-l)!(l!)^2}$.

We can write the equation (2.30) in form of the vector as

$$G^{\alpha}L_a(x) = [S_\alpha(a,0), S_\alpha(a,1), \cdots, S_\alpha(a,N)]\psi(x), \quad a = \lceil\alpha\rceil, \cdots, N. \tag{2.31}$$

Now using Theorem 2.1, we have

$$G^{\alpha}L_a(x) = [0,0,\cdots,0]\psi(x), \quad a = 0,1,\cdots,\lceil\alpha\rceil - 1. \tag{2.32}$$

Using the equations (2.31) – (2.32), we can easily write the following expression of $G^{(\alpha)}$ as

$$G^{(\alpha)} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ S_\alpha(\lceil\alpha\rceil,0) & S_\alpha(\lceil\alpha\rceil,1) & S_\alpha(\lceil\alpha\rceil,2) & \cdots & S_\alpha(\lceil\alpha\rceil,N) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ S_\alpha(i,0) & S_\alpha(i,1) & S_\alpha(i,2) & \cdots & S_\alpha(i,N) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ S_\alpha(N,0) & S_\alpha(N,1) & S_\alpha(N,2) & \cdots & S_\alpha(N,N) \end{bmatrix}, \tag{2.33}$$

where

$$S_\alpha(a,m) = \sum_{\nu=\lceil\alpha\rceil}^{a}\sum_{l=0}^{m}\frac{(-1)^{\nu+l}a!m!\Gamma(\nu-\alpha+l+1)}{(a-\nu)!\Gamma(\nu-\alpha+1)\nu!(m-l)!(l!)^2}. \tag{2.34}$$

Here, it is noticed that in $G^{(\alpha)}$, the starting $\lceil\alpha\rceil$ rows vanish.

Note. If $\alpha = n \in \mathbb{N}$, the result of Theorem 2.2 will be reduced to equation (2.20). □

2.5 The proposed method for Laguerre operational matrix of fractional differentiation

In this section of the chapter, the main goal is to propose the collocation method in a suitable way, based on Laguerre operational matrix, so that the method can be implemented efficiently to solve our concerned model.

Now, we shall approximate $u(x,t)$ in the form of Laguerre polynomials in the following manner as

$$u(x,t) = \sum_{a=0}^{N}\sum_{b=0}^{N} c_{ab} L_a(x) L_b(t), \tag{2.35}$$

where c_{ab}'s are the unknown coefficients and will be calculated later for $a = 1, 2, 3, \cdots$; and $b = 1, 2, 3, \cdots$; The equation (2.35) can be rewritten as

$$u(x,t) = \phi^T(x) \cdot C \cdot \phi(t), \tag{2.36}$$

where $C = [c_{ab}]$ is a matrix of order $(N+1) \times (N+1)$ of arbitrary unknown constant coefficients and $\phi(t) = (L_0(t), L_1(t), \cdots, L_N(t))^T$ is a column vector.

Now, using fractional differentiation of order β with respect to x on (2.35) and using Theorem (2.2), we get

$$\frac{\partial^\beta u}{\partial x^\beta} = G^\beta u(x,t) = G^\beta \phi^T(x) \cdot C \cdot \phi(t). \tag{2.37}$$

Similarly,

$$\frac{\partial^\alpha u}{\partial t^\alpha} = G^\alpha u(x,t) = \phi^T(x) \cdot C \cdot G^\alpha \phi(t). \tag{2.38}$$

Now from boundary conditions (2.4) with the aid of equation (2.3), we get

$$\begin{aligned} &\phi^T(x).C.\phi(0) = x(1-x), \\ &\phi^T(1).C.\phi(t) = 0, \\ &\phi^T(0).C.\phi(t) = 0. \end{aligned} \tag{2.39}$$

Now let us collocate equation (2.3) with the help of equations (2.39) at points $x_a=\frac{a}{N}$ for $a = 0, 1, 2, \cdots, N$ and $t_a=\frac{a}{N}$ for $a = 0, 1, 2, \cdots, N$. After collocating, a system of non-linear equations is obtained. Further by simplifying this system of equations and finding C, we obtain numerical solution of the given problem by substituting C in equation (2.36).

2.6 Convergence analysis of the proposed approximation

The special attention given in this section is to estimate upper bound of the error of proposed approximation.

Theorem 2.3. *The global uniform bound for generalized Laguerre polynomial $L_m^{(\omega)}(t)$ is estimated as*

$$| L_m^{(\omega)}(t) | \leq \begin{cases} \frac{(\omega+1)_m}{m!} e^{\frac{t}{2}}, & \text{if } \omega \geq 0, t \geq 0, m = 0, 1, 2, \cdots; \\ (2 - \frac{(\omega+1)_m}{m!}) e^{\frac{t}{2}}, & \text{if } -1 < \omega \leq 0, t \geq 0, m = 0, 1, 2, \cdots; \end{cases} \tag{2.40}$$

where $(a)_m = a(a+1)(a+2)\cdots(a+m-1)$, is known as Pochhammer symbol.

Proof. These global uniform bounds are given in [71], also the estimate is discussed in [72, 73]. □

Remark. The usual Laguerre polynomial is the special case of the generalized Laguerre polynomial for $\omega = 0$, i.e.,

$$L_m^{(0)}(t) = L_m(t). \tag{2.41}$$

For the Laguerre polynomial the global uniform bounds (2.40) takes the form

$$| L_m(t) | \leq \frac{1}{m!} e^{\frac{t}{2}}, \quad t \geq 0, \quad m = 0, 1, 2, \cdots; \tag{2.42}$$

Theorem 2.4. *Suppose $u(x,t)$ be the sufficiently smooth function on the region A, $(\frac{\partial^\alpha u}{\partial t^\alpha})_N$ be the approximation of $(\frac{\partial^\alpha u}{\partial t^\alpha})$. Then the error in approximating $(\frac{\partial^\alpha u}{\partial t^\alpha})$ by $(\frac{\partial^\alpha u}{\partial t^\alpha})_N$ is bounded by*

$$| E_r(N) | \leq \sum_{a=N+1}^{\infty} \sum_{b=N+1}^{\infty} \frac{c_{ab}}{a! \cdot m!} \chi_{mbN} e^{\frac{x+t}{2}}, \quad x, t \geq 0, \quad a, b = 0, 1, 2, \cdots; \tag{2.43}$$

where $\chi_{mbN} = \sum_{m=0}^{N} S_\alpha(b, m)$.

Proof. In view of equation (2.35), we have

$$u(x,t)=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}c_{ab}L_a(x)L_b(t),\;\; v(x,t)=\sum_{h=0}^{\infty}\sum_{k=0}^{\infty}c_{hk}^{'}L_h(x)L_k(t). \tag{2.44}$$

Truncating it upto $N+1$ term, we have

$$u_N(x,t)=\sum_{a=0}^{N}\sum_{b=0}^{N}c_{ab}L_a(x)L_b(t),\;\; v_N(x,t)=\sum_{h=0}^{N}\sum_{k=0}^{N}c_{hk}^{'}L_h(x)L_k(t). \tag{2.45}$$

We can write α order partial derivative of $u(x,t)$ and $u_N(x,t)$ w.r.to t as

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}c_{ab}L_a(x)\frac{\partial^\alpha L_b(t)}{\partial t^\alpha},\;\; (\frac{\partial^\alpha u}{\partial t^\alpha})_N=\sum_{a=0}^{N}\sum_{b=0}^{N}c_{ab}L_a(x)\frac{\partial^\alpha L_b(t)}{\partial t^\alpha}. \tag{2.46}$$

From the above equation, we can write

$$E_r(N)=\frac{\partial^\alpha u}{\partial t^\alpha}-(\frac{\partial^\alpha u}{\partial t^\alpha})_N=\sum_{a=N+1}^{\infty}\sum_{b=N+1}^{\infty}c_{ab}L_a(x)\frac{\partial^\alpha L_b(t)}{\partial t^\alpha}. \tag{2.47}$$

Using derivative of Laguerre polynomials, above equation reduces to

$$\mid E_r(N)\mid=\mid\sum_{a=N+1}^{\infty}\sum_{b=N+1}^{\infty}c_{ab}L_a(x)(\sum_{m=0}^{N}S_\alpha(b,m)L_m(t))\mid, \tag{2.48}$$

or,

$$\mid E_r(N)\mid=\sum_{a=N+1}^{\infty}\sum_{b=N+1}^{\infty}c_{ab}\cdot\chi_{mbN}\mid L_m(t)\mid\cdot\mid L_a(t)\mid. \tag{2.49}$$

Applying equation (2.42), we get

$$\mid E_r(N)\mid\leq\sum_{a=N+1}^{\infty}\sum_{b=N+1}^{\infty}c_{ab}\cdot\chi_{mbN}\cdot\frac{1}{m!}e^{\frac{t}{2}}\cdot\frac{1}{a!}e^{\frac{x}{2}}, \tag{2.50}$$

or,

$$\mid E_r(N)\mid\leq\sum_{a=N+1}^{\infty}\sum_{b=N+1}^{\infty}\frac{c_{ab}}{a!\cdot m!}\chi_{mbN}e^{\frac{x+t}{2}},\;\; x,t\geq 0,\;\; a,b=0,1,2,\cdots; \tag{2.51}$$

Thus we have the required proof. □

2.7 Error analysis of proposed scheme

In this section of the chapter, the Laguerre polynomial operational matrix method is applied for the fractional order differentiation to solve some fractional order one dimensional partial differential equations (linear/non-linear) to illustrate the accuracy and applicability of the proposed scheme and compared the obtained results with the exact solutions of the given examples. All the numerical commutations are carried out by using the software Mathematica 11.3.

Example 1. Let us consider a linear fractional order partial differential equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 2x, \tag{2.52}$$

under the prescribed initial and boundary conditions given by

$$\begin{aligned} u(x,0) &= 0, \\ u(0,t) &= 0, \\ u(1,t) &= t^2, \end{aligned} \tag{2.53}$$

whose exact solution is $u(x,t) = xt^2$. The absolute error between the exact solution and the numerical solution can be easily analyzed through the Fig. 2.1.

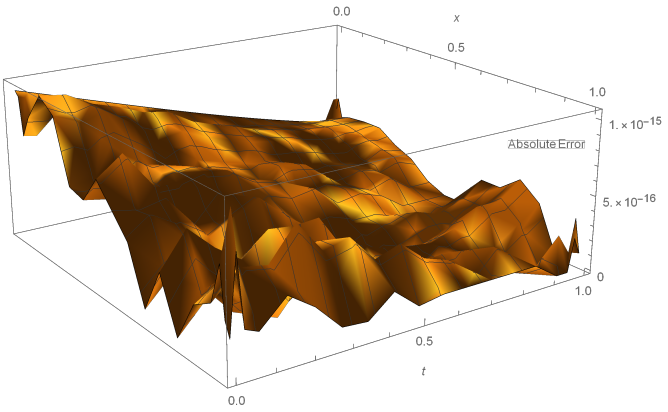

FIGURE 2.1: Plots of the absolute error between the exact and numerical solutions vs. x and t.

The absolute errors between the results obtained by our proposed numerical scheme and the exact solutions are exhibited through Table 2.1 for $t = 0.5$ and $0 \leq x \leq 1$ for the order of the approximation $N = 4, 5, 6$. The tables clearly confirm that the order of convergence for the proposed method increases as the degree of Laguerre polynomials in x increases. Again absolute error decreases with the increase of the order of approximation of the polynomial N, which clearly shows the effectiveness of the proposed numerical scheme.

TABLE 2.1: Efficiency of the numerical method for $u(x,t)$ at t=0.5

t	N	Maximal absolute error $0 \leq x \leq 1$	Order of convergence
0.5	4	1.18×10^{-10}	-
	5	3.40×10^{-11}	5.17
	6	8.21×10^{-12}	7.79

Example 2. Let us consider a linear fractional order partial differential equation

$$\frac{\partial^{\frac{3}{2}} u}{\partial t^{\frac{3}{2}}} + \frac{\partial^{\frac{1}{2}} u}{\partial x^{\frac{1}{2}}} = \frac{4\sqrt{t}x}{\sqrt{\pi}} + \frac{2\sqrt{x}t^2}{\sqrt{\pi}}, \tag{2.54}$$

with the initial and boundary conditions as

$$\begin{aligned} u(x,0) &= 0, \\ u(0,t) &= 0, \\ u(1,t) &= t^2, \end{aligned} \tag{2.55}$$

whose exact solution is $u(x,t) = xt^2$. The absolute error between the solutions is given in Fig. 2.2.

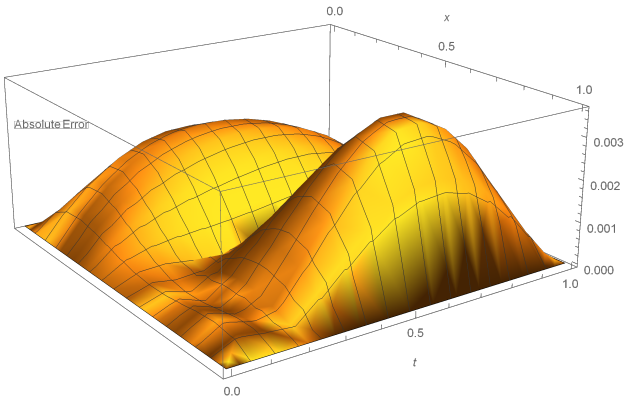

FIGURE 2.2: Plots of the absolute error between the exact and numerical solutions vs. x and t.

The absolute errors between the results obtained by our proposed numerical scheme and the exact solutions are exhibited through Table 2.2 for $t = 0.5$ and $0 \leq x \leq 1$ for the order of the approximation $N = 3, 5, 7$. The tables clearly confirm that the order of convergence for the proposed method increases as the degree of Laguerre polynomials in x increases. Again absolute error decreases with the increase of the order of approximation of the polynomial N, which clearly shows the effectiveness of the proposed numerical scheme.

TABLE 2.2: Efficiency of the numerical method for $u(x,t)$ at t=0.5

t	N	Maximal absolute error $0 \leq x \leq 1$	Order of convergence
0.5	3	2.7×10^{-3}	-
	5	1.8×10^{-4}	5.30
	7	1.0×10^{-5}	8.59

Example 3. The one dimensional non-linear partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \tag{2.56}$$

with the initial and boundary conditions as

$$\begin{aligned} u(x,0) &= \frac{1}{4}(1 - \tanh[\frac{x}{2\sqrt{6}}])^2, \\ u(0,t) &= \frac{1}{4}(1 - \tanh[\frac{1}{2\sqrt{6}}(-\frac{5t}{\sqrt{6}})])^2, \\ u(1,t) &= \frac{1}{4}(1 - \tanh[\frac{1}{2\sqrt{6}}(1 - \frac{5t}{\sqrt{6}})])^2, \end{aligned} \tag{2.57}$$

has the exact solution $u(x,t) = \frac{1}{4}(1 - \tanh[\frac{1}{2\sqrt{6}}(x - \frac{5t}{\sqrt{6}})])^2$. The absolute error between solutions is depicted through Fig. 2.3.

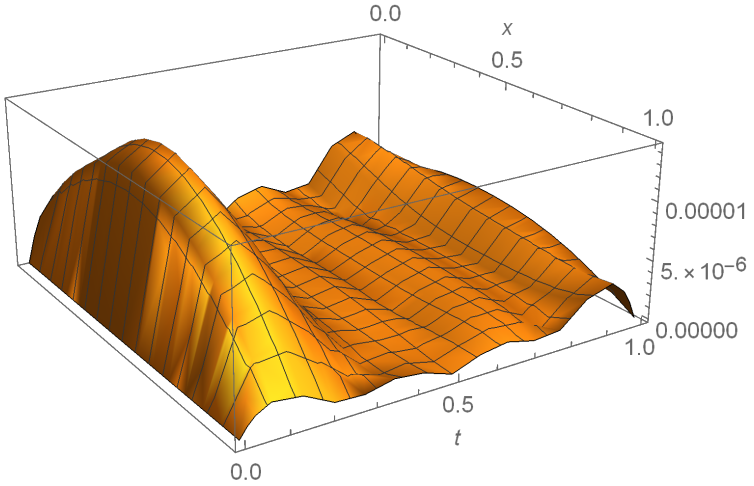

FIGURE 2.3: Plots of the absolute error between the exact and numerical solutions vs. x and t.

The absolute errors between the results obtained by our proposed numerical scheme and the exact solutions are exhibited through Table 2.3 for $t = 0.5$ and $0 \leq x \leq 1$ for the order of the approximation $N = 3, 5, 7$. The tables clearly confirm that the order of convergence for the proposed method increases as the degree of Laguerre polynomials in x increases. Again absolute error decreases with the increase of the order of approximation of the polynomial N, which clearly shows the effectiveness of the proposed numerical scheme.

TABLE 2.3: Efficiency of the numerical method for $u(x,t)$ at t=0.5

t	N	Maximal absolute error $0 \leq x \leq 1$	Order of convergence
0.5	3	4.2×10^{-4}	-
	5	2.7×10^{-5}	5.37
	7	1.9×10^{-6}	7.88

Example 4. The one dimensional non-linear time-fractional diffusion equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}, \tag{2.58}$$

with the initial and boundary conditions as

$$\begin{aligned} u(x,0) &= x^2, \\ u(0,t) &= 0, \\ u(1,t) &= E_\alpha(t^\alpha), \end{aligned} \tag{2.59}$$

has the exact solution $u(x,t) = x^2 E_\alpha(t^\alpha)$ [57]. The absolute errors between the results obtained by our proposed numerical scheme and the exact solutions are exhibited through Table 2.4 for $t = 0.5$ and $0 \leq x \leq 1$ for the order of the approximation $N = 3, 5, 7$ and $\alpha = 0.9$. The tables clearly confirm that the order of convergence for the proposed method increases as the degree of Laguerre polynomials in x increases. Again absolute error decreases with the increase of the order of approximation of the polynomial N, which clearly shows the effectiveness of the proposed numerical scheme.

TABLE 2.4: Efficiency of the numerical method for $u(x,t)$ at t=0.5

t	N	Maximal absolute error $0 \leq x \leq 1$	Order of convergence
0.5	3	3.2×10^{-3}	-
	5	2.1×10^{-4}	5.33
	7	7.8×10^{-6}	9.78

In the Table 2.5 the comparison between the existing method and present method for various values of spatial variable x for $\alpha = 0.9$ at $t = 1$ is given which justify the accuracy and efficiency of the proposed method.

TABLE 2.5: Comparison of the proposed method with existing method at $t = 1$

x	Present Method	Bayrak et al.[57]
0	0	0
0.2	3.27×10^{-4}	1.16×10^{-2}
0.4	6.93×10^{-3}	1.74×10^{-2}
0.6	9.04×10^{-3}	1.74×10^{-2}
0.8	1.03×10^{-4}	1.16×10^{-2}
1	0	0

2.8 Results and discussion

The variations of the solute concentration $u(x,t)$ vs. the column length x at $t = 1$ for various values of the fractional order time parameter for conservative case ($k = 0$) and non-conservative case ($k \neq 0$) in the absence/presence of the advection term are found numerically taking $\gamma = 1$ and the obtained results are displayed through Figs. 2.4-2.7.

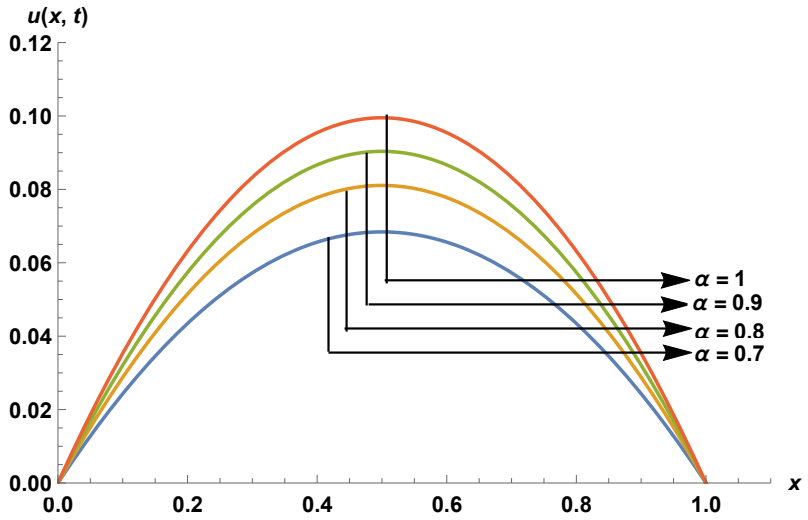

FIGURE 2.4: Plots of field variable $u(x,t)$ vs. x at $t = 1$ for $k = 0, v = 0$ and different values of α.

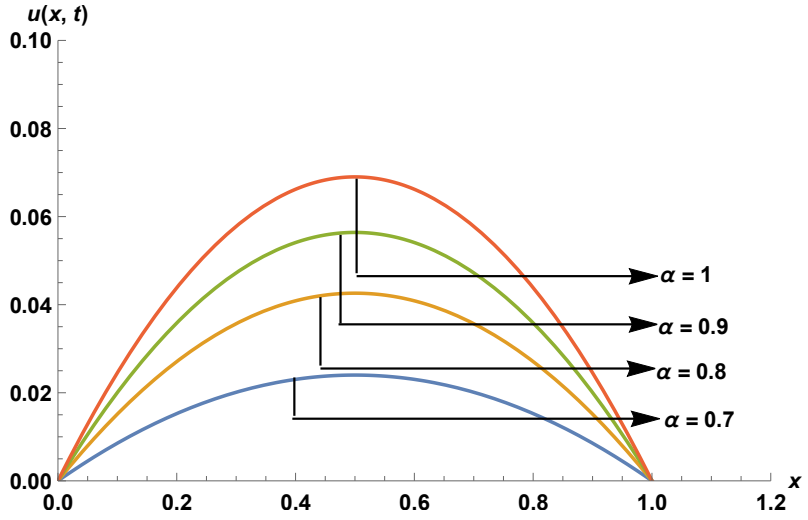

FIGURE 2.5: Plots of field variable $u(x,t)$ vs. x at $t = 1$ for $k = -1, v = 0$ and different values of α.

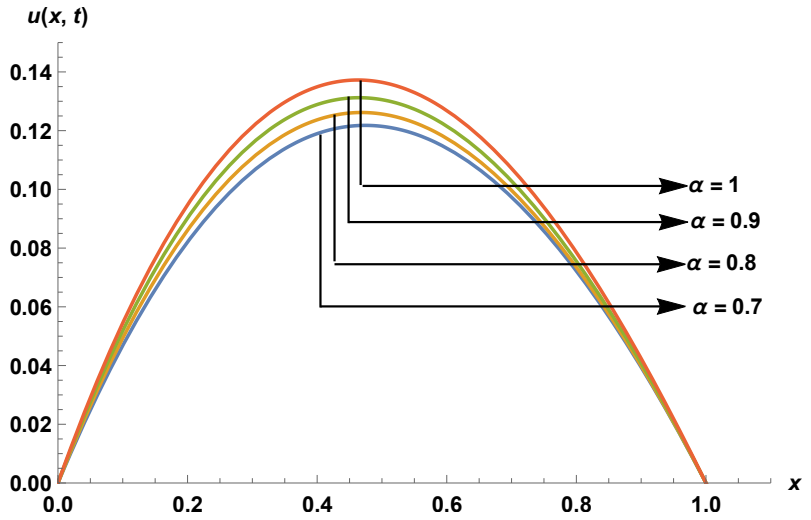

FIGURE 2.6: Plots of field variable $u(x,t)$ vs. x at $t=1$ for $k=0, v=-1$ and different values of α.

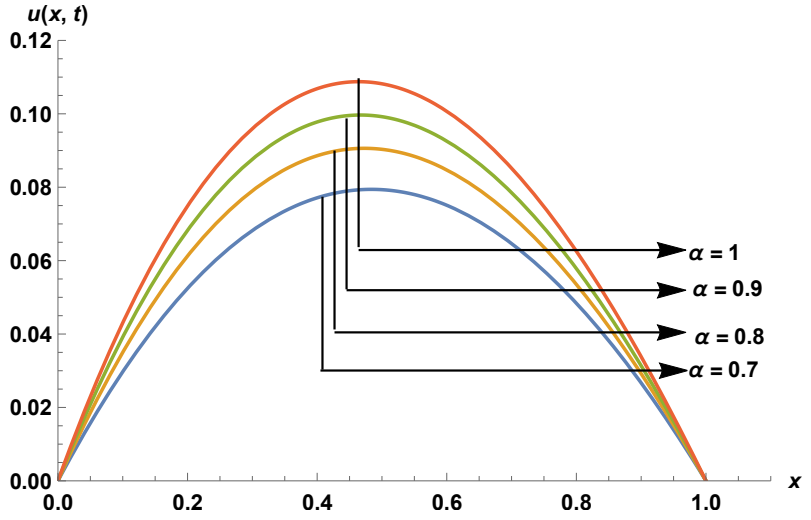

FIGURE 2.7: Plots of field variable $u(x,t)$ vs. x at $t=1$ for $k=-1, v=-1$ and different values of α.

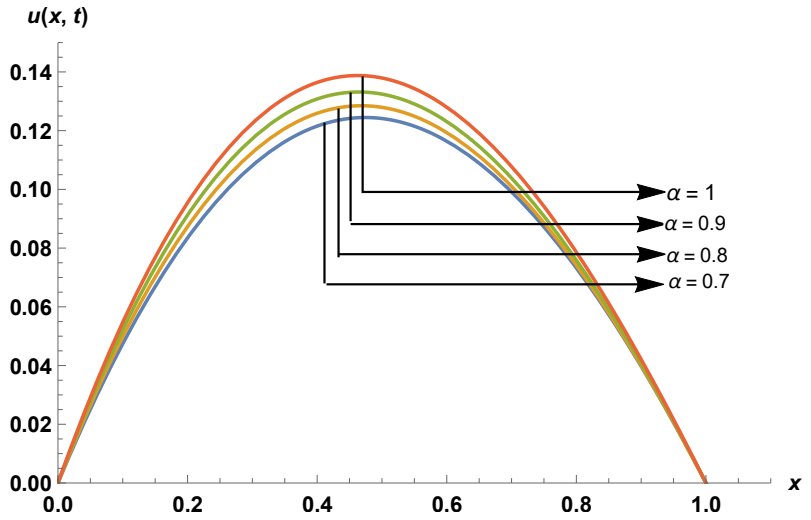

FIGURE 2.8: Plots of field variable $u(x,t)$ vs. x at $t = 1$ for $k = -1, v = -1, \gamma = 0.5$ and different values of α.

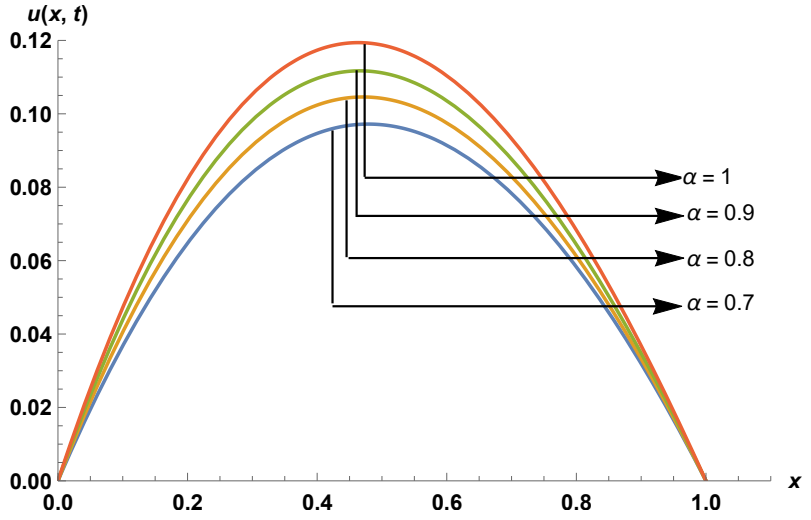

FIGURE 2.9: Plots of field variable $u(x,t)$ vs. x at $t = 1$ for $k = -1, v = -1, \gamma = 0.75$ and different values of α

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The effect of reaction term on the solution profile during the absences of advection term for various values of the fractional order time derivative and also those during the presence of advection term can be found through numerical results given in Figs. 2.4-2.5 and Figs. 2.6-2.7, respectively. It is seen that the overshoots of sub-diffusion are decreased as the system approaches from standard order to fractional order. It is also seen from the figures that damping are found in both cases due to the presence of sink term. Again the overshoots of the probability density function $u(x,t)$ increases due to the presence of advection term. It

is also seen from Figs. 2.8-2.9 that as γ decreases the overshoots of sub-diffusion increases for various α. This can be physically interpreted as the concentration increases with the increase of the separation of the transition regions between the domains.

2.9 Conclusions

The present scientific contribution has achieved three important goals. First one is finding the numerical solution of the solute concentration $u(x,t)$ of famous Cahn-Hilliard equation of integer order as well as fractional order using the powerful and efficient technique Laguerre operational matrix. Second one is the pictorial presentations of the nature of overshoots during sub-diffusion due to presence of advection and reaction terms. The third one is the graphical presentations of the damping nature of the solute concentration when the system approaches from standard order to fractional order in the presence of sink term.